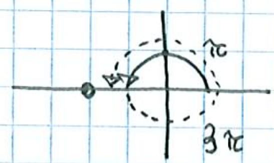
EXTRA EXAMPLES

OCT. 3RD /16 (CALCULUS)

a) FIND ALL POINTS ON THE GRAPH OF $f(x) = x + \sin x$ WHERE THE TANGENT LINE IS HORIZONTAL.

$f(x) = x + \sin x$
$f'(x) = 1 + \cos x$




$\cos x = -1$
$\Rightarrow x = \pi, 3\pi, 5\pi \ldots$
$\Rightarrow x = -\pi, -3\pi, -5\pi \ldots$

$x = (2n+1)\pi \quad n \in \mathbb{Z}$

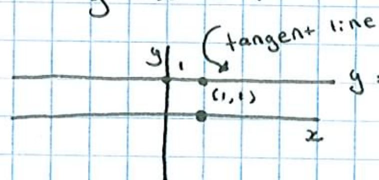
$((2n+1)\pi, (2n+1)\pi) \quad \forall n \in \mathbb{Z}$

b) SHOW THAT THE TANGENT LINE TO THE GRAPH OF $y = x^n$ AT $[1, 1]$ HAS A Y-INTERCEPT OF $1-n$.

$y = x^n$
$y' = nx^{n-1} \quad (n \neq 0)$

$\boxed{\text{IF } n = 0}$
$y = x^0 = 1$

(sub $x = 1$) $\quad m = n(1)^{n-1} = n$
$y = nx + b$




(sub 1,1) $\quad 1 = n(1) + b$
$b = 1 - n$

This has a ~~tangent~~ y-int of
$y = 1 - 0$

∎

# EXAMPLE 11

Find the derivative of the following functions:

a) $f(x) = \frac{3\cos x}{5}$ $\Rightarrow f'(x) \left(\frac{3}{5}\right) \cos x$

$\boxed{\Rightarrow f'(x) \cdot \left(\frac{3}{5}\right) \sin x}$

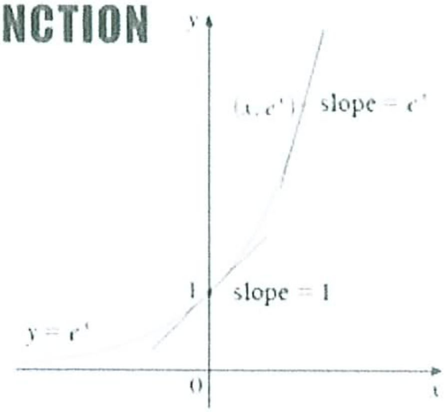
b) $g(x) = \sin x - 3x^4$

$g'(x) = \sin x - 3x^4$

$\boxed{g'(x) = \cos x - 12x^3}$

# DERIVATIVE OF THE NATURAL EXPONENTIAL FUNCTION

$$\boxed{\frac{d}{dx}[e^x] = e^x}$$




$\boxed{\text{Oct. 3RD/16}}$

PROVE THAT IF $f(x) = e^x$ that $f'(x) = e^x$ as well.

Proof: $f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$

$= \lim_{\Delta x \to 0} \frac{e^{x+\Delta x} - e^x}{\Delta x}$

$= \lim_{\Delta x \to 0} \frac{e^x(e^{\Delta x} - 1)}{\Delta x}$

By definition, as $\Delta x \to 0$

$(1+\Delta x)^{1/\Delta x} \to e$

OR

$e^{\Delta x} \to (1+\Delta x)$ Sub in to limit.

as $\Delta x \to 0$

$\lim_{\Delta x \to 0} \frac{e^x(1+\Delta x - 1)}{\Delta x}$

$\lim_{\Delta x \to 0} \frac{\Delta x\, e^x}{\Delta x}$

$= e^x$ ∎

# EXAMPLE 12

Find the derivative of the following functions:

