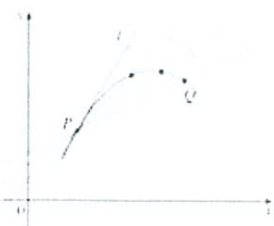
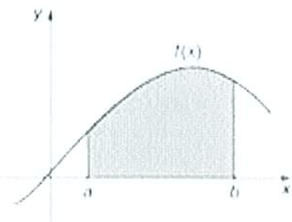
CHAPTER 3
DIFFERENTIATION

RECALL: PREVIEW OF CALCULUS

Tangent Line Problem



Area Problem



EXAMPLE 1

Sept. 27/16

Sacha drains the water from a hot tub. The tub holds 1600L of water. It takes 2 hours for the water to drain completely. The volume of water in the hot tub is modelled by

$$V(t) = 1600 - \frac{t^2}{9}$$

where V is the volume (in litres) and t is the time (in minutes) with $t \in [0, 120]$. → interval between 0 and 120 minutes

a) Verify that the tub is empty after 2 hours.

$$V(120) = 1600 - \frac{(120)^2}{9}$$
$$= 1600 - 1600$$
$$= 0 \text{ L}$$

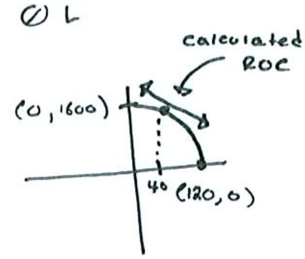
b) Approximate the instantaneous rate of change of the volume at the 40 minute mark.

$$V(40) = 1600 - \frac{(40)^2}{9}$$

$$ROC = \frac{\Delta V(t)}{\Delta t}$$ = NOT NEEDED..

$$= \frac{V(40.01) - V(39.99)}{(40.01) - (39.99)} \Rightarrow -8.89 \text{ L/min}$$

↳ negative because value is decreasing

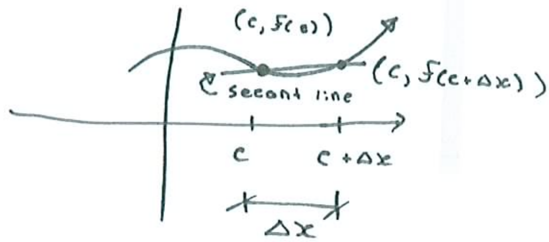


DEFINITION OF A TANGENT LINE WITH SLOPE m

If f is defined on an open interval containing c, and if the limit

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(c + \Delta x) - f(c)}{\Delta x} = m$$

exists, then the line passing through $(c, f(c))$ with slope m is the tangent line to the graph of f at the point $(c, f(c))$.



$$m = \frac{f(c+\Delta x) - f(c)}{\cancel{c} + \Delta x - \cancel{c}}$$

$$m = \frac{f(c+\Delta x) - f(c)}{\Delta x}$$

$$m = \lim_{\Delta x \to 0} \frac{f(c+\Delta x) - f(c)}{\Delta x}$$

So as $\Delta x \to 0$, the secant line goes to the tangent line.

EXAMPLE 2

For $f(x) = -3x - 5$, find the slope of the tangent line at $(1, -8)$

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$$m = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} \qquad x = 1$$

$$= \lim_{\Delta x \to 0} \frac{f(1+\Delta x) - f(1)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{-3(1+\Delta x) - 5 - (-3(1) - 5)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\cancel{-3} - 3\Delta x \cancel{-5} + \cancel{8}}{\Delta x} = 8$$

$$= \lim_{\Delta x \to 0} \frac{-3\Delta x}{\Delta x} \Rightarrow \boxed{-3}$$

EXAMPLE 3

For $f(x) = x^2 - 5$, find the slope of the tangent line at $(2, -1)$

$$m = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} \qquad x = 2$$

$$= \lim_{\Delta x \to 0} \frac{f(2+\Delta x) - f(2)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{(2+\Delta x)^2 - 5 - (2^2 - 5)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} (4 + \Delta x^2) - 5 - 4 + 5 \Rightarrow \frac{4 + 4\Delta x + \Delta x^2 - 5 + 1}{\Delta x}$$

$$= \lim_{x \to 0} \frac{(\Delta x)(4 + \Delta x)}{\Delta x} \Rightarrow \lim_{x \to 0} (4 + \Delta x) \boxed{= 4}$$

DEFINITION OF THE DERIVATIVE

The derivative of f at x is

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

provided the limit exists. For all x for which this limit exists, f' is a function of x.

The derivative can be uses to find the Instantaneous Rate of Change

Notations

$$\boxed{f'(x),\quad \frac{dy}{dx},\quad y',\quad \frac{d}{dx}[f(x)],\quad D_x[y].}$$

EXAMPLE 4

Determine the derivative of the following function

$$f(x) = 2x^3 - 5$$

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$$\lim_{\Delta x \to 0} \frac{2(x+\Delta x)^3 \cancel{-5} - (2x^3 \cancel{-5})}{\Delta x}$$

$$\lim_{\Delta x \to 0} 2(\cancel{x^3} + 3x^2\Delta x + 3x\Delta x^2 + \Delta x^3) - \cancel{2x^3}$$

$$\lim_{\Delta x \to 0} \frac{\Delta x(6x^2 + 6x\Delta x + 2\Delta x^2)}{\Delta x} \Rightarrow \lim_{\Delta x \to 0} (6x^2 + 6x\Delta x + 2\Delta x^2)$$

$$\boxed{= 6x^2}$$

EXAMPLE 5

Determine the slope of the tangent line at the point (7,2) of the following function

$$f(x) = \sqrt{x - 3}$$

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$$\lim_{\Delta x \to 0} \frac{\sqrt{x+\Delta x-3} - \sqrt{x-3}}{\Delta x} \times \frac{\sqrt{x+\Delta x-3} + \sqrt{x-3}}{\sqrt{x+\Delta x-3} + \sqrt{x-3}}$$

$$\lim_{\Delta x \to 0} \frac{x + \Delta x - 3 - x + 3}{\Delta x(\sqrt{x+\Delta x-3} + \sqrt{x-3})}$$

$$\lim_{\Delta x \to 0} \frac{1}{\sqrt{x+\Delta x-3} + \sqrt{x-3}} \Rightarrow \frac{1}{2\sqrt{x-3}}$$

$$f'(7) = \frac{1}{2\sqrt{7-3}}$$

$$= \frac{1}{4}$$

$\therefore$ the slope of the tangent line @ (7, 2) is 1/4.

EXAMPLE 6

Determine the equation of the tangent line at the point $\left(0, \frac{1}{3}\right)$ of the following function

$$f(x) = \frac{1}{x+3}$$

$$\Rightarrow \lim_{\Delta x \to 0} \frac{\frac{1}{x+\Delta x+3} - \frac{1}{x+3}}{\Delta x}$$

$$\Rightarrow \lim_{\Delta x \to 0} \frac{x+3-(x+\Delta x+3)}{\Delta x\,(x+\Delta x+3)(x+3)}$$

$$\Rightarrow \lim_{\Delta x \to 0} \frac{-1}{(x+\Delta x+3)(x+3)}$$

$$= \frac{-1}{(x+3)^2}$$

$$m = f'(0) = \frac{-1}{(0+3)^2} = \frac{-1}{9}$$

$$y = -\tfrac{1}{9}x + b$$

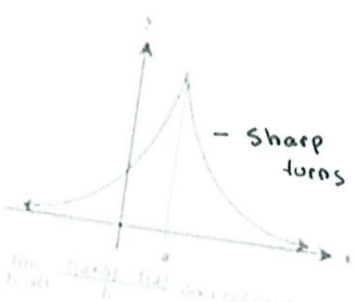
$$\tfrac{1}{3} = -\tfrac{1}{9}(0) + b$$

$$b = \tfrac{1}{3}$$

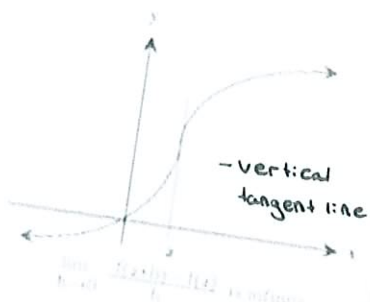
$$\therefore\ y = -\tfrac{1}{9}x + \tfrac{1}{3}$$

(Midterm 10 MP
3 Problems)
(Chapters 1-3)

