CALC

Jan 9th /17

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Office hours : Monday and Wednesday

11:30 - 12:30

Assignments : 15%

Midterm (Wednesday, February 15th) : 30%

Final Exam : 55%

Webwork :

https://ftcourses.webwork.maa.org/webwork2/ft_lakehead_math12[illegible]

Username: University email address

Password: Student ID number

Formulas for anti-derivatives :

$$\underbrace{\int f(x)\,dx}_{\text{Indefinite integral}} = \underbrace{F(x)+C}_{\text{Antiderivatives of } f}, \quad C \in \mathbb{R}$$

$\Updownarrow$

$$F'(x) = f(x)$$

Fundamental Theorem of Calculus

Definite Integrals :

$$\int_a^b f(x)\,dx = F(x)\Big|_a^b = f(b) - f(a)$$

Integration by substitution (Section 5.5)

<u>Thm</u> g with range and interval I

f continuous function on I

F an antiderivative of f on I

Then, $\int f(g(x))\,g'(x)\,dx = F(g(x)) + C$

$C \in \mathbb{R}$

Remark Set $u = g(x)$, then $du = g'(x)\,dx$, and so:

$$\int f(\underbrace{g(x)}_{u})\,\underbrace{g'(x)\,dx}_{du} = \int f(u)\,du$$

$$F(g(x)) = F(u) \;\Rightarrow\; \int f(u)\,du = F(u) + C$$

$C \in \mathbb{R}$

Examples:

1. $\int 2x(x^2+1)^4\,dx = \int u^4\,du = \dfrac{u^5}{5} + c = \dfrac{(x^2+1)^5}{5} + c$ $\quad c \in \mathbb{R}$

   $u = x^2+1 \Rightarrow du = 2x\,dx$

2. $\int x^2 e^{x^3}\,dx = \int e^u \tfrac{1}{3}\,du = \tfrac{1}{3}\int e^u\,du = \tfrac{1}{3}e^u + c = \tfrac{1}{3}e^{x^3} + c$ $\quad c \in \mathbb{R}$

   $u = x^3 \Rightarrow du = 3x^2\,dx$

   $\Rightarrow \tfrac{1}{3}\,du = x^2\,dx$

3. $\int \sec^2 x(\tan^2 x + 5)\,dx = \int (u^2+5)\,du = u^3/3 + 5u + c$

   $= \dfrac{\tan^2 x}{3} + 5\tan x + c$ $\quad c \in \mathbb{R}$

   $u = \tan x \Rightarrow du = \sec^2 x\,dx$

$u = e^{x^3} \Rightarrow du = e^{x^3}(3x^2)\,dx$

$\Rightarrow 1/3\,du = x^2 e^{x^3}\,dx$

$\int x^2 e^{x^3} = \int 1/3\,du = \tfrac{1}{3}u + c$

$= 1/3\, e^{x^3} + c$

$[c \in \mathbb{R}]$

4. $\int \sqrt{3x-2}\,dx = \int \sqrt{u}\cdot \tfrac{1}{3}\,du \Rightarrow 1/3 \int u + du$

   $u = 3x-2 \Rightarrow du = 3\,dx \qquad \Rightarrow 1/3\, u/3 + c$

   $\Rightarrow 1/3\,du = dx \qquad \ldots = 2/9\,(3x-2) + c$

   $c \in \mathbb{R}$

<u>Thm</u> g is a function with continuous derivative on $[a, b]$

f is continuous on the range of g

Then

$$\int_a^b f(g(x))\,g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du$$

Examples

1. $\int_0^1 x(2x^2+3)^4\,dx = \int_3^5 u^4\, 1/4\, du = 1/4\, \frac{u^5}{5}\Big|_3^5 = \frac{5^5}{20} - \frac{3^5}{20}$