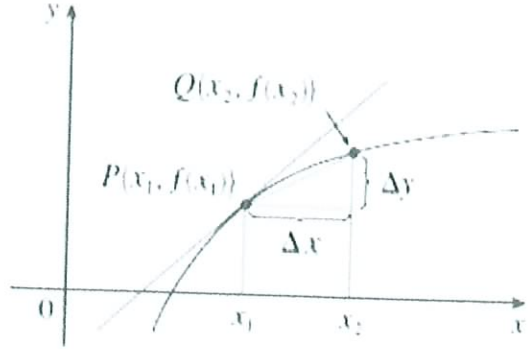
a) $f(x) = -7e^x$ $\Rightarrow f'(x) = (-7)(e^x) = \boxed{-7e^x}$

b) $g(x) = 2x^3 - 6\cos x + \frac{e^x}{2}$

$g'(x) = (2)(3x^2) + 6\sin x + \frac{e^x}{2}$

$\boxed{= 6x^2 + 6\sin x + \frac{e^x}{2}}$

# RATE OF CHANGE




Average ROC

$$\boxed{\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}}$$

$m_{PQ}$ = average rate of change

$m = f'(x_1)$ = instantaneous rate of change

# RATE OF CHANGE

If $s = s(t)$ is the position function (displacement) for an object moving along a straight line, the velocity $v$ of the object at time $t$ is given by:

$$\boxed{v(t) = s'(t)}$$

The position of a free-falling object (neglecting air resistance) under the influence of gravity can be represented by the equation

$$\boxed{s(t) = \frac{1}{2}gt^2 + v_0 t + s_0}$$

$g = -9.8\ m/s^2$ or $-32\ ft/s^2$

$v_0$ -- initial velocity

$s_0$ -- initial position

$_0$ "naught" (initial)

Given:

$V_0 = 300$ ft/s

$S_0 = 65$ ft

$g = -32$ ft/sec$^2$

# EXAMPLE 13

A paintball gun shoots a paint ball 300 ft/s straight up in the air off the top of a 65 ft building.

a) What is the position function for the paintball?

b) What is the paintball's average velocity for $t \in [1,2]$?

c) What is the paintball's velocity at $t = 2$ seconds?

d) When does the paintball hit the ground?

e) What is the maximum height of the paint ball and when does this happen?

a) $S(t) = \frac{1}{2}gt^2 + V_0 t + S_0$

$= \frac{1}{2}(-32\text{ ft/s}^2)(t)^2 + (300\text{ ft/s})(t) + (65)$

$= -16\text{ ft/s}^2\, t^2 + 300\text{ ft/s}\, t + 65\text{ ft}$

b) $ROC_{AVG} = \frac{S(2) - S(1)}{2 - 1}$ $\quad t \in [1,2]$

$= \frac{601 - 349}{1}$

$\boxed{= 252 \text{ ft/s}}$

c) $IROC = S'(2)$

$S'(t) = -32t + 300$

$S'(2) = -32(2) + 300$

$\boxed{= 236 \text{ ft/s}}$

d) $S(t) = 0$

$-16t^2 + 300t + 65 = 0$

$t = \frac{-300 \pm \sqrt{300^2 - 4(16)(65)}}{2(-16)}$

$\therefore t \simeq -0.21s$ or $\boxed{18.96s}$

$\therefore$ The paintball hits the ground @ $\simeq$ 19 seconds.

e) $S'(t) = -32t + 300 = 0$

$\boxed{t = 9\,3/8\ s}$ time

$\boxed{S(9\,3/8) = 1471\,\frac{1}{4}\text{ ft}}$ height.

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## PRODUCT RULE

$$\frac{d}{dx}[f(x)g(x)] = f(x)\frac{d}{dx}[g(x)] + g(x)\frac{d}{dx}[f(x)]$$

$$\boxed{(fg)' = f'g + fg'}$$

$$(fg)' = \lim_{\Delta x \to 0} \frac{f(x+\Delta x)g(x+\Delta x) - f(x)g(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{f(x+\Delta x)g(x+\Delta x) - f(x+\Delta x)g(x) + f(x+\Delta x)g(x) - f(x)g(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \left[ f(x+\Delta x)\left(\frac{g(x+\Delta x) - g(x)}{\Delta x}\right) + g(x)\left(\frac{f(x+\Delta x) - f(x)}{\Delta x}\right)\right]$$

$$= f(x)g'(x) + g(x)f'(x)$$

## EXAMPLE 14

Find the derivative of the following functions:

$f'(x) = 2x\sin x + x^2(\cos x)$

a) $f(x) = x^2 \sin x$

b) $T(x) = (6x^3)(7x^4)$

c) $g(x) = (3x^7 + 1)(x^3 - x)$

d) $h(x) = \sqrt{x} \cdot e^x - e^x \cos x$

$$f'(x) = \boxed{2x\sin x + x^2(\cos x)}$$

$$T'(x) = (6x^3)(28x^3) + (18x^2)(7x^4)$$

$$T'(x) = 126x^6 + 168x^6$$

$$\boxed{= 294x^6}$$

$$g'(x) = \boxed{(3x^7+1)(3x^2-1) + (21x^6)(x^3-x)}$$

GOOD ENOUGH.

$$h(x) = \sqrt{x} \cdot e^x - e^x \cos x$$

$$h'(x) = e^x(x^{1/2} - \cos x)$$

$$h'(x) = e^x\left[\left(\tfrac{1}{2}\right)x^{-1/2} + \sin x\right] + \ldots$$

$$\ldots e^x(x^{1/2} - \cos x)$$

$$\boxed{h'(x) = e^x\left(\sqrt{x} - \cos x + \frac{1}{2\sqrt{x}} + \sin x\right)}$$

## QUOTIENT RULE

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\frac{d}{dx}[f(x)] - f(x)\frac{d}{dx}[g(x)]}{[g(x)]^2}$$

$$\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{(g)^2}$$

PROOF...

# EXAMPLE 15

Determine $y'$ for the following: $y = \dfrac{x^2 + x - 2}{x^3 + 6}$

$$y = \frac{x^2 + x - 2}{x^3 + 6}$$

$$y' = \frac{(2x+1)(x^3+6) - (x^2+x-2)(3x^2)}{(x^3+6)^2}$$

$$y' = \frac{2x^4 + x^3 + 12x + 6 - 3x^4 - 3x^3 + 6x^2}{(x^3+6)^2}$$

$$y' = \frac{-x^4 - 2x^3 + 6x^2 + 12x + 6}{(x^3+6)^2}$$

# EXAMPLE 16

Find the derivative of the following functions:

where $n$ is a positive integer.

a) $f(x) = x^{-n}$, where $n \in \mathbb{Z}^+$

b) $g(x) = \tan x$

c) $h(x) = \dfrac{e^x + x^7 \sin x}{e^x}$

$$f(x) = \frac{1}{x^n}$$

$$f'(x) = \frac{0(x^n) - 1(nx^{n-1})}{(x^n)^2}$$

$$f'(x) = -nx^{n-1-2n}$$

$$\boxed{f'(x) = -nx^{-n-1}}$$

# EXAMPLE 17

Determine the equation to the tangent line at point $\left(1, \frac{1}{2}\right)$ to the curve:

$$y = \frac{\sqrt{x}}{1 + x^2}$$

$$y' = \frac{\frac{1}{2}x^{-1/2}(1+x^2) - \sqrt{x}\,(2x)}{(1+x^2)^2}$$

$$m = \frac{\frac{1}{2}(1)^{-1/2}(1+1^2) - \sqrt{1}\,(2(1))}{(1+1^2)^2}$$

$$= \frac{\frac{1}{2}(2) - 2}{4} \Rightarrow -\frac{1}{4}$$

$y = -\frac{1}{4}x + b$

$\frac{1}{2} = -\frac{1}{4}(1) + b$

$b = \frac{3}{4}$

$y = -\frac{1}{4}x + \frac{3}{4}$

$$\left(\frac{f}{g}\right)' = \lim_{\Delta x \to 0} \frac{\frac{f(x+\Delta x)}{g(x+\Delta x)} - \frac{f(x)}{g(x)}}{\Delta x}$$

$$\lim_{\Delta x \to 0} \frac{f(x+\Delta x)\,g(x) - f(x)\,g(x+\Delta x)}{\Delta x\, g(x+\Delta x)\, g(x)}$$

