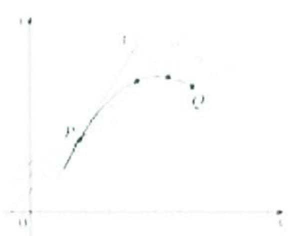
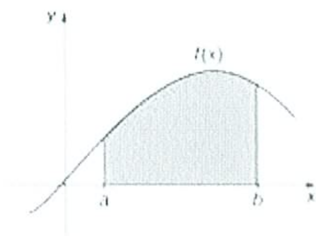
CHAPTER 2
LIMITS AND THEIR PROPERTIES

PREVIEW OF CALCULUS

Tangent Line Problem



Area Problem



LIMITS

Study the following function around $x = 1$:

$$f(x) = \frac{x-1}{x^2-1}$$

0.9
0.99
0.999
0.9999

1.1
1.01
1.001
1.0001

LIMITS

Study the following function around $x = 1$:

$$f(x) = \frac{x-1}{x^2-1}$$

DEFINITION

Definition Suppose $f(x)$ is defined when x is near the number a. (This means that f is defined on some open interval that contains a, except possibly at a itself.) Then we write

$$\lim_{x \to a} f(x) = L$$

and say "the limit of $f(x)$, as x approaches a, equals L"

if we can make the values of $f(x)$ arbitrarily close to L (as close to L as we like) by taking x to be sufficiently close to a (on either side of a) but not equal to a.

EXAMPLE 1

Guess the following limit:

$$\lim_{x \to 0} \frac{\sin x}{x}$$

x	-0.1	-0.01	-0.001	0.001	0.01	0.1
$f(x)$						

LIMITS THAT DO NOT EXIST

$$H(t) = \begin{cases} 0 & \text{if } t < 0 \\ 1 & \text{if } t \geq 0 \end{cases}$$

ONE SIDED LIMITS

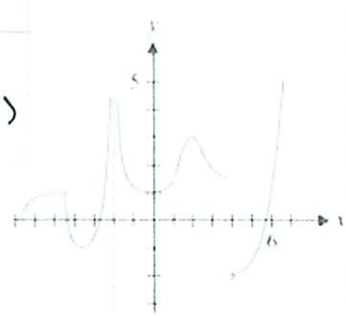
In the previous slide, we actually did one-sided limits:

As $t \to 0$ from the left, $H(t) \to 0$	$t \to 0^-$
As $t \to 0$ from the right, $H(t) \to 1$	$t \to 0^+$

EXAMPLE 2

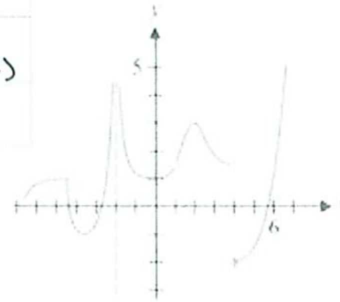
$\lim_{x \to 1^+} f(x) = 1.5$	$\lim_{x \to 1^-} f(x) = 1.5$	$\lim_{x \to 1} f(x) = 1.5$
$\lim_{x \to 4^+} f(x) = -2$	$\lim_{x \to 4^-} f(x) = 1.5$	$\lim_{x \to 4} f(x)$
$\lim_{x \to -2^+} f(x) = +\infty$	$\lim_{x \to -2^-} f(x) = +\infty$	$\lim_{x \to -2} f(x) = +\infty$ (DNE)
$\lim_{x \to +\infty} f(x) = +\infty$	$\lim_{x \to 0} f(x) = 1$	

Determine the limits listed above:



EXAMPLE 2 (ANSWERS)

1.5	1.5	1.5
-2	1.5	DNE
$+\infty$	$+\infty$	$+\infty$ (DNE)
$+\infty$	1	



Determine the limits listed above:

FORMAL DEFINITION OF A LIMIT

Let f be a function defined on an open interval containing c (except possibly at c), and let L be a real number. The statement

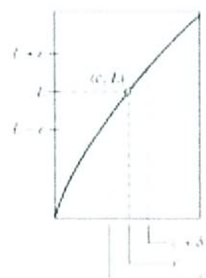
$$\lim_{x \to c} f(x) = L$$

means that for each $\varepsilon > 0$ there exists a $\delta > 0$ such that if

$$0 < |x - c| < \delta$$

then

$$|f(x) - L| < \varepsilon.$$



EXAMPLE 3

Use the formal definition of a limit to prove that:

$$\lim_{x \to 3}(2x - 5) = 1$$

PROPERTIES OF LIMITS

Let b and c be real numbers, and let n be a positive integer.