FUNCTIONS THAT ARE NOT DIFFERENTIABLE EVERYWHERE



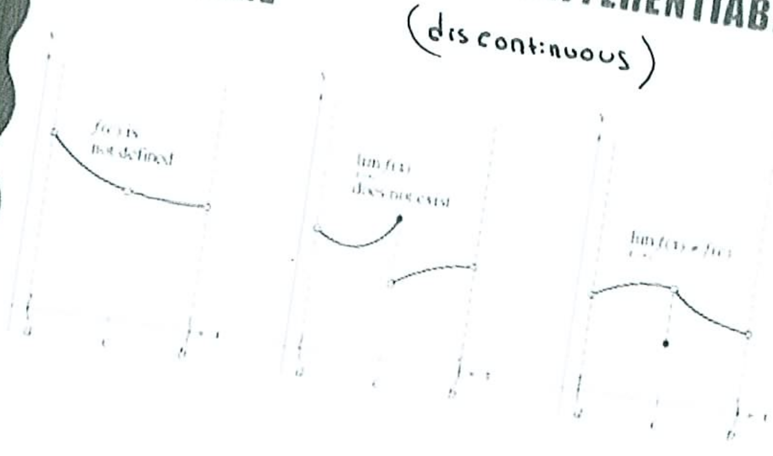
- Sharp turns



- vertical tangent line

FUNCTIONS THAT ARE NOT DIFFERENTIABLE EVERYWHERE

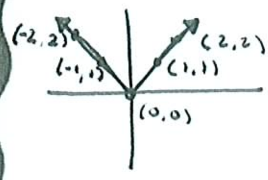
(discontinuous)



EXAMPLE 7

Discuss the differentiability at $x = 0$ of $f(x) = |x|$

$f(x) = |x|$



$$f(x) = |x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

$(@\ x = 0)$

$$\lim_{\Delta x \to 0^+} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0^+} \frac{\cancel{x} + \Delta x - \cancel{x}}{\Delta x} \Rightarrow 1$$

$$\lim_{\Delta x \to 0^-} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0^-} \frac{-(\cancel{x} + \Delta x) - (-\cancel{x})}{\Delta x} \Rightarrow -1$$

THEOREM: DIFF $\Longrightarrow$ CONTINUITY

If f is differentiable at $x = c$, then f is continuous at $x = c$.

differentiable = continuous

but continuous may not be differentiable.

1. If a function is differentiable at $x = c$, then it is continuous at $x = c$. So, differentiability implies continuity.
2. It is possible for a function to be continuous at $x = c$ and not be differentiable at $x = c$. So, continuity does not imply differentiability.

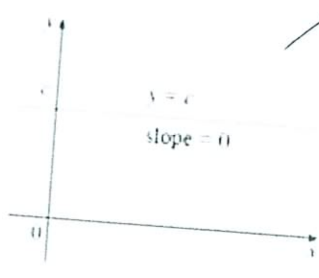
$$\lim_{\Delta x \to 0} \frac{f(0+\Delta x) - f(0)}{\Delta x}$$

$\boxed{\text{DNE}}$

$\therefore f$ is not diff. at $x = 0$

CONSTANT RULE

$$\boxed{\frac{d}{dx}(c) = 0}$$



So if $f(x) = -3$, then $f'(x) = 0$.

$\boxed{\text{Sept. 30/16}}$

PROOF:

$f(x) = c \Rightarrow f'(x) = 0$

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{c - c}{\Delta x} \Rightarrow \frac{0}{\Delta x}$$

$= 0$ ▩

CONSIDER...

$f(x)$	$f'(x)$
x	1
x^2	$2x$
x^3	$3x^2$
x^4	$4x^3$
x^5	$5x^4$
x^6	$6x^5$

etc.

POWER RULE

If n is a positive integer, then

$$\boxed{\frac{d}{dx}(x^n) = nx^{n-1}}$$

So if $f(x) = x^{99}$, then $f'(x) = 99x^{98}$

$f(x) = x^n \Rightarrow f'(x) = nx^{n-1}$

(actually true for all $n \in \mathbb{R} \setminus \{0\}$)

PROOF:

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{(x+\Delta x)^n - x^n}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\binom{n}{0}x^n \Delta x^0 + \binom{n}{1}x^{n-1}\Delta x^1 + \dots + \binom{n}{n}x^0 \Delta x^n - x^n}{\Delta x}$$

$$\lim_{\Delta x \to 0} \frac{\cancel{x^n} + nx^{n-1}\Delta x + \binom{n}{2}x^{n-2}\Delta x^2 + \dots + \binom{n}{n}x^2 \Delta x^n - \cancel{x^n}}{\Delta x}$$

$$\lim_{\Delta x \to 0} \frac{\cancel{\Delta x}\left(nx^{n-1} + \binom{n}{2}x^{n-2}\Delta x + \dots + \binom{n}{n}x^0 \Delta x^{n-1}\right)}{\cancel{\Delta x}}$$

$$\boxed{= nx^{n-1}}$$

CONSTANT MULTIPLE RULE

If c is a constant and f is a differentiable function, then

$$\boxed{\frac{d}{dx}[cf(x)] = c\frac{d}{dx}f(x)}$$

[So if $f(x) = x^{99}$ then $f'(x) = 99x^{98}$]

PROOF:

Let $g(x) = cf(x)$

$$g'(x) = \lim_{\Delta x \to 0} \frac{g(x+\Delta x) - g(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{cf(x+\Delta x) - cf(x)}{\Delta x} \Rightarrow c\,\frac{f(x+\Delta x) - f(x)}{\Delta x} = cf'(x)$$

EXAMPLE 8

Find the derivative of the following functions:

a) $f(x) = \pi^3 \Rightarrow f'(x) = 0$ (just a number..)

b) $g(x) = 12x^4 \Rightarrow 12 \cdot (4x^3) = g'(x) \Rightarrow g'(x) = \boxed{48x^3}$

c) $h(x) = -\dfrac{1}{x^2}$ PROPER CONVENTION.

$h^{1}(x) = (-1)(x^{-2})$

$h'(x) = (-1)(-2x^{-3}) \Rightarrow 2x^{-3}$

$\boxed{\Rightarrow \dfrac{2}{x^3}}$

SUM AND DIFFERENCE RULES

The sum (or difference) of two differentiable functions f and g is itself differentiable. Moreover, the derivative of $f + g$ (or $f - g$) is the sum (or difference) of the derivatives of f and g.

$$\frac{d}{dx}[f(x) + g(x)] = f'(x) + g'(x)$$

$$\frac{d}{dx}[f(x) - g(x)] = f'(x) - g'(x)$$

EXAMPLE 9

Find the derivative of the following functions:

a) $f(x) = 2x^3 + 2x \Rightarrow f'(x) = 2(3x^2) + 2(\overbrace{1x^0}^{1})$

$\Rightarrow f'(x) = \boxed{6x^2 + 2}$

b) $g(x) = 3x^3 - 8 \longrightarrow g'(x) = 9x^2 - 0$

$\boxed{= 9x^2}$

c) $h(x) = 4x^{27} - 3x^8 + 6x^4 - 7\sqrt[3]{x} + e^5$ write as $x^{1/3}$

$h'(x) = 4(27x^{26}) - 3(8x^7) + 6(4x^3) - [7(x^{1/3})] + \cancel{5e^4}$ constant

$[7(\frac{1}{3}x^{-2/3}) + (0)$

$h'(x) = 108x^{2\cancel{6}} - 24x^7 + 24x^3 - \dfrac{7}{3x^{2/3}}$

$$\boxed{h'(x) = 108x^{25} - 24x^7 + 24x^3 \ldots - \frac{7}{3\sqrt[3]{x^2}}}$$

EXAMPLE 10

Find the equation of the tangent line at $x = -1$ for

$$f(x) = 2x^5 - 7$$

$m = f'(-1)$

$f'(x) = 2(5x^4) - 0$

$f'(x) = 10x^4$

$f(-1) = 2(-1)^5 - 7$
$= -9$

@ $(-1, -9)$

$y = mx + b$
$-9 = 10(-1) + b$
$b = 1$

$\boxed{y = 10x + 1}$

RECALL....

$$\lim_{\Delta x \to 0} \frac{\sin \Delta x}{\Delta x} = 1 \quad \text{and} \quad \lim_{\Delta x \to 0} \frac{1 - \cos \Delta x}{\Delta x} = 0$$