$$\lim_{\Delta x \to 0} \frac{f(x+\Delta x)\,g(x) - f(x)\,g(x) + f(x)\,g(x) - f(x)\,g(x+\Delta x)}{\Delta x\, g(x+\Delta x)\, g(x)}$$

$$\lim_{\Delta x \to 0} \left[\frac{g(x)}{g(x+\Delta x)\,g(x)}\left(\frac{f(x+\Delta x) - f(x)}{\Delta x}\right) - \dots\right.$$

$$\left.\dots \frac{f(x)}{g(x+\Delta x)\,g(x)}\left(\frac{g(x+\Delta x) - g(x)}{\Delta x}\right)\right]$$

$$= \frac{g(x)}{[g(x)]^2} \times f'(x) - \frac{f(x)}{[g(x)]^2} \times g'(x)$$

$$= \frac{f'(x)\,g(x) - f(x)\,g'(x)}{[g(x)]^2}$$

$$g(x) = \tan x = \frac{\sin x}{\cos x}$$

$$g'(x) = \frac{\cos x(\cos x) - \sin x(\sin x)}{(\cos x)^2}$$

$$g'(x) = \frac{\cos^2 x + \sin^2 x}{\cos^2 x}$$

$$g'(x) = \frac{1}{\cos^2 x} = \boxed{\sec^2 x}$$

$$h(x) = \frac{e^x + x^7 \sin x}{e^x}$$

$$h'(x) = \frac{(e^x + 7x^6 \sin x + x^7 \cos x)e^x \dots \dots - (e^x + x^7 \sin x)e^x}{(e^x)^2}$$

$$h'(x) = \frac{\cancel{e^x}\left(e^x + 7x^6 \sin x + x^7 \cos x \dots \dots - \cancel{e^x} - x^7 \sin x\right)}{(e^x)^{\cancel{2}}}$$

$$\boxed{= \frac{x^6(7\sin x + x\cos x - x\sin x)}{e^x}}$$

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# DERIVATIVES OF TRIG FUNCTIONS

| | |
|---|---|
| $\frac{d}{dx}(\sin x) = \cos x$ | $\frac{d}{dx}(\csc x) = -\csc x \cot x$ |
| $\frac{d}{dx}(\cos x) = -\sin x$ | $\frac{d}{dx}(\sec x) = \sec x \tan x$ |
| $\frac{d}{dx}(\tan x) = \sec^2 x$ | $\frac{d}{dx}(\cot x) = -\csc^2 x$ |

PROOF:

let $f(x) = \sec x$, then

$$= \frac{1}{\cos x} \quad \left(\left(\frac{f}{g}\right)' = \frac{f'g - g'f}{g^2}\right)$$

$$f'(x) = \frac{(0)\cos x - 1(-\sin x)}{\cos^2 x}$$

$$= \frac{\sin x}{\cos^2 x} \Rightarrow \frac{1\,\cancel{\sin x}}{\cos x} \times \frac{1}{\cos x}$$

$$= \tan x \sec x \quad \blacksquare$$

# EXAMPLE 18

Which point(s) on the following function contain a horizontal tangent line:

$$f(x) = \frac{\sec x}{1 + \tan x}$$

$$f'(x) = \frac{\sec x \tan x (1 + \tan x) - \sec x (\sec^2 x)}{(1 + \tan x)^2}$$

$$f'(x) = \frac{\sec x \tan x + \sec x \tan^2 x - \sec^3 x}{(1 + \tan x)^2}$$

$$= \frac{\sec x \tan x + \sec x (\sec^2 x - 1) - \sec^3 x}{(1 + \tan x)^2}$$

$$= \frac{\sec x \tan x + \sec^3 x - \sec x - \sec^3 x}{(1 + \tan x)^2}$$

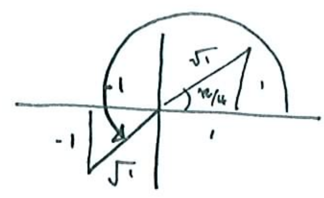
$$= \frac{\sec x (\tan x - 1)}{(1 + \tan x)^2}$$