1. $\lim_{x \to c} b = b$ 2. $\lim_{x \to c} x = c$ 3. $\lim_{x \to c} x^n = c^n$

Sept. 19/16

$(f(x))$ (L)

Proof: $\lim_{x \to c} x = c$

Let $\varepsilon > 0$ be given.

We need to find $\delta > 0$ such that if $|x - c| < \delta$, we have $|x - c| < \varepsilon$

So let $\delta = \varepsilon$

$|x - c| < \delta = \varepsilon$

$\Rightarrow |x - c| < \varepsilon$

Thus, $\lim_{x \to c} x = c$ ▦

EXAMPLE 4

Use the properties on the previous slide to determine the following limits:

a) $\lim_{x \to 4} -5$ $\Rightarrow$ -5

b) $\lim_{x \to -3} x$ $\Rightarrow$ $(-3) = -3$

c) $\lim_{x \to 2} x^3$ $\Rightarrow$ $(2)^3 = 8$

PROPERTIES OF LIMITS

Let b and c be real numbers, let n be a positive integer, and let f and g be functions with the limits $\lim_{x \to c} f(x) = L$ $\lim_{x \to c} g(x) = K$

1. Scalar multiple: $\lim_{x \to c} [b f(x)] = bL$
2. Sum or difference: $\lim_{x \to c} [f(x) \pm g(x)] = L \pm K$
3. Product: $\lim_{x \to c} [f(x)g(x)] = LK$
4. Quotient: $\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{L}{K}$, $K \neq 0$
5. Power: $\lim_{x \to c} [f(x)]^n = L^n$

(Appendix A)

EXAMPLE 5

Determine the following limits using the properties of limits:

a) $\lim_{x\to 2}(4x^2 - 3x + 11)$

b) $\lim_{x\to -1}\left(\frac{3x^2+2}{x+5}\right)$

c) $\lim_{x\to -4}\left(\frac{x+6}{3}\right)^3$

a) $\lim_{x\to 2} 4x^2 - \lim_{x\to 2} 3x + \lim_{x\to 2} 11$

$= 4\lim_{x\to 2} x^2 - 3\lim_{x\to 2} x + \lim_{x\to 2} 11$

$= 4(2)^2 - 3(2) + 11$

$\boxed{= 21}$

b) $\lim_{x\to -1}\left(\frac{3x^2+2}{x+5}\right)$

$= \dfrac{\lim_{x\to -1}(3x^2+2)}{\lim_{x\to -1}(x+5)}$

$= \dfrac{3\lim_{x\to -1} x^2 + \lim_{x\to -1} 2}{\lim_{x\to -1} x + \lim_{x\to -1} 5}$

$\Rightarrow \dfrac{3(-1)^2+2}{-1+5} \Rightarrow \boxed{\frac{5}{4}}$

LIMITS OF POLYNOMIAL FUNCTIONS

If p is a polynomial function and c is a real number, then

$$\lim_{x\to c} p(x) = p(c).$$

If r is a rational function given by $r(x) = p(x)/q(x)$ and c is a real number such that $q(c) \neq 0$, then

$$\lim_{x\to c} r(x) = r(c) = \frac{p(c)}{q(c)}.$$

c) $\lim_{x\to -4}\left(\frac{x+6}{3}\right)^3$

$= \left[\lim_{x\to -4}\left(\frac{x+6}{3}\right)\right]^3$

$= \left[\dfrac{\lim_{x\to -4} x + \lim_{x\to -4} 6}{\lim_{x\to -4} 3}\right]^3$

$= \left(\frac{(-4)+6}{3}\right)^3 \Rightarrow \boxed{\frac{8}{27}}$

EXAMPLE 5

Determine the following limit:

$$\lim_{x\to 1}\left(\frac{2x^3 - 5x^2 + 7x - 13}{x^2 - 5x + 5}\right)$$

$\lim_{x\to 1}\left(\dfrac{2(1)^3 - 5(1)^2 + 7(1) - 13}{(1)^2 - 5(1) + 5}\right)$

$\boxed{= -9}$

LIMIT OF A FUNCTION ~~S~~ INVOLVING A RADICAL

Let n be a positive integer. The limit below is valid for all c when n is odd, and is valid for $c > 0$ when n is even.