$u = 2x^2+3 \Rightarrow du = 4x\,dx^3 \Rightarrow 1/4\,du = x\,dx$

$x = 0 \Rightarrow u = 2\cdot 0^2+3 = 3$

$x = 1 \Rightarrow u = 2\cdot 1^2 + 3 = 5$

2. $\int_0^1 x^4 e^{-x^5}dx = \int_0^{-1} e^u(-1/5)\,du = 1/5\int_{-1}^0 e^u du = 1/5\, e\,\Big|_{-1}^0 = 1/5 \ldots 1/e$

$u = -x^5 \Rightarrow du = 5x^4dx \Rightarrow -1/5\,du = x^4dx$

$x = 0 \Rightarrow u = -0^5 = 0$

$x = 1 \Rightarrow u = -1^5 = -1$

1. $\int (2 + 1/t^2)^4 \, 1/t^3 \, dt \Rightarrow \int u^4 (-1/2) du = -1/2 \int u^4 du$

$\Rightarrow -1/2 \cdot u^5/5 + C$

Let $u = 2 + 1/t^2 \Rightarrow du = -2/t^3 \, dt$

$\Rightarrow -1/2 \, du = 1/t^3 \, dt$

$\boxed{\Rightarrow -1/10 \, (2 + 1/t^2)^5 + C \quad \text{where } C \in \mathbb{R}}$

2. $\int \frac{1 + e^x - e^{2x}}{e^{4x}} dx \Rightarrow \int \frac{1 + e^x - e^{2x}}{e^{4x}} \cdot \frac{1}{e^x} \cdot e^x dx$

Let $t = e^x \Rightarrow dt = e^x dx$

$\Rightarrow \int \frac{1 + t - t^2}{t^5} dt$

$\Rightarrow \int \left(\frac{1}{t^5} + \frac{1}{t^4} - \frac{1}{t^3} \right) dt$

$\Rightarrow \int (t^{-5} + t^{-4} + t^{-3}) dt$

$\Rightarrow \int \frac{t^{-4}}{-4} + \frac{t^{-3}}{-3} - \frac{t^{-2}}{-2} + C$

$\boxed{\Rightarrow -\frac{e^{-4x}}{4} - \frac{e^{-3x}}{3} + \frac{t^{-2x}}{2} + C, \quad C \in \mathbb{R}}$

3. $\int \frac{\sin(\ln x)}{x} dx$ (Let $t = \ln x$, $dt = 1/x \, dx$)