PROOF :

$f(x) = \sin(x) \Rightarrow f'(x) = \cos(x)$

$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$

$\lim_{\Delta x \to 0} \frac{\sin(x+\Delta x) - \sin x}{\Delta x}$

$\lim_{\Delta x \to 0} \frac{\sin x \cos \Delta x + \sin \Delta x \cos x - \sin x}{\Delta x}$

$\lim_{\Delta x \to 0} \left[\sin x \left(\frac{\cos \Delta x - 1}{\Delta x} \right) + \cos x \ldots \right.$

$\left. \ldots \left(\frac{\sin \Delta x}{\Delta x} \right) \right]$

$= \lim_{\Delta x \to 0} \left[\sin x (0) + \cos x (1) \right]$

$= \cos x$ ■

DERIVATIVE OF THE SINE AND COSINE FUNCTIONS

$$\frac{d}{dx}[\sin x] = \cos x \qquad \frac{d}{dx}[\cos x] = -\sin x$$

EXAMPLE 11

Find the derivative of the following functions:

a) $f(x) = \frac{3\cos x}{5}$ $\Rightarrow f'(x)\left(\frac{3}{5}\right)\cos x$

$\boxed{\Rightarrow f'(x)\cdot\left(\frac{3}{5}\right)\sin x}$

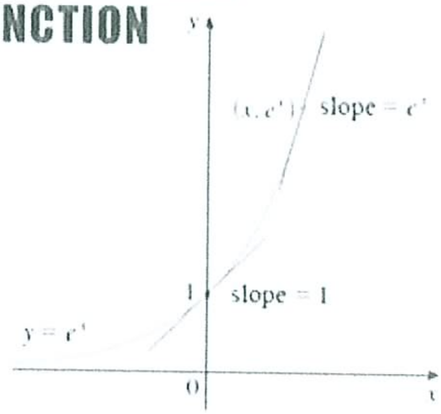
b) $g(x) = \sin x - 3x^4$

$g'(x) = \sin x - 3x^4$

$\boxed{g'(x) = \cos x - 12x^3}$

DERIVATIVE OF THE NATURAL EXPONENTIAL FUNCTION

$$\boxed{\frac{d}{dx}[e^x] = e^x}$$



$\boxed{\text{Oct. } 3^{RD}/16}$

PROVE THAT IF $f(x) = e^x$ that $f'(x) = e^x$ as well.

Proof: $f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$

$= \lim_{\Delta x \to 0} \frac{e^{x+\Delta x} - e^x}{\Delta x}$

$= \lim_{\Delta x \to 0} \frac{e^x(e^{\Delta x} - 1)}{\Delta x}$

By definition, as $\Delta x \to 0$

$(1+\Delta x)^{1/\Delta x} \to e$

OR

$e^{\Delta x} \to (1+\Delta x)$ as $\Delta x \to 0$ Sub into limit.

$\lim_{\Delta x \to 0} \frac{e^x(1+\Delta x - 1)}{\Delta x}$

$\lim_{\Delta x \to 0} \frac{\Delta x\, e^x}{\Delta x}$

$= e^x$ ∎

EXAMPLE 12

Find the derivative of the following functions:

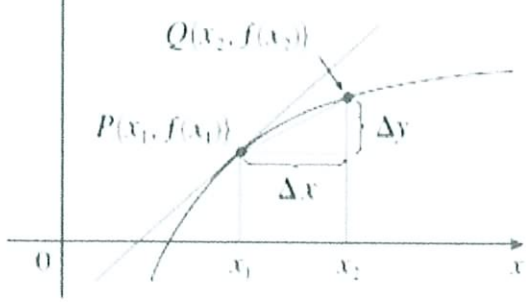
a) $f(x) = -7e^x$ $\Rightarrow f'(x) = (-7)(e^x) = \boxed{-7e^x}$

b) $g(x) = 2x^3 - 6\cos x + \frac{e^x}{2}$

$g'(x) = (2)(3x^2) + 6\sin x + \frac{e^x}{2}$

$\boxed{= 6x^2 + 6\sin x + \frac{e^x}{2}}$

RATE OF CHANGE



Average ROC

$$\boxed{\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}}$$

m_{PQ} = average rate of change
$m = f'(x_1)$ = instantaneous rate of change

RATE OF CHANGE

If $s = s(t)$ is the position function (displacement) for an object moving along a straight line, the velocity v of the object at time t is given by:

$$\boxed{v(t) = s'(t)}$$

The position of a free-falling object (neglecting air resistance) under the influence of gravity can be represented by the equation

$$\boxed{s(t) = \frac{1}{2}gt^2 + v_0 t + s_0}$$

$g = -9.8\ m/s^2$ or $-32\ ft/s^2$
v_0 -- initial velocity
s_0 -- initial position

^{0} "naught" (initial)

Given:
$V_0 = 300$ ft/s
$S_0 = 65$ ft
$g = -32$ ft/sec^2

EXAMPLE 13

A paintball gun shoots a paint ball 300 ft/s straight up in the air off the top of a 65 ft building.

a) What is the position function for the paintball?
b) What is the paintball's average velocity for $t \in [1, 2]$?
c) What is the paintball's velocity at $t = 2$ seconds?
d) When does the paintball hit the ground?
e) What is the maximum height of the paint ball and when does this happen?

a) $S(t) = \frac{1}{2}gt^2 + V_0 t + S_0$

$= \frac{1}{2}(-32\text{ ft/s}^2)(t)^2 + (300\text{ ft/s})(t) + (65$

$= -16\text{ ft/s}^2\, t^2 + 300\text{ ft/s}\, t + 65\text{ ft}$

b) $ROC_{AVG} = \frac{S(2) - S(1)}{2 - 1}$ $\quad t \in [1, 2]$

$= \frac{601 - 349}{1}$

$\boxed{= 252 \text{ ft/s}}$

c) $IROC = S'(2)$

$S'(t) = -32t + 300$

$S'(2) = -32(2) + 300$

$\boxed{= 236 \text{ ft/s}}$

d) $s(t) = 0$

$-16t^2 + 300t + 65 = 0$

$t = \frac{-300\cancel{t} \pm \sqrt{300^2 - 4(16)(65)}}{2(-16)}$

$\therefore t \simeq -0.21s$ or $\boxed{18.96s}$

$\therefore$ The paintball hits the ground @ $\simeq$ 19 seconds.

e) $S'(t) = -32t^2 + 300 = 0$

$\boxed{t = 9\,3/8\ s}$ time

$\boxed{S(9\,3/8) = 1471\tfrac{1}{4}\text{ ft}}$ height.

PRODUCT RULE

OCT. 5/16

$$\frac{d}{dx}[f(x)g(x)] = f(x)\frac{d}{dx}[g(x)] + g(x)\frac{d}{dx}[f(x)]$$

$$\boxed{(fg)' = f'g + fg'}$$

$$(fg)' = \lim_{\Delta x \to 0} \frac{f(x+\Delta x)g(x+\Delta x) - f(x)g(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{f(x+\Delta x)g(x+\Delta x) - f(x+\Delta x)g(x) + f(x+\Delta x)g(x) - f(x)g(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \left[f(x+\Delta x)\left(\frac{g(x+\Delta x) - g(x)}{\Delta x}\right) + g(x)\left(\frac{f(x+\Delta x) - f(x)}{\Delta x}\right)\right]$$

$$= f(x)g'(x) + g(x)f'(x)$$

EXAMPLE 14

Find the derivative of the following functions:

$f'(x) = 2x \sin x + x^2(\cos x)$

a) $f(x) = x^2 \sin x$

b) $T(x) = (6x^3)(7x^4)$

c) $g(x) = (3x^7 + 1)(x^3 - x)$

d) $h(x) = \sqrt{x} \cdot e^x - e^x \cos x$

$$f'(x) = \boxed{2x \sin x + x^2(\cos x)}$$

$$T'(x) = (6x^3)(28x^3) + (18x^2)(7x^4)$$

$$T'(x) = 126x^6 + 168x^6$$

$$\boxed{= 294x^6}$$

$$g'(x) = \boxed{(3x^7 + 1)(3x^2 - 1) + (21x^6)(x^3 - x)}$$

GOOD ENOUGH.

$$h(x) = \sqrt{x} \cdot e^x - e^x \cos x$$

$$h'(x) = e^x(x^{1/2} - \cos x)$$

$$h'(x) = e^x\left[(1/2)x^{-1/2} + \sin x\right] + \ldots$$

$$\ldots e^x(x^{1/2} - \cos x)$$

$$\boxed{h'(x) = e^x\left(\sqrt{x} - \cos x + \frac{1}{2\sqrt{x}} + \sin x\right)}$$

QUOTIENT RULE

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\frac{d}{dx}[f(x)] - f(x)\frac{d}{dx}[g(x)]}{[g(x)]^2}$$

$$\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{(g)^2}$$

PROOF...