$$\lim_{x \to c} \sqrt[n]{x} = \sqrt[n]{c}$$

For example,

$$\lim_{x \to 32} \left(\sqrt[5]{x}\right) = \sqrt[5]{32} \to \boxed{2}$$

LIMITS OF COMPOSITE FUNCTIONS

If f and g are functions such that $\lim_{x \to c} g(x) = L$ and $\lim_{x \to L} f(x) = f(L)$, then

$$\lim_{x \to c} f(g(x)) = f\left(\lim_{x \to c} g(x)\right) = f(L).$$

Because

$$\lim_{x \to 3} (2x^2 - 10) = 2(3^2) - 10 = 8 \quad \text{and} \quad \lim_{x \to 8} \sqrt[3]{x} = \sqrt[3]{8} = 2$$

you can conclude that

$$\lim_{x \to 3} \sqrt[3]{2x^2 - 10} = \sqrt[3]{8} = 2.$$

LIMITS OF TRANSCENDENTAL FUNCTIONS

Let c be a real number in the domain of the given trigonometric function.

1. $\lim_{x \to c} \sin x = \sin c$
2. $\lim_{x \to c} \cos x = \cos c$
3. $\lim_{x \to c} \tan x = \tan c$
4. $\lim_{x \to c} \cot x = \cot c$
5. $\lim_{x \to c} \sec x = \sec c$
6. $\lim_{x \to c} \csc x = \csc c$
7. $\lim_{x \to c} a^x = a^c,\ a > 0$
8. $\lim_{x \to c} \ln x = \ln c$

FUNCTIONS THAT AGREE ON ALL BUT ONE POINT

Let c be a real number, and let $f(x) = g(x)$ for all $x \neq c$ in an open interval containing c. If the limit of $g(x)$ as x approaches c exists, then the limit of $f(x)$ also exists and

$$\lim_{x \to c} f(x) = \lim_{x \to c} g(x).$$

This is very useful when you end up with 0/0 situations.

EXAMPLE 6

Determine the following limit:

$$\lim_{x \to 2} \left(\frac{x^2 + x - 6}{x^2 - 4} \right) \Rightarrow \left(\frac{(2)^2 + (2) - 6}{(2)^2 - 4} \right) \Rightarrow \frac{0}{0}$$

(more work is to be done)

Can't conclude anything yet.

(Note, if it was just $x/0$, where $x \neq 0$, the limit does not exist.)

$f(x) \mid g(x)$ @ $x = 2$, $f(x) \neq g(x)$

$$\Rightarrow \frac{(x+3)\cancel{(x-2)}}{(x+2)\cancel{(x-2)}}$$

(allowed to cancel)

$$\Rightarrow \lim_{x \to 2} \left(\frac{x+3}{x+2} \right) = \frac{(2+3)}{2+2} \Rightarrow \boxed{\frac{5}{4}}$$

EXAMPLE 7

Determine the following limit:

$$\lim_{x \to 6} \left(\frac{\sqrt{x+3} - 3}{x - 6} \right) \Rightarrow \left(\frac{\sqrt{6+3} - 3}{6 - 6} \right) \Rightarrow \frac{0}{0}$$

More work

(This is called the "dividing out" method)

$$\lim_{x \to 6} \left(\frac{\sqrt{x+3} - 3}{x - 6} \right) \times \frac{\sqrt{x+3} + 3}{\sqrt{x+3} + 3}$$

conjugate

$$\lim_{x \to 6} \left(\frac{(x+3-9)}{(x-6)(\sqrt{x+3}+3)} \right)$$

$$\lim_{x \to 6} \frac{(x-6)}{(x-6)(\sqrt{x+3}+3)} \Rightarrow \frac{1}{\sqrt{6+3}+3} \Rightarrow \boxed{\frac{1}{6}}$$

(This is the rationalizing technique.)

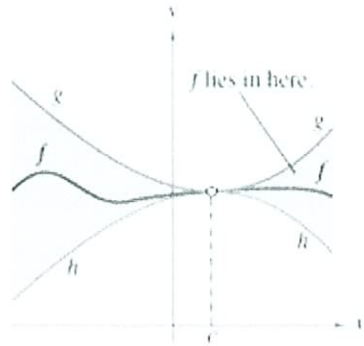
SEPT. 21/16

SQUEEZE THEOREM (AKA SANDWICH THEOREM)

If $h(x) \le f(x) \le g(x)$ for all x in an open interval containing c, except possibly at c itself, and if

$$\lim_{x\to c} h(x) = L = \lim_{x\to c} g(x)$$

then $\lim_{x\to c} f(x)$ exists and is equal to L.



