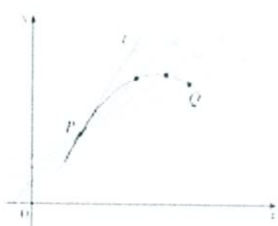
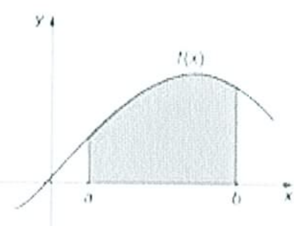
CHAPTER 3
DIFFERENTIATION

RECALL: PREVIEW OF CALCULUS

Tangent Line Problem



Area Problem



EXAMPLE 1

Sept. ~~28~~ 27/16

Sacha drains the water from a hot tub. The tub holds 1600L of water. It takes 2 hours for the water to drain completely. The volume of water in the hot tub is modelled by

$$V(t) = 1600 - \frac{t^2}{9}$$

where V is the volume (in litres) and t is the time (in minutes) with $t \in [0, 120]$. → interval between 0 and 120 minutes

a) Verify that the tub is empty after 2 hours.

$$V(120) = 1600 - \frac{(120)^2}{9}$$
$$= 1600 - 1600$$
$$= 0 \text{ L}$$

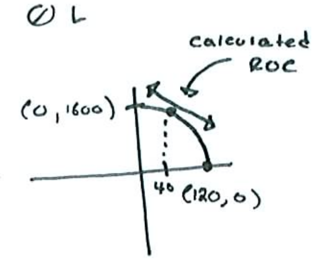
b) Approximate the instantaneous rate of change of the volume at the 40 minute mark.

$$V(40) = 1600 - \frac{(40)^2}{9}$$

$$ROC = \frac{\Delta V(t)}{\Delta t} = \text{NOT NEEDED..}$$

$$= \frac{V(40.01) - V(39.99)}{(40.01) - (39.99)} \Rightarrow -8.89 \text{ L/min}$$

↳ negative because value is decreasing

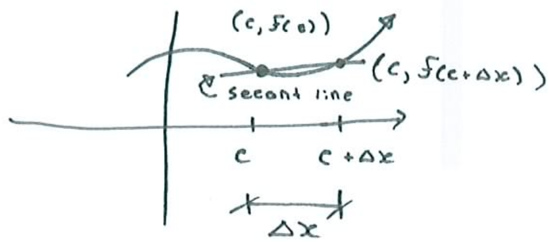


DEFINITION OF A TANGENT LINE WITH SLOPE m

If f is defined on an open interval containing c, and if the limit

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(c + \Delta x) - f(c)}{\Delta x} = m$$

exists, then the line passing through $(c, f(c))$ with slope m is the tangent line to the graph of f at the point $(c, f(c))$.



$$m = \frac{f(c+\Delta x) - f(c)}{\cancel{c} + \Delta x - \cancel{c}}$$

$$m = \frac{f(c+\Delta x) - f(c)}{\Delta x}$$

$$m = \lim_{\Delta x \to 0} \frac{f(c+\Delta x) - f(c)}{\Delta x}$$

So as $\Delta x \to 0$, the secant line goes to the tangent line.

EXAMPLE 2

For $f(x) = -3x - 5$, find the slope of the tangent line at $(1, -8)$

$\boxed{\text{Sept. 28/16}}$

$$m = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} \qquad x = 1$$

$$= \lim_{\Delta x \to 0} \frac{f(1+\Delta x) - f(1)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{-3(1+\Delta x) - 5 - (-3(1) - 5)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\cancel{-3} - 3\Delta x - \cancel{5} + \cancel{8}}{\Delta x} = 8$$

$$= \lim_{\Delta x \to 0} \frac{-3\Delta x}{\Delta x} \Rightarrow \boxed{-3}$$

EXAMPLE 3

For $f(x) = x^2 - 5$, find the slope of the tangent line at $(2, -1)$

$$m = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} \qquad x = 2$$

$$= \lim_{\Delta x \to 0} \frac{f(2+\Delta x) - f(2)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{(2+\Delta x)^2 - 5 - (2^2 - 5)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} (4 + \Delta x^2) - 5 - 4 + 5 \Rightarrow \frac{4 + 4\Delta x + \Delta x^2 - 5 + 1}{\Delta x}$$

$$= \lim_{x \to 0} \frac{(\Delta x)(4 + \Delta x)}{\Delta x} \Rightarrow \lim_{x \to 0} (4 + \Delta x) \boxed{= 4}$$

DEFINITION OF THE DERIVATIVE

The derivative of f at x is

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

provided the limit exists. For all x for which this limit exists, f' is a function of x.