EXAMPLE 15

Determine y' for the following: $y = \dfrac{x^2 + x - 2}{x^3 + 6}$

$$y = \frac{x^2 + x - 2}{x^3 + 6}$$

$$y' = \frac{(2x+1)(x^3+6) - (x^2+x-2)(3x^2)}{(x^3+6)^2}$$

$$y' = \frac{2x^4 + x^3 + 12x + 6 - 3x^4 - 3x^3 + 6x^2}{(x^3+6)^2}$$

$$y' = \frac{-x^4 - 2x^3 + 6x^2 + 12x + 6}{(x^3+6)^2}$$

EXAMPLE 16

Find the derivative of the following functions:

where n is a positive integer.

a) $f(x) = x^{-n}$, where $n \in \mathbb{Z}^+$

b) $g(x) = \tan x$

c) $h(x) = \dfrac{e^x + x^7 \sin x}{e^x}$

$$f(x) = \frac{1}{x^n}$$

$$f'(x) = \frac{0(x^n) - 1(n x^{n-1})}{(x^n)^2}$$

$$f'(x) = -n x^{n-1-2n}$$

$$\boxed{f'(x) = -n x^{-n-1}}$$

EXAMPLE 17

Determine the equation to the tangent line at point $\left(1, \frac{1}{2}\right)$ to the curve:

$$y = \frac{\sqrt{x}}{1 + x^2}$$

$$y' = \frac{\frac{1}{2} x^{-1/2}(1+x^2) - \sqrt{x}(2x)}{(1+x^2)^2}$$

$$m = \frac{\frac{1}{2}(1)^{-1/2}(1+1^2) - \sqrt{1}(2(1))}{(1+1^2)^2}$$

$$= \frac{\frac{1}{2}(2) - 2}{4} \Rightarrow -\frac{1}{4}$$

$y = -\frac{1}{4}x + b$

$\frac{1}{2} = -\frac{1}{4}(1) + b$

$b = 3/4$

$y = -1/4\, x + 3/4$

$$\left(\frac{f}{g}\right)' = \lim_{\Delta x \to 0} \frac{\frac{f(x+\Delta x)}{g(x+\Delta x)} - \frac{f(x)}{g(x)}}{\Delta x}$$

$$\lim_{\Delta x \to 0} \frac{f(x+\Delta x)\, g(x) - f(x) g(x+\Delta x)}{\Delta x\, g(x+\Delta x) g(x)}$$

$$\lim_{\Delta x \to 0} \frac{f(x+\Delta x)\, g(x) - f(x) g(x) + f(x) g(x) - f(x) g(x+\Delta x)}{\Delta x\, g(x+\Delta x) g(x)}$$

$$\lim_{\Delta x \to 0} \left[\frac{g(x)}{g(x+\Delta x) g(x)} \left(\frac{f(x+\Delta x) - f(x)}{\Delta x} \right) - \ldots \right.$$

$$\left. \ldots \frac{f(x)}{g(x+\Delta x) g(x)} \left(\frac{g(x+\Delta x) - g(x)}{\Delta x} \right) \right]$$

$$= \frac{g(x)}{[g(x)]^2} \times f'(x) - \frac{f(x)}{[g(x)]^2} \times g'(x)$$

$$= \frac{f'(x) g(x) - f(x) g'(x)}{[g(x)]^2}$$

$$g(x) = \tan x = \frac{\sin x}{\cos x}$$

$$g'(x) = \frac{\cos x(\cos x) - \sin x(\sin x)}{(\cos x)^2}$$

$$g'(x) = \frac{\cos^2 x + \sin^2 x}{\cos^2 x}$$

$$g'(x) = \frac{1}{\cos^2 x} = \boxed{\sec^2 x}$$

$$h(x) = \frac{e^x + x^7 \sin x}{e^x}$$

$$h'(x) = \frac{(e^x + 7x^6 \sin x + x^7 \cos x)e^x \ldots - (e^x + x^7 \sin x)e^x}{(e^x)^2}$$

$$h'(x) = \frac{\cancel{e^x}(e^x + 7x^6 \sin x + x^7 \cos x \ldots - e^x - x^7 \sin x)}{(e^x)^{\cancel{2}}}$$

$$\boxed{= \frac{x^6(7 \sin x + x \cos x - x \sin x)}{e^x}}$$

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DERIVATIVES OF TRIG FUNCTIONS

$\frac{d}{dx}(\sin x) = \cos x$	$\frac{d}{dx}(\csc x) = -\csc x \cot x$
$\frac{d}{dx}(\cos x) = -\sin x$	$\frac{d}{dx}(\sec x) = \sec x \tan x$
$\frac{d}{dx}(\tan x) = \sec^2 x$	$\frac{d}{dx}(\cot x) = -\csc^2 x$

PROOF:

let $f(x) = \sec x$, then

$= \frac{1}{\cos x}$ $\quad \left(\left(\frac{f}{g}\right)' = \frac{f'g - g}{g^2}\right)$

$f'(x) = \frac{(0)\cos x - 1(-\sin x)}{\cos^2 x}$

$= \frac{\sin x}{\cos^2 x} \Rightarrow \frac{1}{\cos x} \times \frac{1}{\cos x}$

$= \tan x \sec x$ ■

EXAMPLE 18

Which point(s) on the following function contain a horizontal tangent line:

$$f(x) = \frac{\sec x}{1 + \tan x}$$

$f'(x) = \frac{\sec x \tan x (1 + \tan x) - \sec x (\sec^2 x)}{(1 + \tan x)^2}$

$f'(x) = \frac{\sec x \tan x + \sec x \tan^2 x - \sec^3 x}{(1 + \tan x)^2}$

$= \frac{\sec x \tan x + \sec x (\sec^2 x - 1) - \sec^3 x}{(1 + \tan x)^2}$

$= \frac{\sec x \tan x + \sec^3 x - \sec x - \sec^3 x}{(1 + \tan x)^2}$

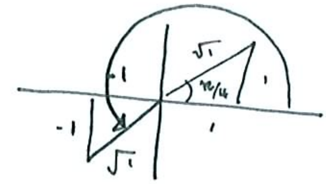
$= \frac{\sec x (\tan x - 1)}{(1 + \tan x)^2}$

$= 0$

$\Rightarrow \sec x = 0$ or $\tan x = 1$

×

$\frac{1}{\cos x} \neq 0$



$x = \pi/4, \frac{9\pi}{4}, \frac{17\pi}{4}$

$\frac{5\pi}{4}, \frac{13\pi}{4}$

i.e. $x = \frac{(4n+1)\pi}{4}, n \in \mathbb{R}$

HIGHER ORDER DERIVATIVES

First derivative:	y',	$f'(x)$,	$\frac{dy}{dx}$,	$\frac{d}{dx}[f(x)]$.
Second derivative:	y'',	$f''(x)$,	$\frac{d^2y}{dx^2}$,	$\frac{d^2}{dx^2}[f(x)]$.
Third derivative:	y''',	$f'''(x)$,	$\frac{d^3y}{dx^3}$,	$\frac{d^3}{dx^3}[f(x)]$.
Fourth derivative:	$y^{(4)}$,	$f^{(4)}(x)$,	$\frac{d^4y}{dx^4}$,	$\frac{d^4}{dx^4}[f(x)]$.
⋮				
nth derivative:	$y^{(n)}$,	$f^{(n)}(x)$,	$\frac{d^ny}{dx^n}$,	$\frac{d^n}{dx^n}[f(x)]$.