THREE SPECIAL LIMITS

1. $\lim_{x\to 0} \dfrac{\sin x}{x} = 1$ PROOF
2. $\lim_{x\to 0} \dfrac{1-\cos x}{x} = 0$ (PROOF ON NEXT PAGE.)
3. $\lim_{x\to 0} (1+x)^{1/x} = e$

Reciprocal is not true for this function, ie:

NOTE:

$$\left(\lim_{x\to 0} \frac{x}{\sin x} = 1\right)$$

EXAMPLE 8

Determine the following limits:

a) $\lim_{x\to 0} \dfrac{\sin 3x}{x} \Rightarrow \dfrac{\sin 3x}{x} \cdot \dfrac{3}{3} \Rightarrow \lim_{x\to 0}\left[\dfrac{\sin 3x}{3}\cdot 3\right] \rightarrow (1)(3) = \boxed{3}$

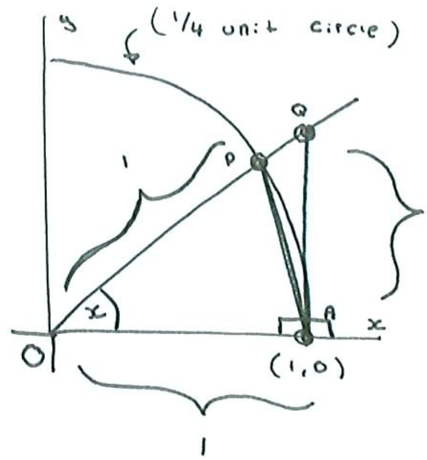
b) $\lim_{x\to 0} \dfrac{1-\cos 7x}{x} \Rightarrow \dfrac{1-\cos 7x}{x}\cdot\dfrac{7}{7} \Rightarrow (0)(7) \rightarrow \boxed{7}$

c) $\lim_{x\to 0} \dfrac{x^2}{\sec x - 1} \Rightarrow \dfrac{x^2}{\left(\frac{1}{\cos x}\right) - 1} \Rightarrow \lim_{x\to 0} \dfrac{x^2}{\frac{(1-\cos x)}{\cos x}}$

$$= \lim_{x\to 0} \frac{x^2 \cos x}{1-\cos x} \times \frac{1+\cos x}{1+\cos x}$$

$$= \lim_{x\to 0} \frac{x^2 \cos x\,(1+\cos x)}{\sin^2 x} \Rightarrow \lim_{x\to 0}\left[\frac{x}{\sin x} \times \frac{x}{\sin x} \cdots * \cos x(1+\cos x)\right]$$

$$= (1)(1)(1)(1+2) = \boxed{2}$$



$\underset{\Delta(OAP)}{\text{Area}} = \frac{1}{2}bh = \frac{1}{2}(1)(\sin x) \rightarrow \left(\frac{\sin x}{2}\right)$

$\underset{◇(OAP)}{\text{Area}} = \dfrac{r^2\theta}{2} = \dfrac{r^2 x}{2} \Rightarrow \left(\dfrac{x}{2}\right)$

$\underset{\Delta(QAO)}{\text{Area}} = \frac{1}{2}bh = \frac{1}{2}(1)(\tan x) \rightarrow \left(\frac{\tan x}{2}\right)$

$\left(\underset{\Delta OAP}{\text{Area}}\right) \le ◇\underset{OAP}{\text{Area}} \le \underset{\Delta QAO}{\text{Area}}$

$\therefore \dfrac{\sin x}{2} < \dfrac{x}{2} < \dfrac{\tan x}{2}$

(sin x is > 0 in Quad I)

$$\frac{2}{\sin x}\left[\frac{\sin x}{2} < \frac{x}{2} < \frac{\tan x}{2}\right]$$

$$\Rightarrow 1 \le \frac{x}{\sin x} \le \frac{1}{\cos x}$$

$$\Rightarrow 1 \ge \frac{\sin x}{x} \ge \cos x$$

CONT'D...

CONTINUITY

(can you draw a graph without lifting a pencil?)