$$= 0$$

$$\Rightarrow \sec x = 0 \quad \text{or} \quad \tan x = 1$$

$\times$

$$\frac{1}{\cos x} \neq 0$$




$$x = \frac{\pi}{4}, \frac{9\pi}{4}, \frac{17\pi}{4}$$

$$\frac{5\pi}{4}, \frac{13\pi}{4}$$

i.e. $x = \frac{(4n+1)\pi}{4}, n \in \mathbb{R}$

# HIGHER ORDER DERIVATIVES

| | | | | |
|---|---|---|---|---|
| *First derivative:* | $y'$, | $f'(x)$, | $\frac{dy}{dx}$, | $\frac{d}{dx}[f(x)]$ |
| *Second derivative:* | $y''$, | $f''(x)$, | $\frac{d^2y}{dx^2}$, | $\frac{d^2}{dx^2}[f(x)]$ |
| *Third derivative:* | $y'''$, | $f'''(x)$, | $\frac{d^3y}{dx^3}$, | $\frac{d^3}{dx^3}[f(x)]$ |
| *Fourth derivative:* | $y^{(4)}$, | $f^{(4)}(x)$, | $\frac{d^4y}{dx^4}$, | $\frac{d^4}{dx^4}[f(x)]$ |
| ⋮ | | | | |
| *nth derivative:* | $y^{(n)}$, | $f^{(n)}(x)$, | $\frac{d^ny}{dx^n}$, | $\frac{d^n}{dx^n}[f(x)]$ |

Points:

$$f\left(\frac{\pi}{4}\right) = \frac{\sec(\pi/4)}{1 + \tan(\pi/4)} = \frac{\frac{1}{\cos \pi/4}}{1+1} = \frac{\frac{2}{\sqrt{2}}}{2} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} \ldots$$

$$\ldots f\left(\frac{5\pi}{4}\right) = \frac{-\sqrt{2}}{2} = \left(\frac{(4n+1)\pi}{4}, \frac{\sqrt{2}}{2}\right) \; n \text{ is even}$$

## EXAMPLE 19

Determine is the second derivative of:

$$y = 3x^5 - 6x^2 + 2x - 5$$

$y' = (5)3x^4 - (2)6x + 2 - 0$

$y' = 15x^4 - 12x$

$y'' = (4)15x^3 - 12$

$y'' = 60x^3 - 12$

## EXAMPLE 20

Determine the 97th derivative of $f(x) = \cos x$ ☺

$f'(x) = -\sin(x) = f^5 = f^9 \ldots f^{97}$

$f''(x) = -\cos(x) = f^6 = f^{10}$

$f'''(x) = \sin(x) = f^7 = f^{11}$

$f^{IV}(x) = \cos(x) = f^8 = f^{12} \ldots f^{96}$

so...

$97/4 = 24\tfrac{1}{4}$

## ACCELERATION

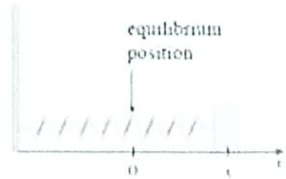
| | |
|---|---|
| $s(t)$ | Position function |
| $v(t) = s'(t)$ | Velocity function |
| $a(t) = v'(t) = s''(t)$ | Acceleration function |

## EXAMPLE 21

A mass on a spring vibrates horizontally on a smooth level surface (see the figure). Its equation of motion is $x(t) = 8 \sin t$, where $t$ is in seconds and $x$ in centimeters.

(a) Find the velocity and acceleration at time $t$.

(b) Find the position, velocity, and acceleration of the mass at time $t = 2\pi/3$. In what direction is it moving at that time?




a) $v(t) = x'(t)$

$\boxed{= 8 \cos t}$

$a(t) = v'(t)$

$\boxed{= -8 \sin t}$

## CONSIDER...

How would differentiate functions of the following form:

$$f(x) = \sqrt{x^2 + 1} \quad ?$$

## CHAIN RULE

If $y = f(u)$ is a differentiable function of $u$ and $u = g(x)$ is a differentiable function of $x$, then $y = f(g(x))$ is a differentiable function of $x$ and

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

or, equivalently,

$$\boxed{\frac{d}{dx}[f(g(x))] = f'(g(x))g'(x)}$$