Points:

$f(\pi/4) = \frac{\sec(\pi/4)}{1 + \tan(\pi/4)} = \frac{\frac{1}{\cos \pi/4}}{1+1} = \frac{\frac{2}{\sqrt{2}}}{2} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} \ldots$

$\ldots f(\frac{5\pi}{4}) = \frac{-\sqrt{2}}{2} = \left(\frac{(4n+1)\pi}{4}, \frac{\sqrt{2}}{2}\right)$ n is even

EXAMPLE 19

Determine is the second derivative of:

$$y = 3x^5 - 6x^2 + 2x - 5$$

$y' = (5)3x^4 - (2)6x + 2 - 0$

$y' = 15x^4 - 12x$

$y'' = (4)15x^3 - 12$

$y'' = 60x^3 - 12$

EXAMPLE 20

Determine the 97th derivative of $f(x) = \cos x$ $\therefore$

$f'(x) = -\sin(x) = f^5 = f^9 \ldots f^{97}$

$f''(x) = -\cos(x) = f^6 = f^{10}$

$f'''(x) = \sin(x) = f^7 = f^{11}$

$f^{IV}(x) = \cos(x) = f^8 = f^{12} \ldots f^{96}$

so...

$\frac{97}{4} = 24\frac{1}{4}$

ACCELERATION

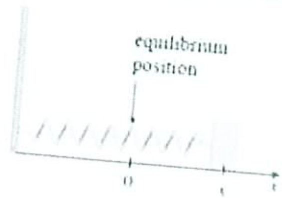
$s(t)$	Position function
$v(t) = s'(t)$	Velocity function
$a(t) = v'(t) = s''(t)$	Acceleration function

EXAMPLE 21

A mass on a spring vibrates horizontally on a smooth level surface (see the figure). Its equation of motion is $x(t) = 8\sin t$, where t is in seconds and x in centimeters.

(a) Find the velocity and acceleration at time t.

(b) Find the position, velocity, and acceleration of the mass at time $t = 2\pi/3$. In what direction is it moving at that time?



a) $v(t) = x'(t)$

$\boxed{= 8\cos t}$

$a(t) = v'(t)$

$\boxed{= -8\sin t}$

CONSIDER...

How would differentiate functions of the following form:

$$f(x) = \sqrt{x^2+1}\ ?$$

let $g(x) = x^2+1$

let $h(x) = \sqrt{x}$ } $f(x) = h(g(x))$

$f'(x) = h'(g(x))\, g'(x)$

$h'(x) = \dfrac{1}{2\sqrt{x}}$ $\qquad g'(x) = 2x$

$f'(x) = h'(g(x))\, g'(x)$

$= \dfrac{1}{2\sqrt{x^2+1}} \times 2x \quad \boxed{= \dfrac{x}{\sqrt{x^2+1}}}$

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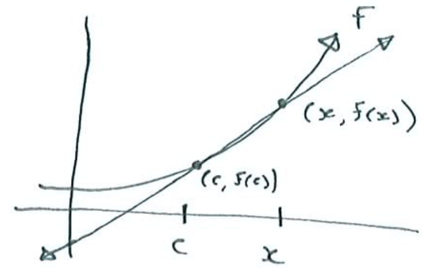
Alternative Form of the derivative



$F'(x) = \lim\limits_{x \to c} \dfrac{f(x) - f(c)}{x - c}$

using this

CHAIN RULE

If $y = f(u)$ is a differentiable function of u and $u = g(x)$ is a differentiable function of x, then $y = f(g(x))$ is a differentiable function of x and

$$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx}$$

or, equivalently,

$$\boxed{\frac{d}{dx}[f(g(x))] = f'(g(x))g'(x)}$$

Proof:

$f(g(x))' = \lim\limits_{x \to c} \dfrac{f(g(x)) - f(g(c))}{x - c} \times \dfrac{g(x) - g(c)}{g(x) - g(c)}$

$\lim\limits_{x \to c} \dfrac{f(g(x)) - f(g(c))}{g(x) - g(c)} \times \dfrac{g(x) - g(c)}{x - c}$

$= f'(g(x))\, g'(x)$

OTHER METHOD

$f(x) = \sqrt{x^2+1} = (x^2+1)^{\frac{1}{2}}$

$f'(x) = \frac{1}{2}(x^2+1)^{-\frac{1}{2}}(2x)$

$f'(x) = \boxed{\dfrac{x}{\sqrt{x^2+1}}}$

EXAMPLE 22

Differentiate the following function:

$$f(x) = \sqrt{x^2+1}\ ?$$

(see previous)

EXAMPLE 23

Find the derivative of the following functions:

a) $f(x) = \sin x^2$ $\longrightarrow$ $\left.\begin{matrix} h(x) = \sin x \\ g(x) = x^2 \end{matrix}\right\}$ $f(x) = h(g(x))$ $\quad f'(x) = h'(g(x))\,g'(x)$

$h'(x) = \cos x$

$g'(x) = 2x$

$f'(x) = \cos(x^2)(2x)$

$\boxed{= 2x\cos x^2}$

OR

$f(x) = \sin x^2$

$\boxed{2x\cos x^2}$

b) $g(x) = \sin^2 x$

$g(x) = \sin^2 x = (\sin x)^2$

$\boxed{g'(x) = 2\sin x\cos x}$

EXAMPLE 24

$f'(x) = h'(g(x))\,g'(x)$

Find the derivative of the following functions:

a) $f(x) = (x^3-1)^{100}$ $\Rightarrow$ $100(x^3-1)^{99}(3x^2)$

$\boxed{= 300(x^2)(x^3-1)^{99}}$

b) $g(x) = \dfrac{1}{\sqrt[3]{x^2+x+1}}$ $\Rightarrow$ $(x^2+x+1)^{-1/3}$ $\Rightarrow$ $g'(x) = -(1/3)(x^2+x+1)^{-4/3}(2x+1)$

$$\boxed{= \frac{-(2x+1)}{3(\sqrt[3]{x^2+x+1})^4}}$$

c) $h(t) = \left(\dfrac{t-2}{2t+1}\right)^9$

$h'(t) = 9\left(\dfrac{t-2}{2t+1}\right)^8 \cdot \left(\dfrac{5}{(2t+1)^2}\right)$

using quotient rule: $\left[\left(\frac{f}{g}\right)' = \dfrac{(f')(g)-(f)(g')}{g^2}\right]$

$$= \frac{(2t+1)-(t-2)(2)}{(2t+1)^2}$$

$$= \frac{5}{(2t+1)^2}$$

$h'(t) = 9\left(\dfrac{t-2}{2t+1}\right)^8 \cdot \left(\dfrac{5}{(2t+1)^2}\right)$

EXAMPLE 24

Find the derivative of the following functions:

use product rule, chain rule

a) $y = (2x+1)^5(x^3 - x + 1)^4$ $\Rightarrow$ ~~$y' = 5(2x+1)^4 (x^3-x+1)^4 + (2x+1)^5(3x^2-1$~~

$y' = 5(2x+1)^4(2)(x^3-x+1)^4 + (2x+1)^5(4)(x^3-x+1)^3(3x^2-1)$

$= (2x+1)^4(x^3-x+1)^3 \cdot 10(x^3-x+1) + 4(2x+1)(3x^2-1)$

$= (2x+1)^4(x^3+x+1)^3(34x^3+12x^2-18x+6)$

b) $f(x) = \sin(\cos(\tan x))$

$a(x) = \sin(x)$
$b(x) = \cos(x)$
$c(x) = \tan(x)$

$f(x) = a(b(c(x)))$

$f'(x) = a'(b(c(x)))\, b'(c(x))\, c'(x)$

$a'(x) = \cos(x)$
$b'(x) = -\sin(x)$
$c'(x) = (\sec^2 x)$

$f'(x) = a'(b(c(x)))\, b'(c(x))\, c'(x)$
$= \cos(\cos(\tan(x)))(-\sin(\tan x))(\sec^2 x)$

WITHOUT "TRAINING WHEELS."

