Lecture Series and Convergence (Section 9.2)
The Integral Test (Section 9.3)

Thm Properties of Series

IF $\sum_{n=1}^{\infty} a_n = A$ and $\sum_{n=1}^{\infty} b_n = B$ THEN

1. $\sum_{n=1}^{\infty}(a_n \pm b_n) = A \pm B$

2. $\sum_{n=1}^{\infty} Ca_n = CA$ for all $C \in \mathbb{R}$

Prop

$\sum_{n=1}^{\infty} a_n$ Converges if and only if $\sum_{n=k}^{\infty} a_n$ Converges

$(a_n + a_2 + \dots)$ $\qquad$ $\sum_{n=k}^{\infty} a_n = (a_k + a_{k+1} + \dots)$

Example

$$\sum_{n=3}^{\infty} \frac{2^{2n} + 3^{n-2}}{6^{n+1}} = \sum_{n=3}^{\infty} \frac{4^n + 3^{-2}\cdot 3^n}{6 \cdot 6^n}$$

$$= \sum_{n=3}^{\infty}\left(\frac{1}{6}\left(\frac{4}{6}\right)^n + \frac{3^{-2}}{6}\left(\frac{3}{6}\right)^n\right)$$

$$= \sum_{n=3}^{\infty}\left(\frac{1}{6}\left(\frac{2}{3}\right)^n + \frac{3^{-2}}{6}\left(\frac{1}{2}\right)^n\right)$$

$$\frac{1}{6}\sum_{n=3}^{\infty}\left(\frac{2}{3}\right)^n + \frac{1}{54}\sum_{n=3}^{\infty}\left(\frac{1}{2}\right)^n$$

$$\Rightarrow \sum_{n=3}^{\infty}\left(\frac{2}{3}\right)^n = \sum_{n=0}^{\infty}\left(\frac{2}{3}\right)^n - \left(\frac{2}{3}\right)^0 - \left(\frac{2}{3}\right)^1 - \left(\frac{2}{3}\right)^2 \dots$$

OR

$$\Rightarrow \sum_{n=3}^{\infty}\left(\frac{2}{3}\right)^n = \left(\frac{2}{3}\right)^3 + \left(\frac{2}{3}\right)^4 \dots$$

$$\Rightarrow \left(\frac{2}{3}\right)^3 + \left[1 + \left(\frac{2}{3}\right) + \left(\frac{2}{3}\right)^2 \dots\right]$$

$$\Rightarrow \left(\frac{2}{3}\right)^2 \sum_{n=0}^{\infty}\left(\frac{2}{3}\right)^n$$

Consider :

$= \frac{1}{6} \cdot (2/3)^3 \sum_{n=0}^{\infty} (2/3)^n + 1/54 \cdot 1/2^3 \sum_{n=0}^{\infty} (1/2)^n$

$= 1/6 \cdot 8/27 \cdot \frac{1}{1-2/3} + 1/54 \cdot 1/8 \cdot \frac{1}{1-1/2}$

$\hookrightarrow$ etc.

<u>Thm</u> (n^{th} term test for divergence)

If $\lim_{n\to\infty} a_n \neq 0$ Then $\sum_{n=1}^{\infty} a_n$ diverges

Examples:

$\sum_{n=1}^{\infty} 1$ $\quad a_n = 1 \Rightarrow \lim_{n\to\infty} a_n \Rightarrow \lim_{n\to\infty} 1 = 1 \neq 0$

$\Rightarrow \sum 1$ div by the n^{th} term test for div

(2) $\sum_{n=1}^{\infty} \frac{n}{n+1}$

$a_n = \frac{n}{n+1}$

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n}{n+1} \cdot \frac{1/n}{1/n}$

$= \lim_{n\to\infty} \frac{1}{1 + \cancel{1/n}^{\,0}} = 1 \neq 0$

So, $\sum \frac{n}{n+1}$ div

(3) $\sum_{n=1}^{\infty} n \sin(1/n)$

$= 1 \neq 0$

$\Rightarrow \sum_{n=1}^{\infty} n \sin 1/n$ div.