The derivative can be uses to find the Instantaneous Rate of Change

Notations

$$\boxed{f'(x),\quad \frac{dy}{dx},\quad y',\quad \frac{d}{dx}[f(x)],\quad D_x[y].}$$

EXAMPLE 4

Determine the derivative of the following function

$$f(x) = 2x^3 - 5$$

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$$\lim_{\Delta x \to 0} \frac{2(x+\Delta x)^3 \cancel{-5} - (2x^3 \cancel{-5})}{\Delta x}$$

$$\lim_{\Delta x \to 0} 2(\cancel{x^3} + 3x^2\Delta x + 3x\Delta x^2 + \Delta x^3) - \cancel{2x^3}$$

$$\lim_{\Delta x \to 0} \frac{\Delta x(6x^2 + 6x\Delta x + 2\Delta x^2)}{\Delta x} \Rightarrow \lim_{\Delta x \to 0} (6x^2 + 6x\Delta x + 2\Delta x^2)$$

$$\boxed{= 6x^2}$$

EXAMPLE 5

Determine the slope of the tangent line at the point (7,2) of the following function

$$f(x) = \sqrt{x - 3}$$

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$$\lim_{\Delta x \to 0} \frac{\sqrt{x+\Delta x-3} - \sqrt{x-3}}{\Delta x} \times \frac{\sqrt{x+\Delta x-3} + \sqrt{x-3}}{\sqrt{x+\Delta x-3} + \sqrt{x-3}}$$

$$\lim_{\Delta x \to 0} \frac{x + \Delta x - 3 - x + 3}{\Delta x(\sqrt{x+\Delta x-3} + \sqrt{x-3})}$$

$$\lim_{\Delta x \to 0} \frac{1}{\sqrt{x+\Delta x-3} + \sqrt{x-3}} \Rightarrow \frac{1}{2\sqrt{x-3}}$$

$$f'(7) = \frac{1}{2\sqrt{7-3}}$$

$$= \frac{1}{4}$$

$\therefore$ the slope of the ~~triangle~~ tangent line @ (7,2) is 1/4.

EXAMPLE 6

Determine the equation of the tangent line at the point $\left(0, \frac{1}{3}\right)$ of the following function

$$f(x) = \frac{1}{x+3}$$

$$\Rightarrow \lim_{\Delta x \to 0} \frac{\frac{1}{x+\Delta x+3} - \frac{1}{x+3}}{\Delta x}$$

$$\Rightarrow \lim_{\Delta x \to 0} \frac{x+3-(x+\Delta x+3)}{\Delta x\,(x+\Delta x+3)(x+3)}$$

$$\Rightarrow \lim_{\Delta x \to 0} \frac{-1}{(x+\Delta x+3)(x+3)} = \frac{-1}{(x+3)^2}$$

$$m = f'(0) = \frac{-1}{(0+3)^2} = \frac{-1}{9}$$

$$y = -\tfrac{1}{9}x + b$$

$$\tfrac{1}{3} = -\tfrac{1}{9}(0) + b$$

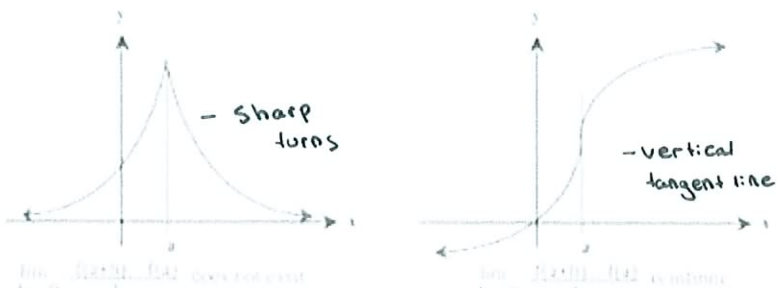
$$b = \tfrac{1}{3}$$

$$\therefore\ y = -\tfrac{1}{9}x + \tfrac{1}{3}$$

(Midterm 10 MP
3 Problems)

(Chapters 1-3)

FUNCTIONS THAT ARE NOT DIFFERENTIABLE EVERYWHERE



- sharp turns

- vertical tangent line

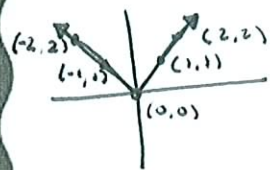
FUNCTIONS THAT ARE NOT DIFFERENTIABLE EVERYWHERE

(discontinuous)

