CALC

1

Sept. 26/16

Extra example 1

Find any vertical asymptotes on the graph of
$f(x) = \tan x \quad [0 \le x \le 2\pi]$

$$f(x) = \tan x = \frac{\sin x}{\cos x}$$

denom. $= 0 \Rightarrow \cos x = 0$
$x = \pi/2$ or $3\pi/2$

numer. $= \sin(\pi/2) = 1 \neq 0$
$= \sin(3\pi/2) = -1 \neq 0$

$\therefore x = \pi/2$ and $3\pi/2$ are V.A.

Extra example 2

Show that $\cos(\pi/2\, x) = x^2$ has a solution in $[0, 1]$

Let $f(x) = \cos(\pi/2\, x) - x^2$

Use intermediate value theorem, let $x = 0$, $x = 1$

$f(0) = \cos(\pi/2\, 0) - (0)^2$
$= \underbrace{\cos(0) - 0}_{1} \Rightarrow 1 > 0$

$f(1) = \cos(\pi/2\,(1)) - (1)^2$
$= \underbrace{\cos(\pi/2)}_{0} - 1 \Rightarrow -1 < 0$

Since f is continuous on $[0, 1]$ and $f(0) > 0$, $f(1) < 0$,
then by IVT, $\exists\, c \in [0, 1]$ s.t. $f(c) = 0$
$\therefore \cos(\pi/2\, x) = x^2$ has at least one solution in $[0, 1]$

Extra examples (Example 1) Sept. 30/16

$x = -2$ $\lim_{x \to -2} f(x)$ DNE

Intervals

b) $[-4,-2) \cup (-2,2) \cup (2,4) \cup (4,6) \cup (6,8)$

$x = 2$ $\lim_{x \to 2} f(x)$ DNE

$x = 4$ $f(4)$ DNE

OR

$\lim_{x \to 4} f(x)$ DNE

$x = 6$ $\lim_{x \to 6} f(x)$ DNE

$x = 8$ $f(8)$ DNE

(Example 2)

$x = 0$, $f(0) = 1$

$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} (1 + x^2) = 1$

$= \lim_{x \to 0^+} (2 - x) = 2$

Since $\lim_{x \to 0} f(x)$ DNE, we get that f is discontinuous at $x = 0$

Also, f is not differentiable at $x = 0$

(all differentiable are continuous. Because this is not, it's not diff.)

$x = 2$, $f(2) = 2 - 2 = 0$

$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} (2 - x) = 0$

$\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} (x - 2)^2 = 0$

Since $\lim_{x \to 2} f(x) = 0 = f(2)$ we get that f is continuous @ $x = 2$

$$\lim_{\Delta x \to 0^-} \frac{f(2+\Delta x) - f(2)}{\Delta x} = \lim_{\Delta x \to 0^-} \frac{(2-(2+\Delta x)-2-2)}{\Delta x} = -1$$

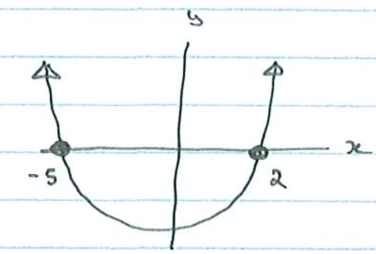
$$\lim_{\Delta x \to 0^+} \frac{f(2+\Delta x) - f(2)}{\Delta x} = \lim_{\Delta x \to 0^+} \frac{(\cancel{2}+\Delta x-\cancel{2})^2 - (2-2)^2}{\Delta x} = 0$$

Since $-1 \neq 0$, we get $\lim_{\Delta x \to 0} \frac{f(2+\Delta x) - f(2)}{\Delta x}$ DNE.

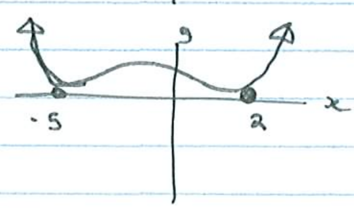
Thus, f is not differentiable at $x = 2$.

Example 3

$$g(x) = x^2 + 3x - 10 = (x+5)(x-2)$$



$$f(x) = \left| x^2 + 3x - 10 \right|$$



$$f(x) = \begin{cases} x^2 + 3x - 10 & x \leq -5 \text{ or } x \geq 2 \\ -(x^2+3x-10) & -5 < x \; 2 \end{cases}$$

$x = -5$ $\quad \lim_{\Delta x \to 0^-} \frac{f(-5+\Delta x) - f(-5)}{\Delta x} \Rightarrow \frac{(-5+\Delta x)^2 + 3(-5+\Delta x) - 10 \ldots}{\Delta x}$

$\Rightarrow 7$

$$\lim_{\Delta x \to 0^+} \frac{f(-5+\Delta x) - f(-5)}{\Delta x} \Rightarrow -((-5+\Delta x)^2 + 3(-5+\Delta x) - 10) - (-5)^2 \ldots$$

$\Rightarrow -7$

So $\lim_{\Delta x \to 0} \frac{f(-5+\Delta x) - f(-5)}{\Delta x}$ DNE

$\Rightarrow$ f is not differentiable @ $x = 5$.

Similarly, the $\lim_{\Delta x \to 0} \frac{f(2+\Delta x) - f(2)}{\Delta x}$ DNE

$\Rightarrow$ f is not differentiable @ $x = 2$

Example 5

1. Prove true for $n = 1$

$f(x) = x^1 \Rightarrow f'(x) = 1x^0$?

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{x + \Delta x - x}{\Delta x} = 1$$
$$= 1x^0$$

2. Assume it is true for $n = k$

$f(x) = x^k \rightarrow f'(x) = kx^{k-1}$

ie $$\lim_{\Delta x \to 0} \frac{(x+\Delta x)^k - x^k}{\Delta x} = kx^{k-1}$$

3. Prove that it is true for $n = k+1$

$f(x) = x^k \rightarrow f'(x) = (k+1)x^k$

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$
$$= \lim_{\Delta x \to 0} \frac{(x+\Delta x)^{k+1} - x^{k+1}}{\Delta x}$$
$$= \lim_{\Delta x \to 0} \frac{(x+\Delta x)(x+\Delta x)^k - x \cdot x^k}{\Delta x}$$
$$= \lim_{\Delta x \to 0} \frac{x(x+\Delta x)^k + \Delta x(x+\Delta x)^k - x \cdot x^k}{\Delta x}$$
$$= \lim_{\Delta x \to 0} \left\{ \frac{x[(x+\Delta x)^k - x^k]}{\Delta x} + \frac{\Delta x(x+\Delta x)^k}{\Delta x} \right\}$$
$$= x \cdot kx^{k-1} + x^k$$
$$= kx^k + x^k$$
$$= (k+1)x^k$$

$\therefore f(x) = x^n \Rightarrow f'(x) = nx^{n-1}$

for $n = 1, 2, 3 \ldots$

By MI (mathematical induction)

Sept. 30/16

Extra Example

Find the points on the graph of $f(x) = 4x^3 - 12x^2 + 2$ where f has a horizontal tangent line.

$$f'(x) = 12x^2 - 24x = 0$$
$$12x(x-2) = 0$$
$$\Rightarrow x = 0, \quad x = 2$$

$f(0) = 2$ $\quad\therefore$ $(0, 2)$, $(2, -14)$ have horizontal tangent lines.

$f(2) = -14$