$a_n = n \sin(1/n)$

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} n \sin(1/n) = \lim_{n\to\infty} \frac{\sin(1/n)}{(1/n)}$

$\lim_{x\to\infty} \frac{\sin(1/x)}{1/x} = \lim_{x\to\infty} \frac{\cos(1/x)\cdot(-1/x^2)}{(-1/x^2)} = \cos 0 = 1$

* CAN ONLY APPLY L'HOSPITAL'S RULE TO FUNCTION OF x

Thm (The Integral Test)

Let f be positive and decreasing on $[1, \infty)$, and let $a_n = f(n)$. Then

$$\sum_{n=1}^{\infty} a_n \quad \text{and} \quad \int_1^{\infty} f(x)\,dx$$

Either both converge or both diverge

Examples

1. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{1+n^2}$

$$a_n = \frac{1}{1+n^2}$$

$$f(x) = \frac{1}{1+x^2} \quad = \quad f(x) \geq 0 \quad \text{for all } x$$

$$f'(x) = \frac{-2x}{(1+x^2)^2} \quad \Rightarrow \quad f'(x) \leq 0 \quad \text{for } x \geq 0$$

$$\Rightarrow \quad f \text{ decreasing for } x \geq 0$$

$$\int_1^{\infty} f(x)\,dx = \int_1^{\infty} \frac{1}{1+x^2}\,dx = \lim_{b\to\infty} \int_1^b \frac{1}{1+x^2}\,dx$$

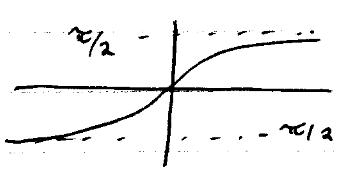
NOTE: $\displaystyle\int \frac{1}{1+x^2}\,dx = \boxed{\arctan x + C}$ $\Rightarrow$ $\displaystyle\lim_{b\to\infty} \left(\arctan x \Big|_1^b \right)$

$\Rightarrow \displaystyle\lim_{b\to\infty} (\arctan b - \arctan 1)$

$\pi/2 - \pi/4$

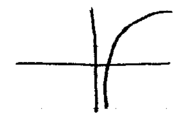
$\Rightarrow \pi/4$ converges

$\Rightarrow \sum \frac{1}{1+n^2}$ conv. by the integral test



Examples :

① $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$; $a_n = \frac{1}{n \ln n}$



$f(x) = \frac{1}{x \ln x} \Rightarrow f(x) \geq 0$ for $x \geq 1$

$f'(x) = \frac{-1}{(x \ln x)^2} (1 \cdot \ln x + x \cdot 1/x) = \frac{-1}{(x \ln x)^2} (\ln x + 1)$

$\Rightarrow f$ decreasing for $x \geq 0$

$$\int_2^{\infty} f(x)\,dx = \int_2^{\infty} \frac{1}{x \ln x}\,dx = \lim_{b \to \infty} \int_2^b \frac{1}{x \ln x}\,dx$$

$$\boxed{\begin{aligned} u &= \ln x \\ du &= 1/x \; dx \\ dx &= du \cdot x \end{aligned}}$$

$$= \lim_{b \to \infty} \int_2^b \frac{1}{u}\,du \quad (\Rightarrow \ln|u| + c)$$

$$= \lim_{b \to \infty} \left(\ln|\ln x| \Big|_2^b \right)^2$$

$$= \lim_{b \to \infty} \left(\ln|\ln b| - \ln|\ln 2| \right)$$

$$= \infty$$

L'Hospital Rule: (Section 8.7)

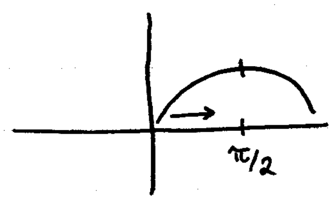
1) $$\lim_{x \to 0} \frac{\arctan x}{\tan x} = \frac{0}{0}$$

$$\lim_{x \to 0} \frac{\frac{1}{1+x^2}}{\sec^2 x} = \frac{1}{1} = 1$$

2) $$\lim_{x \to 1^-} \frac{\cos(\pi/2\, x)}{x-1} = \frac{0}{0}$$

$$\lim_{x \to 1^-} \cancel{\cos} \frac{-\sin(\pi/2\, x) \cdot \pi/2}{1} \Rightarrow -\pi/2$$