$\Rightarrow \sin t \, dt \Rightarrow -\cos t + C$

$\boxed{\Rightarrow -\cos(\ln x) + C, \quad C \in \mathbb{R}}$

4. $\int 2^{x/3} dx \Rightarrow$

Let $u = 2^{x/3}$

$du = 2^{x/3} \cdot \ln 2 \cdot 1/3 \, dx$

$\Rightarrow 3/\ln 2 \cdot du = 2^{x/3} dx$

$\Rightarrow \int 3/\ln 2 \cdot du \Rightarrow 3/\ln 2 \int 1 \cdot du$

$\Rightarrow 3/\ln 2 \cdot u + C$

$\Rightarrow 3/\ln 2 \cdot 2^{x/3} + C, \quad C \in \mathbb{R}$

OR

Let $u = x/3$

$du = 1/3 \, dx$

$\int 2^{x/3} dx = \int 2^u \cdot 3 du = 3 \int 2^u du = 3 \, 2^u / \ln 2 + C$

$\boxed{= 3/\ln 2 \cdot 2^{x/3} + C, \quad C \in \mathbb{R}}$

(5) $\int_0^1 (x+1)e^{x^2+2x+3}\,dx$

$t = x^2 + 2x + 3 \longrightarrow$

$dt = (2x+2)\,dx \qquad x = 0, \quad t = 3$

$dt = 2(x+1)\,dx \qquad x = 1, \quad t = 6$

$\frac{dt}{2} = (x+1)\,dx$

$\Rightarrow \int_0^1 (x+1)e^{x^2+2x+3}\,dx$

$\Rightarrow \int_3^6 e^t \, ½ \, dt \Rightarrow ½ \int_3^6 e^t\,dt$

$\Rightarrow ½ e^t \Big|_3^6$

$\boxed{\Rightarrow ½(e^6 - e^3)}$

(6) $\int_0^{\pi/2} \frac{\cos x}{e^{\sin x}}\,dx$

$u = \sin x \longrightarrow$

$du = \cos x\,dx \qquad x = 0, \quad u = 0$

$x = \pi/2, \quad u = 1$

$\Rightarrow \int_0^1 \frac{1}{e^u}\,du \Rightarrow \int_0^1 e^{-u}\,du$

$t = -u \longrightarrow$

$dt = -du \qquad u = 0, \quad t = -0, \text{ (or } 0)$

$u = 1, \quad t = -1$

$\Rightarrow \int_0^{-1} e^t(-1)\,dt$

$\Rightarrow -\int_0^{-1} e^t\,dt \Rightarrow -e^t \Big|_0^{-1} \Rightarrow -e^{-1} - (-e^0)$

$\boxed{\Rightarrow -e^{-1} + 1}$

7 $\displaystyle\int_1^2 \frac{x^2+1}{x^3+3x}\,dx$

$u = x^3 + 3x$

$du = 3x^2 + 3$

$du = 3(x^2+1)$

$du/3 = x^2+1$

$x = 1, \quad u = 4$

$x = 2, \quad u = 14$

$$\int_4^{14} \frac{1}{u}\,du \Rightarrow \left(\ln|u|\right)\Big|_4^{14} \Rightarrow \frac{1}{3}(\ln 14 - \ln 4)$$

$$\boxed{\Rightarrow \ln\left(\left(\frac{14}{4}\right)^{1/3}\right)}$$

Lecture 2

Integration by substitution (Section 5.5, cont.)

The natural logarithmic function (Section 5.7, cont.)

$$\int f(x)\,dx = F(x) + C, \quad C \in \mathbb{R}$$

$$(F(x)+c)' = f(x)$$

$\int_a^b f(x)\,dx$ number

$$\int_0^1 \frac{e^{1/x^2}}{x^2}\,dx = \frac{\sqrt{\pi}}{2}$$

$\int \frac{e^{-1/x^2}}{x^2}\,dx$ (Can not be written in terms of elementary row operations.)

EXAMPLE:

Find the equation of the function f if

$f'(x) = x \cdot 3^{x^2}$ and the graph of f passes through the point $(1, 2)$

Sol. for f

$f'(x) = x \cdot 3^{x^2} = f$ is on anti-derivative

$f(x) = \int x \cdot 3^{x^2}\,dx$ ↪ set $t = x^2$

$dt = 2x\,dx$

and so, $x\,dx = \frac{1}{2}\,dt$

Hence,

$f(x) = \int x \cdot 3^{x^2}\,dx = \int 3^t (1/2)\,dt$

$\Rightarrow (1/2)\int 3^t\,dt$

$\Rightarrow \frac{3^t}{2\ln 3} + C$

$\left|\frac{(3^t)'}{\ln 3}\right| = 3^t$

Since $f(1) = 2$

so, $2 = \frac{3}{2\ln(3)} + C$

So, $C = 2 - 3/2\ln 3$

Hence $f(x) = \frac{3^{x^2}}{2\ln 3} + 2 \cdot \frac{3}{2\ln 3}$

Thm (Integration of even and odd Functions)
Let f be an integrable Function on the interval $[-a, a]$ $(a > 0)$. Then,

1. f even $\Rightarrow \int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$
2. f odd $\Rightarrow \int_{-a}^{a} f(x)\,dx = 0$

f even $\Leftrightarrow f(-x) = f(x)$

Examples

1. $f(x) = x^2$
2. $f(x) = \cos x$

f odd $\Leftrightarrow f(-x) = -f(x)$

Examples

1. $f(x) = x^3$
2. $f(x) = \sin x$
3. $f(x) = \tan x$

Example

$$\int_{-1}^{1} \frac{\sin(x^3 + x)}{x^8 + 6x^4 + x^2 + 1}$$

Let $f(x) = \dfrac{\sin(x^3 + x)}{x^8 + 6x^4 + x^2 + 1}$

$f(-x) = \dfrac{\sin(-x^3 + (-x))}{(-x)^8 + 6(-x)^4 + (-x)^2 + 1}$ ...