$f(x) = \sin(\cos x \cdot \tan x))$
$f'(x) = \cos(\cos x \tan x)(\sin x(\tan x) \ldots$
$\ldots(\sec^2 x)$

c) $d(p) = \sqrt{\sec p^3}$

$\Rightarrow d(p) = \sqrt{\sec p^3}$

$\Rightarrow (\sec p^3)^{1/2}$

$\Rightarrow (\tfrac{1}{2})(\sec p^3)^{-1/2}(\sec p^3 \tan p^3)(3p^2)$

$\Rightarrow \dfrac{(\sec p^3 \tan p^3)(3p^2)}{2\sqrt{\sec p^3}} \Rightarrow \tfrac{3}{2}p^2\sqrt{\sec p^3} \cdot \tan p^3$

DERIVATIVE OF THE NATURAL LOGARITHMIC FUNCTION

$$\frac{d}{dx}[\ln x] = \frac{1}{x}, \quad x > 0$$

Proof:

Let $y = \ln x$

$e^y = x$

$\frac{d}{dx}[e^y] = \frac{d}{dx}[x]$

$e^y \cdot \frac{dy}{dx} = 1 \Rightarrow \frac{dy}{dx} = \frac{1}{e^y} = \frac{1}{x}$

$\frac{d}{dx}[\ln x] = \frac{1}{x}$

EXAMPLE 25

Find the derivative of the following functions:

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a) $y = \ln(x^3 + 1)$ $\Rightarrow y' = \frac{1}{x^3+1}(3x^2) \Rightarrow \boxed{\dfrac{3x^2}{x^3+1}}$

b) $f(x) = \ln(\sin x)$ $\Rightarrow f'(x) = \frac{1}{\sin x}(\cos x) \Rightarrow \frac{\cos x}{\sin x} \Rightarrow \boxed{\cot x}$

c) $d(p) = \sqrt{\ln p}$ $\Rightarrow (\ln p)^{1/2} \Rightarrow (\tfrac{1}{2})(\ln p)^{-1/2} \cdot (\tfrac{1}{p})$

$\Rightarrow \boxed{\dfrac{1}{2p\sqrt{\ln p}}}$

EXAMPLE 26

Differentiate the following function:

$$q(x) = \ln \frac{x\sqrt[4]{1-\tan x}}{(x^2-5)^6}$$

$\Rightarrow \ln x + \ln(1-\tan x)^{1/4} - \ln(x^2-5)^6$

$= \ln x + \frac{1}{4}\ln(1-\tan x) - 6\ln(x^2-5)$

$q'(x) = \frac{1}{x} + \frac{1}{4}\left(\frac{1}{1-\tan x}\right)(-\sec^2 x) - 6\left(\frac{1}{x^2-5}\right)(2x)$

$q'(x) = \frac{1}{x} - \frac{\sec^2 x}{4(1-\tan x)} - \frac{12x}{x^2-5}$

DERIVATIVE INVOLVING ABSOLUTE VALUE

If u is a differentiable function of x such that $u \neq 0$, then

$$\frac{d}{dx}[\ln|u|] = \frac{u'}{u}.$$

DERIVATIVES FOR BASES OTHER THAN e

Let a be a positive real number ($a \neq 1$) and let u be a differentiable function of x.

PROOF:

1. $\frac{d}{dx}[a^x] = (\ln a)a^x$	2. $\frac{d}{dx}[a^u] = (\ln a)a^u \frac{du}{dx}$
3. $\frac{d}{dx}[\log_a x] = \frac{1}{(\ln a)x}$	4. $\frac{d}{dx}[\log_a u] = \frac{1}{(\ln a)u} \frac{du}{dx}$

$\frac{d}{dx}[a^x] = (\ln a)a^x$

Note $a^x = e^{\ln a^x} = e^{(\ln a)x}$

$\frac{d}{dx}[a^x] = \frac{d}{dx}\left[e^{x(\ln a)}\right]$

$= (\ln a) \cdot a^x(\ln a)$

$\blacksquare$

EXAMPLE 27

Find the derivative of the following functions:

a) $y = e^x$ (using the 1st property) $\Rightarrow y' = (\ln e)e^x$

$= (1)e^x$

b) $f(x) = 3^{x^2-x-12}$ $\Rightarrow f'(x)\ \ln 3(3^{x^2-x-12})(2x-1)$

$\Rightarrow \ln 3\,(2x-1)(3^{x^2-x-12})$

c) $h(n) = \log_9 \frac{(x+1)^5}{\sqrt[3]{x-1}}$

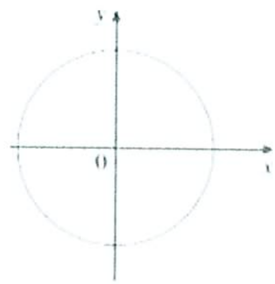
$h(x) = \log_9 (x+1)^5 - \log_9 (x-1)^{1/3}$

$= 5\log_9(x+1) - \frac{1}{3}\log_9(x-1) \Rightarrow 5\left(\frac{1}{\ln 9(x+1)}(1)\right) - \frac{1}{3}\left(\frac{1}{\ln 9(x-1)}(1)\right)$

$\frac{5}{\ln 9(x+1)} - \frac{1}{3\ln 9(x-1)}$

IMPLICIT DIFFERENTIATION

How would we find the slopes of tangent lines of the following graph:



(a) $x^2 + y^2 = 25$

EXAMPLE 28

a) If $x^2 + y^2 = 25$, find $\frac{dy}{dx}$

b) Determine the equation of the tangent line at the point (3,4)

a) $x^2 + y^2 = 25$

$2x + 2y\frac{dy}{dx} = 0$

$\frac{dy}{dx} = \frac{-x}{y}$

b) @ $(\overset{x}{3}, \overset{y}{4})$

$m = -3/4$

$y = mx + b$

$4 = (-3/4)3 + b$

$b = 25/4$

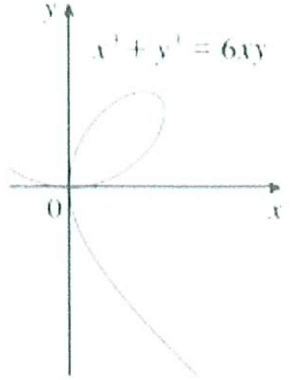
$y = -3/4\,x + 25/4$

EXAMPLE 29

a) Find y' if $x^3 + y^3 = 6xy$

b) Find the equation of the tangent line at $(3,3)$

c) At what point in the 1st quadrant is the tangent line horizontal?



The folium of Descartes

a) $3x^2 + 3y^2 \frac{dy}{dx} = 6y + 6x \frac{dy}{dx}$

$(3y^2 - 6x) \frac{dy}{dx} = 6y - 3x^2$

$\frac{dy}{dx} = \frac{6y - 3x^2}{(3y^2 - 6x)}$

$\Rightarrow \frac{2y - x^2}{y^2 - 2x}$

b) @ $(3,3)$

$m = \frac{2(3) - 3^2}{3^2 - 2(3)} = -1$

$y = -x + b$

$3 = -3 + b \Rightarrow b = 6$

$y = -x + 6$

c) $\frac{dy}{dx} = 0$

$\Rightarrow 2y - x^2 = 0 \quad (y^2 - 2x \neq 0)$

$y = \frac{1}{2}x^2$

$x^3 + (\frac{1}{2}x^2)^3 = (6x(\frac{1}{2}x^2))$

$x^3 + \frac{1}{8}x^6 = 3x^3$

$\frac{1}{8}x^6 = 2x^3$

$\frac{x^6}{x^3} = \frac{16x^3}{x^3} \quad (x \neq 0 \text{ in } Q1)$

$x^3 = 16$

$x = 2^{4/3}$

$\therefore (2^{4/3}, 2^{5/3})$ has a horizontal tangent line.

EXAMPLE 30

Determine y':

a) $y^5 + 3x^2y^2 + 5x^4 = 12 \Rightarrow 5y^4 \frac{dy}{dx} + 6xy^2 + 3x^2 2y \frac{dy}{dx} \ldots + 20x^3 = 0$

$(5y^4 + 6x^2y) \frac{dy}{dx} = -6xy^2 - 20x^3$

$\frac{dy}{dx} = \frac{-6xy^2 - 20x^3}{5y^4 + 6x^2y}$

b) $\sin(x + y) = y^2 \cos x$

$\cos(x+y)(1 + \frac{dy}{dx}) = \ldots$

$\ldots = 2y \frac{dy}{dx} \cos x + y^2(-\sin x)$

$\cos(x+y) + \cos(x+y) \frac{dy}{dx} = 2y \frac{dy}{dx} \cos x - y^2 \sin x$

$(\cos(x+y) - 2y) \frac{dy}{dx} = -\cos(x+y) - y^2 \sin x$

$$\boxed{\frac{dy}{dx} = \frac{-\cos(x+y) - y^2 \sin x}{\cos(x+y) - 2y}}$$

EXAMPLE 31

Determine y'' if

$x^4 + y^4 = 16$

$4x^3 + 4y^3 \frac{dy}{dx} = 0$

$\frac{dy}{dx} = -\frac{x^3}{y^3}$

$$\frac{d^2y}{dx^2} = -\left(\frac{3x^2y^3 - x^3(3y^2)\frac{dy}{dx}}{y^6}\right)$$

$$= -\left(\frac{3x^2y^3 - 3x^3y^2(-\frac{x^3}{y^3})}{y^6}\right)$$

$$\boxed{= -\left(\frac{3x^2y^3 + 3x^6(\frac{1}{y})}{y^6}\right)}$$

x

EXAMPLE 32

Midterm will cover up until the chain rule.