3) $$\lim_{x \to 1^-} \frac{\sin(\pi/2\, x)}{x-1} = -\infty$$



4) $$\lim_{x \to \infty} x^{1/3x} \Rightarrow \infty^0 \quad \text{(indeterminate)}$$

$$y = \lim_{x \to \infty} x^{1/3x}$$

$$\ln y = \lim_{x \to \infty} \ln\left(x^{1/3x}\right) = \lim_{x \to \infty} \cancel{\ln 3} \frac{\ln 3}{3x} \left(\frac{\infty}{\infty}\right) \Rightarrow \lim_{x \to \infty} \frac{1/x}{3} = 0$$

$$\ln y = 0 \Rightarrow e^0 = 1$$

$$\Rightarrow \lim_{x \to \infty} x^{1/3x} = 1$$

Improper Integrals

1) $$\int_0^\infty x e^{-2x}\, dx = \lim_{b \to \infty} \int_0^b x e^{-2x}\, dx \Rightarrow \int x e^{-2x}\, dx$$

where:

$u = x \Rightarrow u' = x$

$v' = e^{-2x} \Rightarrow v = -\frac{1}{2} e^{-2x}$

$$\boxed{\ldots \Rightarrow \frac{-x}{2e^{2x}} - \frac{1}{4e^{2x}} + C}$$

$$\lim_{b \to \infty} \cancel{\int_0^b} \left[\frac{-x}{2e^{2x}} - \frac{1}{4e^{2x}} \cancel{+C}\right]\Bigg|_0^b$$

$$\lim_{b \to \infty} \left[\frac{-x}{2e^{2b}} - \frac{1}{4e^{2b}} + 1/4\right] \Rightarrow \ldots 1/4$$

$$\int_{-\infty}^{\infty} \frac{1}{e^x+e^{-x}}\,dx \Rightarrow \int_{-\infty}^{0} \frac{1}{e^x+e^{-x}}\,dx + \int_0^{\infty} \frac{1}{e^x+e^{-x}}\,dx$$

$$\int_{-\infty}^{0} \frac{1}{e^x+e^{-x}}\,dx = \lim_{a\to-\infty} \int_a^0 \frac{1}{e^x+e^{-x}}\,dx$$

$$\int_0^{\infty} \frac{1}{e^x+e^{-x}}\,dx = \lim_{b\to\infty} \int_0^b \frac{1}{e^x+e^{-x}}\,dx \qquad \int \frac{e^x}{e^{2x}+1}\,dx$$

$$\int \frac{1}{e^x+e^{-x}}\,dx = \int \frac{1}{e^x+e^{-x}}\,dx = \int \frac{1}{e^x+\frac{1}{e^x}}\,dx \Rightarrow \cancel{\int \frac{1}{u+\frac{1}{u}}}$$

$$u = e^x \qquad \Rightarrow \int \frac{1}{u^2+1}\,du$$

$$du = e^x\,dx$$

$$dx = \frac{du}{e^x} \qquad = \arctan u + C$$

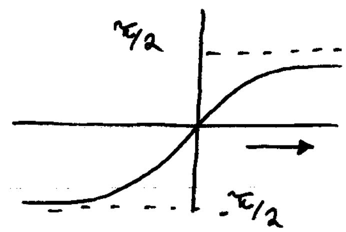
$$= \arctan(e^x) + C$$

$$\lim_{a\to\infty} \arctan(e^x)\Big|_a^0$$

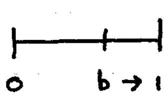
$$= \lim_{a\to-\infty} \left(\arctan(e^0) - \arctan(e^a)\right)$$

$$\lim_{a\to\infty} = \lim_{b\to\infty} \left(\arctan(e^\infty) - \arctan(e^0)\right)$$

$$= \pi/2 - \pi/4 = \pi/4$$



2 $$\int_0^1 \frac{1}{\sqrt[5]{x-1}}\,dx = \lim_{b\to 1^-} \int_0^b \frac{1}{\sqrt[5]{x-1}}\,dx$$



THEN:

$$\lim_{b\to1^-} \left(\tfrac{5}{4}(x-1)^{4/5}\right)\Big|_0^b \rightarrow \lim_{b\to1^-} \left(\tfrac{5}{4}(b-1)^{4/5} - \tfrac{5}{4}(-1)^{4/5}\right)$$

$$= -\tfrac{5}{4}(-1)^{4/5}$$

$$\int \frac{1}{\sqrt[5]{x-1}}\,dx = \int \frac{1}{\sqrt[5]{u}}\,du = \int u^{-1/5}\,du = \frac{5}{4}u^{4/5} + C$$

$$= \tfrac{5}{4}(x-1)^{4/5}$$

$$u = x-1$$

$$du = dx$$

$$\boxed{(-1)^{4/5} = (1-0^4)^{1/5} = 1^{1/5} = 1}$$

For home:

$$\int_0^4 \frac{1}{4-x^2}\,dx$$

Lecture: - Integral Test (Section 9.3)
- Comparisons of Series (Section 9.4)