EXAMPLE 7

Discuss the differentiability at $x = 0$ of $f(x) = |x|$

$f(x) = |x|$



$$f(x) = |x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

$(@\ x = 0)$

$$\lim_{\Delta x \to 0^+} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0^+} \frac{\cancel{x} + \Delta x - \cancel{x}}{\Delta x} \Rightarrow 1$$

$$\lim_{\Delta x \to 0^-} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0^-} \frac{-(\cancel{x} + \Delta x) - (-\cancel{x})}{\Delta x} \Rightarrow -1$$

∴ ~~the limit as~~

$$\lim_{\Delta x \to 0} \frac{f(0+\Delta x) - f(0)}{\Delta x}$$

$\boxed{\text{DNE}}$

∴ f is not diff. at $x = 0$

THEOREM: DIFF ⟹ CONTINUITY

If f is differentiable at $x = c$, then f is continuous at $x = c$.

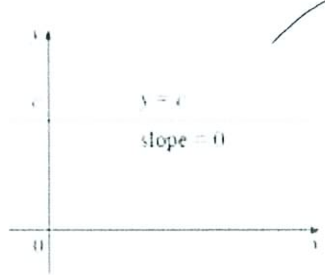
differentiable ⇒ continuous
but continuous may not be differentiable.

1. If a function is differentiable at $x = c$, then it is continuous at $x = c$. So, differentiability implies continuity.
2. It is possible for a function to be continuous at $x = c$ and not be differentiable at $x = c$. So, continuity does not imply differentiability.

$\boxed{\text{Sept. 30/16}}$

CONSTANT RULE

$$\boxed{\frac{d}{dx}(c) = 0}$$



So if $f(x) = -3$, then $f'(x) = 0$.

Proof:

$f(x) = c \Rightarrow f'(x) = 0$

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$$= \lim_{\Delta x \to} \frac{c - c}{\Delta x} \Rightarrow \frac{0}{\Delta x}$$

$= 0$

CONSIDER...

$f(x)$	$f'(x)$
x	1
x^2	$2x$
x^3	$3x^2$
x^4	$4x^3$
x^5	$5x^4$
x^6	$6x^5$

etc.

POWER RULE

If n is a positive integer, then

$$\boxed{\frac{d}{dx}(x^n) = nx^{n-1}}$$

So if $f(x) = x^{99}$, then $f'(x) = 99x^{98}$

& $f(x) = x^n \Rightarrow f'(x) = nx^{n-1}$

(actually true for all $n \in \mathbb{R} \setminus \{0\}$)

PROOF:

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{(x+\Delta x)^n - x^n}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \dots$$

$$\frac{\dots \binom{n}{0} x^n \Delta x^0 + \binom{n}{1} x^{n-1} \Delta x^1 + \dots + \binom{n}{n} x \dots \Delta x^n - x^n}{\Delta x}$$

$$\lim_{\Delta x \to 0} \frac{\cancel{x^n} + nx^{n-1}\Delta x + \binom{n}{2} x^{n-2} \Delta \dots + \binom{n}{n} x^2 \Delta x^n - \cancel{x^n}}{\Delta x}$$

$$\lim_{\Delta x \to 0} \frac{\cancel{\Delta x}\left(nx^{n-1} + \binom{n}{2} x^{n-2} \Delta x + \dots + \binom{n}{n} x^0 \Delta x^{n-1}\right)}{\cancel{\Delta x}}$$

$$\cancel{\lim_{\Delta x \to 0}} \boxed{= nx^{n-1}}$$

CONSTANT MULTIPLE RULE

If c is a constant and f is a differentiable function, then

$$\boxed{\frac{d}{dx}[cf(x)] = c\frac{d}{dx} f(x)}$$

So if $f(x) = x^{99}$
then $f'(x) = 99x^{98}$

PROOF:

Let $g(x) = cf(x)$

$$g'(x) = \lim_{\Delta x \to 0} \frac{g(x+\Delta x) - g(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{cf(x+\Delta x) - cf(x)}{\Delta x} \Rightarrow c\,\frac{f(x+\Delta x) - f(x)}{\Delta x} = cf'(x)$$

EXAMPLE 8

Find the derivative of the following functions:

a) $f(x) = \pi^3$ $\Rightarrow f'(x) = 0$ (Just a number..)

b) $g(x) = 12x^4$ $\Rightarrow 12 \cdot (4x^3) = g'(x) \Rightarrow g'(x) = \boxed{48x^3}$

c) $h(x) = -\frac{1}{x^2}$ PROPER CONVENTION.