- 10 multiple choice
- 3 problems (1 2-part question)
- 45 minutes.
  - no calculator
  - no cell phones, etc.

Differentiate the following: $y = \dfrac{x^{3/4}\sqrt{x^2+1}}{(3x+2)^5}$

$$y = \frac{x^{3/4}(x^2+1)^{1/2}}{(3x+2)^5} \Rightarrow \ln y = \ln\left[\frac{x^{3/4}(x^2+1)^{1/2}}{(3x+2)^5}\right]$$

$$\ln y = (3/4)\ln x + (1/2)\ln(x^2+1) - 5\ln(3x+2)$$

$$\frac{1}{y}\cdot\frac{dy}{dx} = \frac{3}{4x} + \frac{1}{\cancel{2}(x^2+1)}(\cancel{2}x) - \frac{5}{(3x+2)}(3)$$

$$\frac{dy}{dx} = y\left[\frac{3}{4x} + \frac{x}{x^2+1} - \frac{15}{3x+2}\right] \Rightarrow \frac{dy}{dx} = \frac{x^{3/4}\sqrt{x^2+1}}{(3x+2)^5}\left[\frac{3}{4x} + \frac{x}{x^2+1} - \frac{15}{3x+2}\right]$$

(logarithmic differentiation)

CONTINUITY AND DIFFERENTIABILITY OF INVERSE FUNCTIONS

Let f be a function whose domain is an interval I. If f has an inverse function, then the following statements are true.

1. If f is continuous on its domain, then f^{-1} is continuous on its domain.
2. If f is differentiable on an interval containing c and $f'(c) \neq 0$, then f^{-1} is differentiable at $f(c)$.

DERIVATIVE OF AN INVERSE FUNCTION

Let f be a function that is differentiable on an interval I. If f has an inverse function g, then g is differentiable at any x for which $f'(g(x)) \neq 0$. Moreover,

$$\boxed{g'(x) = \frac{1}{f'(g(x))}, \quad f'(g(x)) \neq 0.}$$

EXAMPLE 33

Let $f(x) = 3x - 5$.

a) What is the value of $f^{-1}(x)$ when $x = -2$?

b) What is the value of $(f^{-1})'(x)$ when $x = -2$?

Horizontal line test
↳ is Function 1-1 (PASSES,
↳ IF 1-1, then has inver

$F(x) = 3x - 5$ has inverse (Passes HLT)

a) $x = 3y - 5$

$y = \frac{x+5}{3} = f^{-1}(x) = \frac{x+5}{3}$

So $f^{-1}(-2) = \frac{(-2)+5}{3}$ $\boxed{\Rightarrow 1}$

b) $y^{-1} = \frac{x+5}{3} \Rightarrow \left[\frac{x}{3} + \frac{5}{3}\right]$

$(f^{-1})'(x) = \frac{1}{3}$

$(f^{-1})'(x) = \frac{1}{3}$ (oops)

$\boxed{(f^{-1})'(-2) = \frac{1}{3}}$

EXAMPLE 33

Let $f(x) = 2x^3 + 1$.

a) What is the value of $f^{-1}(x)$ when $x = -15$?

b) What is the value of $(f^{-1})'(x)$ when $x = -15$?

LET'S SAY $f(a) = f(b)$

$2a^3 + 1 = 2b^3 + 1$

$2a^3 = 2b^3$

$a^3 = b^3$

$a = b$

(because it's odd)

Thus, f is 1-1 has inv.

$f(x) = 2x^3 + 1$

$y = 2x^3 + 1$

$x = 2y^3 + 1$

$y^3 = \frac{-1 + x}{2}$

$y = \sqrt[3]{\frac{-1+x}{2}}$

a) $f^{-1}(-15)$

$= \sqrt[3]{\frac{(-15-1)}{2}}$

$\boxed{= -2}$

b)

$(f^{-1})'(-15) = \frac{1}{(f'(f^{-1}(-15))}$

$= \frac{1}{(f'(-2))}$

$f'(x) = 6x^2 \Rightarrow \frac{1}{24}$

OR

$(f^{-1}(x))' = \frac{1}{f'(f^{-1}(x))}$

$(f^{-1}(-2))' = \frac{1}{f'(f^{-1}(-2))}$

$= \frac{1}{f'(1)} = \frac{1}{3}$

DERIVATIVES OF INVERSE TRIGONOMETRIC FUNCTIONS

$\frac{d}{dx}[\arcsin u] = \frac{u'}{\sqrt{1-u^2}}$

$\frac{d}{dx}[\arccos u] = \frac{-u'}{\sqrt{1-u^2}}$

$\frac{d}{dx}[\arctan u] = \frac{u'}{1+u^2}$

$\frac{d}{dx}[\text{arccot}\, u] = \frac{-u'}{1+u^2}$

$\frac{d}{dx}[\text{arcsec}\, u] = \frac{u'}{|u|\sqrt{u^2-1}}$

$\frac{d}{dx}[\text{arccsc}\, u] = \frac{-u'}{|u|\sqrt{u^2-1}}$

EXAMPLE 34

If $f(x) = \arcsin(x^2 - 1)$, then find:

a) The domain of f

b) $f'(x)$

c) The domain of f'

The domain of a function is the range of the inverse.

a) $\text{dom} f = ?$

The range of $\sin x$ is $[-1, 1]$
thus, the domain of $\arcsin x$ is also $[-1, 1]$

$$\text{dom} f = \{x \in \mathbb{R} \mid -1 \leq x^2 - 1 \leq 1\}$$
$$= \{x \in \mathbb{R} \mid 0 \leq x^2 \leq 2\}$$
$$= \{x \in \mathbb{R} \mid -\sqrt{2} \leq x \leq \sqrt{2}\}$$
$$\therefore \text{dom} f = [-\sqrt{2}, \sqrt{2}]$$

b) $f(x) = \arcsin(\underbrace{x^2 - 1}_{u})$

$$f'(x) = \frac{2x}{\sqrt{1-(x^2-1)^2}}$$
$$= \frac{2x}{\sqrt{1-(x^4-2x^2+1)}}$$
$$= \frac{2x}{\sqrt{x^2(2-x^2)}}$$

c) $\text{dom} f' = ?$

$\sqrt{x^2(2-x^2)} = 0$?

$x^2(2-x^2) = 0$

$x \neq 0, \pm\sqrt{2}$

also, $x^2(2-x^2) > 0$

$2 - x^2 > 0$

$2 > x^2$

$\Rightarrow -\sqrt{2} < x < \sqrt{2}$

$\therefore \text{dom} f'$
$= (-\sqrt{2}, 0) \cup (0, \sqrt{2})$

EXAMPLE 35

Let $f(x) = x \cdot \arctan\sqrt{x}$. Determine the equation of the tangent line at the point $\left(1, \frac{\pi}{4}\right)$.

$m = f'(1)$

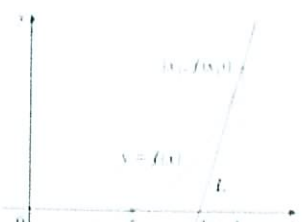
$$F'(x) = \arctan\sqrt{x} + x \times \frac{\frac{1}{2\sqrt{x}}}{1+(\sqrt{x})^2}$$
$$= \arctan\sqrt{x} + \frac{\sqrt{x}}{2(1+x)}$$

$$f'(1) = \arctan\sqrt{1} + \frac{\sqrt{1}}{2(1+1)} = \frac{\pi}{4} + \frac{1}{4} = \frac{\pi+1}{4}$$

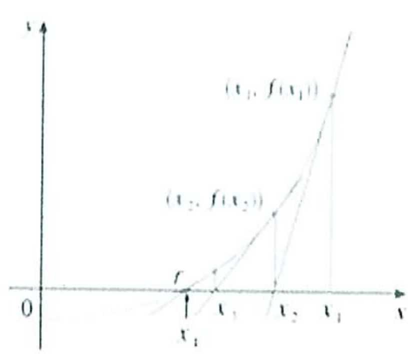
$$y = \left(\frac{\pi+1}{4}\right)x + b \Rightarrow \frac{\pi}{4} = \frac{(\pi+1)}{4}(1) + b$$

$$b = \frac{-1}{4} \qquad y = \frac{\pi+1}{4}x - \frac{1}{4}$$

NEWTON'S METHOD



$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$



EXAMPLE 36

Starting with $x_1 = 2$, find the third approximation x_3 to the root of the equation $x^3 - 2x - 5 = 0$.