Integral Test

Let $f(x)$ be a positive decreasing Function $[1, \infty)$ and let $a_n = f(n)$. Then:

$$\sum_{n=1}^{\infty} a_n \text{ conv. if and only if } \int_1^{\infty} f(x)\,dx \text{ conv.}$$

Thm: (P-series)

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} \cdots$$

Converges if $p > 1$ and diverges if $p \leq 1$

Examples:

1. $$\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n^2}} = \sum_{n=1}^{\infty} \frac{1}{n^{2/3}}$$

P-series
$p = 2/3 \leq 1$
(Hence it diverges.)

2. $$\sum_{n=1}^{\infty} \frac{1}{n^4\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{n^4 \cdot n^{1/2}} = \sum_{n=1}^{\infty} \frac{1}{n^{4/2}}$$

P-series
$p = 4/2 = 2 > 1$
(hence it converges)

Proof Let $a_n = \frac{1}{n^p}$ If $p \leq 0$, Then:

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} 1/n^p = \lim_{n\to\infty} n^{-p} = \begin{cases} \infty & p < \infty \\ 1 & p = 0 \end{cases}$$

Hence the series div for $p \le 0$ by the n^{th} term div. test

If $p > 0$, then $f(x) = 1/x^p$ is positive decreasing for $x > 0$

we have :

$$\int_1^\infty f(x)dx = \int_1^\infty \frac{1}{x^p}\,dx = \lim_{b\to\infty}\int_1^b \frac{1}{x^p}\,dx$$

$$= \begin{cases} \lim_{b\to\infty} \left.\frac{x^{1-p}}{1-p}\right|_1^b = \lim_{b\to\infty}\left(\frac{b^{1-p}}{1-p} - \frac{1}{1-p}\right) \begin{cases} \infty \\ -\frac{1}{1-p} \end{cases} \\ \lim_{b\to\infty} \ln|x|\,\Big|_1^b = \boxed{\lim_{b\to\infty}(\ln|b|) = \infty \quad \text{For } p = 1 \text{ div.}} \end{cases}$$

For $p < 1$, div

For $p > 1$, conv.

=> Hence, by the integral test

$\sum a_n = \sum \frac{1}{n^p}$ conv. for $p > 1$

and div. for $p \le 1$

Thm. (Direct Comparison Test)

Let $0 \le a_n \le b_n$ for all n. Then:

1. IF $\sum_{n=1}^{\infty} b_n$ Conv. then $\sum_{n=1}^{\infty} a_n$ conv.
2. IF $\sum_{n=1}^{\infty} a_n$ div. then $\sum_{n=1}^{\infty} b_n$ div.

Example:

1. $\sum_{n=1}^{\infty} \frac{1}{2^n + 5^n}$

Comparison Test

Compare with a p-series

$\hookrightarrow \sum 1/n^p$

Or with a geometric series

$\hookrightarrow \sum ar^n$

We have:

1 cont.
$$a_n = \frac{1}{2^n + 5^n} \le \frac{1}{2^n} = b_n$$

and: $\displaystyle\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{2^n} = \sum_{n=1}^{\infty} \left(\tfrac{1}{2}\right)^n$

Geometric Series
$r = 1/2 \Rightarrow |r| < 1$
(so it is conv.)

Hence, by the comparison test, $\displaystyle\sum \frac{1}{2^n + 5^n}$ conv.

2 $\displaystyle\sum_{n=9}^{\infty} \frac{1}{\sqrt[3]{n} - 2}$

$\displaystyle\sum_{n=9}^{\infty} \frac{1}{\sqrt[3]{n}} = \sum_{n=9}^{\infty} \frac{1}{n^{1/3}}$

$p = 1/3 \le 1$
div.

Let $b_n = \dfrac{1}{\sqrt[3]{n} - 2}$

$a_n = \dfrac{1}{\sqrt[3]{n}}$

then $b_n = \dfrac{1}{\sqrt[3]{n} - 2} \ge \dfrac{1}{\sqrt[3]{n}} = a_n$

Since $\sum a_n$ div the series $\sum b_n = \sum \frac{1}{\sqrt[3]{n} - 2}$

div. by the comparison test.