$h'(x) = (-1)(x^{-2})$

$h'(x) = (-1)(-2x^{-3}) \Rightarrow 2x^{-3}$

$\boxed{\Rightarrow \frac{2}{x^3}}$

SUM AND DIFFERENCE RULES

The sum (or difference) of two differentiable functions f and g is itself differentiable. Moreover, the derivative of $f + g$ (or $f - g$) is the sum (or difference) of the derivatives of f and g.

$$\frac{d}{dx}[f(x) + g(x)] = f'(x) + g'(x)$$

$$\frac{d}{dx}[f(x) - g(x)] = f'(x) - g'(x)$$

EXAMPLE 9

Find the derivative of the following functions:

a) $f(x) = 2x^3 + 2x$ $\Rightarrow f'(x) = 2(3x^2) + 2(\overbrace{1x^0}^{1})$

$\Rightarrow f'(x) = \boxed{6x^2 + 2}$

b) $g(x) = 3x^3 - 8$ $\longrightarrow$ $g'(x) = 9x^2 - 0$

$\boxed{= 9x^2}$

c) $h(x) = 4x^{27} - 3x^8 + 6x^4 - 7\sqrt[3]{x} + e^5$ write as $x^{1/3}$

$h'(x) = 4(27x^{26}) - 3(8x^7) + 6(4x^3) - [7(x^{1/3})] + \cancel{5e^4}$ constant

$[7(\frac{1}{3}x^{-2/3}) + (0)$

$h'(x) = 108x^{26} - 24x^7 + 24x^3 - \frac{7}{3x^{2/3}}$

$$\boxed{h'(x) = 108x^{26} - 24x^7 + 24x^3 \ldots \ldots - \frac{7}{3\sqrt[3]{x^2}}}$$

EXAMPLE 10

Find the equation of the tangent line at $x = -1$ for
$$f(x) = 2x^5 - 7$$

$m = f'(-1)$

$f'(x) = 2(5x^4) - 0$

$f'(x) = 10x^4$

$f(-1) = 2(-1)^5 - 7$
$= -9$

@ $(-1, -9)$

$y = mx + b$
$-9 = 10(-1) + b$
$b = 1$

$\boxed{y = 10x + 1}$

RECALL....

$$\lim_{\Delta x \to 0} \frac{\sin \Delta x}{\Delta x} = 1 \quad \text{and} \quad \lim_{\Delta x \to 0} \frac{1 - \cos \Delta x}{\Delta x} = 0$$

PROOF:

$f(x) = \sin(x) \Rightarrow f'(x) = \cos(x)$

$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x}{\Delta x}$

$\lim_{\Delta x \to 0} \frac{\sin(x + \Delta x) - \sin x}{\Delta x}$

$\lim_{\Delta x \to 0} \frac{\sin x \cos \Delta x + \sin \Delta x \cos x - \sin x}{\Delta x}$

$\lim_{\Delta x \to 0} \left[\sin x \left(\frac{\cos \Delta x - 1}{\Delta x} \right) + \cos x \ldots \right.$

$\left. \ldots \times \left(\frac{\sin \Delta x}{\Delta x} \right) \right]$

$= \lim_{\Delta x \to 0} \left[\sin x (0) + \cos x (1) \right]$

$= \cos x$

DERIVATIVE OF THE SINE AND COSINE FUNCTIONS

$$\frac{d}{dx}[\sin x] = \cos x \qquad \frac{d}{dx}[\cos x] = -\sin x$$

EXAMPLE 11

Find the derivative of the following functions:

a) $f(x) = \dfrac{3\cos x}{5}$ $\Rightarrow f'(x) \left(\frac{3}{5}\right) \cos x$

$\boxed{\Rightarrow f'(x) \cdot \left(\tfrac{3}{5}\right) \sin x}$

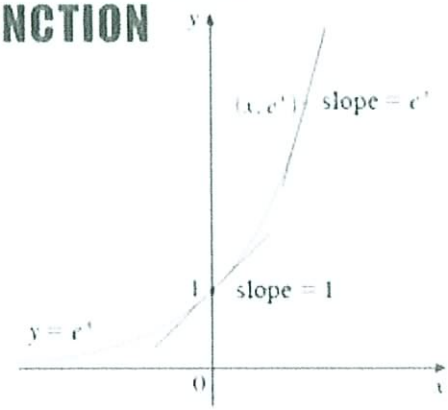
b) $g(x) = \sin x - 3x^4$

$g'(x) = \sin x - 3x^4$

$\boxed{g'(x) = \cos x - 12x^3}$

DERIVATIVE OF THE NATURAL EXPONENTIAL FUNCTION

$$\boxed{\frac{d}{dx}\left[e^x\right] = e^x}$$



EXAMPLE 12

Find the derivative of the following functions:

a) $f(x) = -7e^x$

b) $g(x) = 2x^3 - 6\cos x + \dfrac{e^x}{2}$