Hence, f is odd.
The integral is equal to zero.

Proof:

f odd $\iff \int_{-a}^{a} f(x)\,dx = 0$

Let $A = \int_{-a}^{a} f(x)\,dx$

Put $t = -x$, then $dt = -dx$
For $x = a$, we have $t = -a$
$x = -a$, we have $t = a$

So, $A = \int_{-a}^{a} f(x)\,dx = \int_{a}^{-a} f(-t)(-1)\,dt = \int_{a}^{-a} -f(t)(-1)\,dt$
$= \int_{a}^{-a} f(t)\,dt$
$= -\int_{-a}^{a} f(t)\,dt = -A$

So, $A = -A$
$\Rightarrow 2A = 0$
$\Rightarrow A = 0$
$\Rightarrow \int_{-a}^{a} f(x)\,dx = 0$ ∎

Thus, (Log route for integration)

① $\int 1/x\,dx = \ln|x| + c$, $c \in \mathbb{R}$

② $\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + c$, $c \in \mathbb{R}$

Proof:

② $t = f(x)$
$dt = f'(x)\,dx$

$\int \frac{f'(x)}{f(x)}\,dx = \int \frac{1}{t}\,dt = \ln|t| + c$
$= \ln|f(x)| + c$
$c \in \mathbb{R}$ ∎

Examples :

1. $$\int \frac{2}{3x-2}\,dx = \int \frac{2}{3}\cdot\frac{3}{3x-2}\,dx$$
$$= \frac{2}{3}\int \frac{3}{3x-2}\,dx$$
$$= \frac{2}{3}\ln|3x-2| + c, \quad c \in \mathbb{R}$$

2. $$\int \frac{x^3+1}{x^4+4x}\,dx = \int \frac{1}{4}\cdot\frac{4(x^3+1)}{x^4+4x}\,dx$$
$$= \frac{1}{4}\int \frac{4x^3+4}{x^4+4x}\,dx$$
$$= \frac{1}{4}\ln|x^4+4x| + c$$
$$c \in \mathbb{R}$$

3. $$\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx = \int \frac{(-1)(-\sin x)}{\cos x}\,dx$$
$$= -\ln|\sin x| + c, \quad c \in \mathbb{R}$$

4. $$\int \sec x\,dx = \int \sec x \cdot \frac{\sec x + \tan x}{\sec x + \tan x}\,dx$$
$$\Rightarrow \int \frac{\sec^2 x + \sec x\tan x}{\sec x + \tan x}\,dx$$
$$\Rightarrow \ln|\sec x + \tan x| + c, \quad c \in \mathbb{R}$$

since

$$(\sec x + \tan x)' = \sec x\tan x + \sec^2 x$$

5. $$\int \frac{x^3+x^2+3x+2}{x^2+2}\,dx$$

$$\begin{array}{l}
\qquad\qquad\qquad\qquad\quad x+1 \\
x^3 + x^2 + 3x + 2 \;\big)\; x^2+2 \\
\underline{-x^3 \qquad\quad -2x} \\
\qquad x^2 + x + 2 \\
\quad\;\; \underline{-x^2 \qquad -2} \\
\qquad\qquad x
\end{array}$$

$$x^3 + x^2 + 3x + 2 = (x^2+2)(x+1) + x$$

$$\frac{x^3+x^2+2x+2}{x^2+2} = \frac{(x^2+2)(x+1)+x}{(x^2+2)} = x+1 + \frac{x}{x^2+2}$$