<u>Thm</u> (Limit Comparison Test)

Let $a_n > 0$ and $b_n > 0$ for all n

and $\displaystyle\lim_{n\to\infty} \frac{a_n}{b_n} = L > 0$ (Finite)

Then

$\sum a_n$ conv. if and only if $\sum b_n$ conv.

EXAMPLES:

1. $\sum_{n=1}^{\infty} \frac{1}{2n^2+3n+5}$

$a_n = \frac{1}{2n^2+3n+5}$ $\qquad b_n = \frac{1}{n^2}$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{1}{2n^2+3n+5}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{\frac{1}{2n^2+3n+5}}{\frac{1}{n^2}} \cdot \frac{\frac{1}{n^2}}{1/n^2}$$

$$\boxed{\frac{n^2}{2n^2+3n+5}}$$

$$= \lim_{n\to\infty} \frac{1}{2+3/n+5/n^2} = \frac{1}{2} > 0$$

and $\sum b_n = \sum \frac{1}{n^2}$ p-series with $p = 2 > 1$

so, it is conv.

Hence by the limit comparison test

$\sum a_n = \sum \frac{1}{2n^2+3n+5}$ Conv.

2. $\sum_{n=1}^{\infty} \frac{n}{n^2+\sqrt{n}}$

Let $a_n = \frac{n}{n^2+\sqrt{n}}$

$b_n = \frac{n}{n^2} = \frac{1}{n}$

$$\frac{n}{n^2+\sqrt{n}}$$

THEN $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{n}{n^2+\sqrt{n}}}{1/n} = 1 > 0$

& $\sum a_n = \sum \frac{n}{n^2+\sqrt{n}}$ div.

Examples:

3 $\sum_{n=1}^{\infty} \frac{2^n}{3^n - 1}$

Let $a_n = \frac{2^n}{3^n - 1}$

$b_n = \frac{2^n}{3^n} = \left(\frac{2}{3}\right)^n$

Then $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{2^n}{3^n-1}}{2^n/3^n}$

$\Rightarrow \lim_{n\to\infty} \frac{3^n}{3^n-1} \cdot \frac{1/3^n}{1/3^n}$

$\Rightarrow \lim_{n\to\infty} \frac{1}{1 - 1/3^n} = 1 > 0$

and $\sum b_n = \sum_{n=1}^{\infty} (2/3)^n \longrightarrow$ Geometric Series with $r = 2/3$

$\hookrightarrow |r| = 2/3 < 1$

Hence it's Conv.

~~by~~ ~~LT~~

Lecture : Alternating Series (section 9.5)

$\sum_{n=1}^{\infty} a_n$; $a_n > 0$ for all n

Alternating Series

$\sum_{n=1}^{\infty} (-1)^n a_n = -a_{n_1} + a_{n_2} + a_{n_3} + a_{n_4} \cdots$

$\sum_{n=1}^{\infty} (-1)^{n+1} a_n = a_{n_1} - a_{n_2} + a_{n_3} - a_{n_4} \cdots$

Thm : Alternating Series Test

Let $\{a_n\}$ be such that

1. $a_n \geq 0$ For all n
2. $\lim_{n\to\infty} a_n = 0$
3. $a_{n+1} \leq a_n$ For all n

Then the series

$\sum_{n=1}^{\infty} (-1)^n a_n$ and $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$

Converge.

Examples

1. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - 1/2 + 1/3 - 1/4 + \ldots$

Let $a_n = 1/n$, Then

~~$a_n = 1/n$ Then~~

1. $a_n \geq 0$, for all n
2. $\lim_{n\to\infty} a_n = \lim_{n\to\infty} 1/n = 0$
3. $a_n - a_{n+1} = 1/n - 1/(n+1) = \frac{n+1-n}{n(n+1)} = \frac{1}{n(n+1)} > 0$

so, $a_n > a_{n+1}$ $\forall n$ For all

How to show that :
$\{a_n\}$ decreases
$\{a_n \geq 0\}$

1. $a_n - a_{n+1} \geq 0 \;\; \forall n$
2. $\frac{a_{n+1}}{a_n} \leq 1 \;\; \forall n$

Hence $\sum \frac{(-1)^{n+1}}{n}$ conv. by the alternating series test.