Note: $f\left(2\frac{531}{5615}\right) = 0.000185723\ldots$

(very close to 0)

Let $f(x) = x^3 - 2x - 5$, then

$f'(x) = 3x^2 - 2$.

$x_1 = 2$

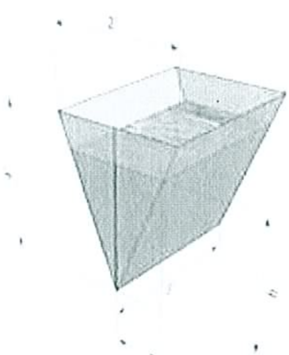
$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 2 - \frac{f(2)}{f'(2)} = \frac{21}{10}$

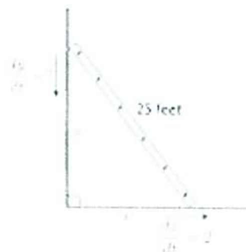
$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = \frac{21}{10} - \frac{f(\frac{21}{10})}{f'(\frac{21}{10})} =$

$2\left(\frac{531}{5615}\right)$

$\boxed{\text{OCT. } 24/16}$

RELATED RATES

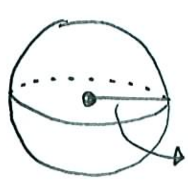




EXAMPLE 37

Air is being pumped into a spherical balloon so that its volume increases at a rate of $100\ \mathrm{cm}^3/s$. How fast is the radius of the balloon increasing when the diameter is 50 cm?

Volume of a sphere = $V = \frac{4}{3}\pi r^3$



$\frac{dv}{dt} = 100\ \mathrm{cm}^3/s$

$r = 25\mathrm{cm}$ ($d = 50\mathrm{cm}$)

$V = \frac{4}{3}\pi r^3$

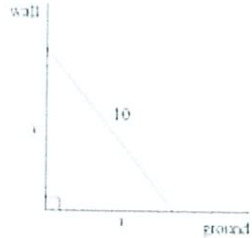
$\frac{dv}{dt} = \frac{4}{3}\pi \cdot 3r^2 \cdot \frac{dr}{dt}$

$\frac{dr}{dt} = \frac{1}{4\pi r^2} \cdot \frac{dv}{dt} = \frac{1}{4\pi (25)^2} \cdot 100$

$= \frac{1}{25\pi}$ cm/s

EXAMPLE 38

A ladder 10 ft long rests against a vertical wall. If the bottom of the ladder slides away from the wall at a rate of 1 ft/s, how fast is the top of the ladder sliding down the wall when the bottom of the ladder is 6 ft from the wall?



$$\frac{dx}{dt} = 1\,\text{ft/s}$$

Find $\frac{dy}{dt}$ when $x = 6$ ft.

$6^2 + y^2 = 10^2$

$\therefore y = 8$ (Pythagorean triple)

→ x is positive, magnitude increasing
y is decreasing, magnitude decreasing

$x^2 + y^2 = 10^2$

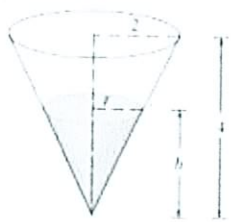
$$2x \cdot \frac{dx}{dt} + 2y \cdot \frac{dy}{dt} = 0$$

$$\frac{dy}{dt} = \frac{-x}{y} \cdot \frac{dx}{dt} \Rightarrow -\frac{6}{8} \cdot 1\,\text{ft/s}$$

$$= -\frac{3}{4}\,\text{ft/s}$$

EXAMPLE 39

A water tank has the shape of an inverted circular cone with base radius 2 m and height 4 m. If water is being pumped into the tank at a rate of 2 m^3/min, find the rate at which the water level is rising when the water is 3 m deep.



$$\frac{dV}{dt} = 2\,\text{m}^3/\text{min}$$

Volume of Cone

$V = \left(\frac{1}{3}\right)\pi r^2 h$

↳ use ratios to put 'r' in terms of 'h'

$r/2 = h/4$

$r = \frac{h}{2}$

→ $V = \frac{1}{3}\pi\left(\frac{h}{2}\right)^2 h$

$$V = \frac{1}{3}\pi \cdot \frac{h^2}{4} \cdot h = \frac{\pi}{12} \cdot h^3$$

$$\frac{dV}{dt} = \frac{\pi}{12} \cdot 3h^2 \cdot \frac{dh}{dt}$$

$$\frac{dh}{dt} = \frac{4}{\pi h^2} \cdot \frac{dV}{dt}$$

$$= \frac{4}{\pi (3)^2} \cdot (2)$$

$$= \frac{8}{9\pi}\,\text{m/min}$$

EXAMPLE 40

Car A is traveling west at 50 mph and car B is traveling north at 60 mph. Both are headed for the intersection of the two roads. At what rate are the cars approaching each other when car A is 0.3 mi and car B is 0.4 mi from the intersection?

$$\frac{dx}{dt} = -50\,\text{mph}$$

→ z (distance between two points is decreasing in magnitude)

$$\frac{dy}{dt} = -60\,\text{mph}$$

$x^2 + y^2 = z^2$

$$\cancel{2}x \cdot \frac{dx}{dt} + \cancel{2}y \cdot \frac{dy}{dt} = \cancel{2}z \cdot \frac{dz}{dt}$$

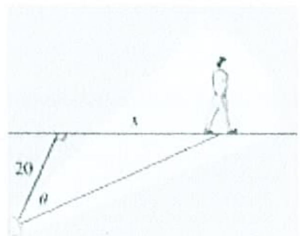
$$\frac{dz}{dt} = \frac{1}{z} \cdot \left[x\frac{dx}{dt} + y\frac{dy}{dt}\right]$$

$\Rightarrow \frac{1}{0.5\,\text{mi}} \left[0.3\,\text{m}(-50\,\text{mph}) \ldots\right.$

$\Rightarrow -18$ mph $\ldots \left. + 0.4(-60\,\text{mph})\right]$

EXAMPLE 41

A man walks along a straight path at a speed of 4 ft/s. A searchlight is located on the ground 20 ft from the path and is kept focused on the man. At what rate is the searchlight rotating when the man is 15 ft from the point on the path closest to the searchlight?



$x = 15\text{ ft}$

$y = 20\text{ ft}$

$\dfrac{dx}{dt} = 4\text{ ft/s}$

$\Rightarrow \cos\theta = \dfrac{20\text{ft}}{25\text{ft}} = \dfrac{4}{5}$

$\tan\theta = x/20$

$x = 20\tan\theta$

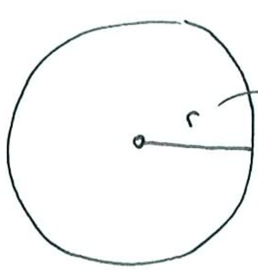
$\dfrac{dx}{dt} = 20\sec^2\theta \cdot \dfrac{d\theta}{dt}$

$\dfrac{d\theta}{dt} = \dfrac{\cos^2\theta}{20} \cdot \dfrac{dx}{dy}$

$\Rightarrow \dfrac{d\theta}{dt} = \dfrac{(4/5)^2}{20} \cdot (4)$

$= \dfrac{16}{125}\text{ rad/s}$

A rock is thrown into a pond of still water, causing a circular ripple. If the radius of the circle increases at a constant rate of 0.5 m/s, how fast is the area of the circle increasing when the radius of the ripple is 2m?



$r = 2\text{m}$

$\dfrac{dr}{dt} = 0.5\text{ m/s}$

$\dfrac{dA}{dt} = ?$

$A = \pi r^2$

$\dfrac{dA}{dt} = 2\pi r \cdot \dfrac{dr}{dt}$

$= 2\pi(2) \cdot (0.5)$

$= 2\pi\text{ m}^2/\text{s}$