$$= \int \left(x+1 + \frac{x}{x^2+2} \right) dx = \frac{x^2}{2} + x + \int \frac{x}{x^2+2}\, dx$$

$$= \frac{x^2}{2} + x + \int \frac{1}{2} \cdot \frac{2x}{x^2+2}\, dx$$

$$= \frac{x^2}{2} + x + \frac{1}{2} \ln |x^2+2| + c$$

$$c \in \mathbb{R}$$

$$\int \sin x\, dx = -\cos x + c$$

$$\int \cos x\, dx = \sin x + c$$

$$\int \tan x\, dx = -\ln |\cos x| + c$$

$$\int \cot x\, dx = \ln |\sin x| + c$$

$$\int \sec x\, dx = \ln |\sec x + \tan x| + c$$

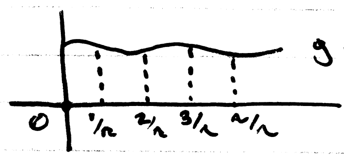
$$\int \csc x\, dx = -\ln |\csc x + \cot x| + c$$

Lecture 3 - Inverse Trigonometric Functions Integration (sec. 5.8)

(before that...)
Average value of $f(x)$ on $[a, b]$

$$\text{Average} = \frac{x_1 + x_2 + x_3 + \dots x_n}{n}$$

$$\text{Average value} = \frac{1}{b-a}\int_a^b f(x)\,dx$$



↳ $$\frac{1}{1-0}\int_0^1 f(x)\,dx = \lim_{n\to\infty} \frac{f(1/n) + f(2/n) + f(3/n) + \dots + f(n/n)}{n}$$

Example: Find the average value of $f(x) = \frac{\ln x}{x}$ on $[1, e]$

Solution: Average value $= \frac{1}{e-1}\int_1^e \frac{\ln x}{x}\,dx \Rightarrow \frac{1}{e-1}\,dx\,\frac{t^2}{2}\Big|_0^1$

where $t = \ln x \Rightarrow dt = 1/x\,dx$ $\quad \Rightarrow \frac{1}{2(e-1)}$
$x = e \Rightarrow t = \ln e = 1$
$x = 1 \Rightarrow t = \ln 1 = 0$

Lecture 3:
$$(\arcsin x)' = \frac{1}{\sqrt{1-x^2}}$$
$$(\arccos x)' = -\frac{1}{\sqrt{1-x^2}}$$
$$(\arctan x)' = \frac{1}{1+x^2}$$
$$(\operatorname{arcsec} x)' = \frac{1}{x\sqrt{x^2-1}}$$

What is $\arcsin x$?
↳ inverse function of $\sin x$

What does inverse mean?
$f(x) = y \iff f^{-1}(y) = x$

Thm: Integrals Involving Inverse Trigonometric Functions

① $\displaystyle\int \frac{dx}{\sqrt{a^2 - x^2}} = \arcsin(x/a) + C$

② $\displaystyle\int \frac{dx}{a^2 + x^2} = 1/a \arctan(x/a) + c$

③ $\displaystyle\int \frac{dx}{x\sqrt{x^2 - a^2}} = 1/a \, \text{arcsec}\left(|x|/a\right) + c$

where $c > \emptyset$

① $\displaystyle\int \frac{u'\,dx}{\sqrt{a^2 - u^2}} = \arcsin(u/a) + c$

② $\displaystyle\int \frac{u'\,dx}{a^2 + u^2} = 1/a \arctan(u/a) + c$

③ $\displaystyle\int \frac{u'\,dx}{u\sqrt{u^2 - a^2}} = 1/a \, \text{arcsec}\left(|u|/a\right) + c$

Examples:

① $\displaystyle\int \frac{1}{\sqrt{2 - x^2}}\,dx \Rightarrow \int \frac{1}{\sqrt{(\sqrt{2})^2 - x^2}}\,dx \Rightarrow \arcsin\left(\frac{x}{\sqrt{2}}\right) + c$

$c \in \mathbb{R}$

② $\displaystyle\int \frac{dx}{3 + 4x^2} \Rightarrow \int \frac{dx}{(\sqrt{3})^3 + (2x)^2} \Rightarrow \frac{1}{2}\int \frac{2\,dx}{(\sqrt{3})^3 + (2x)^2}$

where
$a = \sqrt{3}$
$u = 2x$

$\displaystyle\Rightarrow \frac{1}{2} \cdot \frac{1}{\sqrt{3}} \arctan\left(\frac{2x}{\sqrt{3}}\right) + c$

$c \in \mathbb{R}$

③ $\displaystyle\int \frac{dx}{x\sqrt{25x^2 - 9}}$

$\displaystyle= \int \frac{dx}{x\sqrt{(5x)^2 - 3^2}} \Rightarrow \int \frac{5\,dx}{5x\sqrt{(5x)^2 - 3^2}} \Rightarrow \frac{1}{3}\,\text{arcsec}\left(\frac{|5x|}{3}\right) + c$

where
$a = 3$
$u = 5x$

$c \in \mathbb{R}$

(4) $\int \frac{e^x}{\sqrt{1-e^{2x}}}\,dx \Rightarrow \int \frac{dt}{\sqrt{1-t^2}} \Rightarrow \arcsin t + c$

$\Rightarrow \arcsin(e^x) + c$

$t = e^x \Rightarrow dt = e^x\,dx$ $\quad c \in \mathbb{R}$

where $\int \frac{e^x}{\sqrt{1-(e^x)^2}}\,dx \Rightarrow \arcsin(e^x) + c$ $\quad c \in \mathbb{R}$

$u = e^x$

(5) $\int \frac{x^2+2}{x\sqrt{x^2-9}}\,dx \Rightarrow \int \frac{x^2}{x\sqrt{x^2-9}}\,dx + 2\int \frac{1}{x\sqrt{x^2-9}}\,dx$

$\Rightarrow \int \frac{x}{\sqrt{x^2-9}}\,dx + 2\,(1/3)\,\text{arcsec}\,(|x|/3) \Rightarrow \sqrt{x^2-9} + \frac{2}{3}\,\text{arcsec}\left(\frac{|x|}{3}\right) + c$ $\quad c \in \mathbb{R}$

$t = x^2 - 9 \Rightarrow dt = 2x\,dx$

$\Rightarrow (1/2)\,dt = x\,dx$

$$\int \frac{x}{\sqrt{x^2-9}}\,dx = \int \frac{(1/2)}{\sqrt{t}}\,dt = \frac{1}{2}\int \frac{1}{\sqrt{t}}\,dt$$

$$= \frac{1}{2}\int t^{-1/2}\,dt$$

$$= 1/2 \int t^{-1/2}\,dt$$

$$= 1/2 \cdot \frac{t^{1/2}}{1/2} + c$$

$$= t^{1/2} + c$$

$$= \sqrt{x^2-9} + c$$

(6) $\int \frac{dx}{x^2-6x+12} = \int \frac{dx}{(x-3)^2+(\sqrt{3})^2} = \frac{1}{(\sqrt{3})}\arctan\left(\frac{x-3}{\sqrt{3}}\right) + c$ $\quad c \in \mathbb{R}$

$x^2 - 6x + 12 = x^2 - 6x + 9 - 9 + 12$

$= (x-3)^2 + 3$

$(x-b)^2 = x^2 - 2bx + b$

$2b = 6$

$\Rightarrow b = 3$

(7) Find the area of the region under the graph of:

$f(x) = \frac{1}{\sqrt{3x - x^2}}$

From $x = 3/2$ to $x = 9/4$

Sol.