2. $\sum_{n=1}^{\infty} (-1)^n \frac{1}{n!}$ $\boxed{n! = 1 \cdot 2 \cdot 3 \cdot 4 \cdots n}$

Let $a_n = \frac{1}{n!}$; Then

1. $a_n > 0 \quad \forall n$
2. $0 < \frac{1}{n!} = \frac{1}{1 \cdot 2 \cdot 3 \cdots (n-1) n} < \frac{1}{(n-1) \cdot n} = \frac{1}{n}$

$\lim_{n \to \infty} 0 = 0 = \lim_{n \to \infty} \frac{1}{n}$. Hence $\lim_{n \to \infty} \frac{1}{n!} = 0$

So $\lim_{n \to \infty} a_n = 0$

3. We have

$$\frac{a_{n+1}}{a_n} = \frac{\frac{1}{(n+1)!}}{\frac{1}{n!}} = \frac{n!}{(n+1)!} = \frac{\cancel{(1 \cdot 2 \cdot 3 \cdots n)}}{\cancel{(1 \cdot 2 \cdot 3 \cdots n)}(n+1)}$$

$$= \frac{1}{n+1} < 1$$

So, $a_{n+1} < a_n \quad \forall n$

Hence $\sum (-1)^n \frac{1}{n!}$ conv. by the alternating series test.

4. $\sum_{n=1}^{\infty} (-1)^n \frac{n}{3^n}$

Let $a_n = \frac{n}{3^n}$. Then

1. $a_n \geq 0$
2. $\lim_{n \to \infty} a_n = \lim_{n \to \infty} \frac{n}{3^n} = 0$

$\left(\frac{\infty}{\infty}\right) \quad \lim_{x\to\infty} \frac{x}{3^x} = \lim_{x\to\infty} \frac{1}{3^x \ln 3} = 0$

3 $a_n - a_{n+1} = \frac{n}{3^n} - \frac{n+1}{3^{n+1}}$

$= \frac{3n - n - 1}{3^{n+1}}$

$= \frac{3n-1}{3^{n+1}} \geq 0$

So, $a_n \geq a_{n+1} \quad \forall n$

Hence $\sum (-1)^n \frac{n}{3^n}$ Conv. by the alternating series test.

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \tfrac{1}{5} \cdots = \ln 2$$

$$(+)\ 0 + \tfrac{1}{2} + 0 - \tfrac{1}{4} + 0 \cdots = \tfrac{1}{2}\ln 2$$

$$+1 \quad 0 + \tfrac{1}{3} \quad \cancel{-\tfrac{1}{2}} + \tfrac{1}{5} \cdots = \tfrac{3}{2}\ln 2$$

(Changing the order changes the resulting infinite sum)

Def'n : (Absolute and Conditional convergence)

We say that the series $\sum_{n=1}^{\infty} a_n$

1. converges absolutely if $\sum_{n=1}^{\infty} |a_n|$ converges
2. Converges conditionally if $\sum a_n$ converges but $\sum |a_n|$ diverges

Thm (Absolute Convergence Test)

If $\sum |a_n|$ converges the $\sum a_n$ converges.

Examples

1. $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ Conv. by the alternating series test

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^n}{n} \right| = \sum_{n=1}^{\infty} \frac{1}{n}$$ p-series $p = 1 \leq 1$ diverges by the p-series test

So, $\sum \frac{(-1)^{n+1}}{n}$ converges conditionally

2. $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^3}$

we have

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^n}{n^3} \right| = \sum_{n=1}^{\infty} \frac{1}{n^3}$$ p-series $p = 3 > 1$

Hence by the absolute conv. test

$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^3}$ converges

So, it converges absolutely.

3. $\sum_{n=1}^{\infty} \frac{\sin n}{n^3}$

Consider the series

$$\sum_{n=1}^{\infty} \left| \frac{\sin n}{n^3} \right| = \sum_{n=1}^{\infty} \frac{|\sin n|}{n^3}$$

we have

$$0 \leq \frac{|\sin n|}{n^3} \leq 1/n^3 \quad \forall n$$

$\sum_{n=1}^{\infty} 1/n^3$ p-Series $p = 3 > 1$ so, it is convergent.

Hence $\sum \left| \frac{\sin n}{n^3} \right|$ conv.; by the comparison test.

Thus by the absolute conv. test

$\sum \frac{\sin n}{n^3}$ Conv.

(so it is absolutely convergent).