A function f is continuous at a number a if

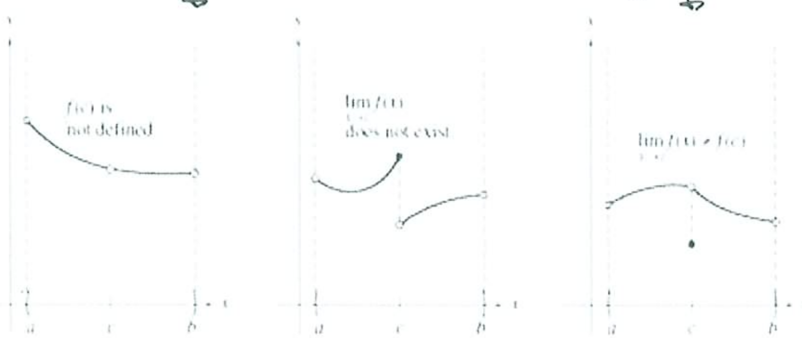
$$\lim_{x \to a} f(x) = f(a)$$

Notice that the above definition implicitly requires three things if f is continuous at a:

1. $f(a)$ is defined (that is, a is in the domain of f)
2. $\lim_{x \to a} f(x)$ exists
3. $\lim_{x \to a} f(x) = f(a)$

NOT CONTINUOUS AT $x = c$:

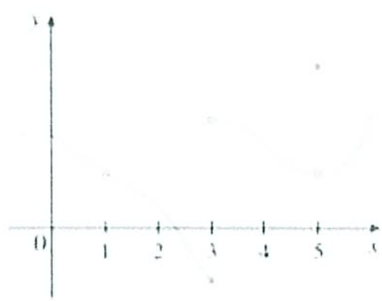
removable discontinuity



also true for asymptote @ c

EXAMPLE 9

The following shows a graph for a function f. For what values of x is f discontinuous and justify your answer.



discontinuous

$x = 1$, $f(1)$ DNE

$x = 3$, ~~$f(3)$~~

$\lim_{x \to 3} f(x)$ DNE

$x = 5$, $\lim_{x \to 5} \neq f(5)$

... From back

$$\lim_{x \to 0} \cos x \leq \lim_{x \to 0} \frac{\sin x}{x} \leq \lim_{x \to 0}$$

$$1 \leq \lim_{x \to 0} \frac{\sin x}{x} \leq \lim_{x \to 0} 1$$

$$\lim_{x \to 0} \frac{\sin x}{x} = 1 \quad \blacksquare$$

(For $\lim_{x \to 0^-}$, use $\frac{\sin(-x)}{-x} = \frac{-\sin x}{x}$ $\boxed{= \frac{\sin x}{x}}$)

Proof (2) (potential exam proof)

$$\lim_{x \to 0} \frac{1 - \cos x}{x} \times \frac{1 + \cos x}{1 + \cos x}$$

$$= \lim_{x \to 0} \frac{1 - \cos^2 x}{x(1 + \cos x)}$$

$$= \lim_{x \to 0} \frac{\sin^2 x}{x(1 + \cos x)}$$

$$= \lim_{x \to 0} \left[\frac{\sin x}{x} \times \frac{\sin x}{1 + \cos x} \right]$$

$$= (1)\left(\frac{0}{1+1}\right) \quad \blacksquare$$

$$= 0$$

EXAMPLE 10

Discuss the continuity of the following function:

$$f(x) = \begin{cases} \dfrac{x^2 - x - 2}{x - 2} & \text{if } x \neq 2 \\ 1 & \text{if } x = 2 \end{cases}$$

First look @ domain

$\text{dom} f = x \in (-\infty, \infty)$

look at $x = 2$

1. $f(2) = 1$

2. $\lim\limits_{x \to 2} \dfrac{x^2 - x - 2}{x - 2} \Rightarrow \dfrac{0}{0}$ MORE WORK

   $\Rightarrow \lim\limits_{x \to 2} \dfrac{(x-2)(x+1)}{(x-2)} \Rightarrow 3$

3. $\lim\limits_{x \to 2} f(x) = f(2)$?

   No - $3 \neq 1$

   $\therefore f$ is discontinuous @ $x = 2$

RECALL (FORMALLY)

Let f be a function, and let c and L be real numbers. The limit of $f(x)$ as x approaches c is L if and only if

$$\lim_{x \to c^-} f(x) = L \quad \text{and} \quad \lim_{x \to c^+} f(x) = L.$$

(between two points, including end-points)?

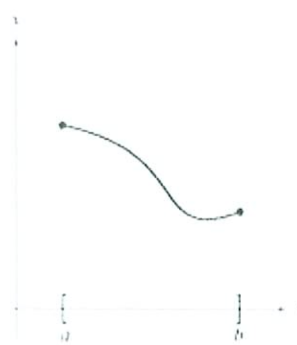
CONTINUITY ON A CLOSED INTERVAL

A function f is continuous on the closed interval $[a, b]$ when f is continuous on the open interval (a, b) and

$$\lim_{x \to a^+} f(x) = f(a)$$

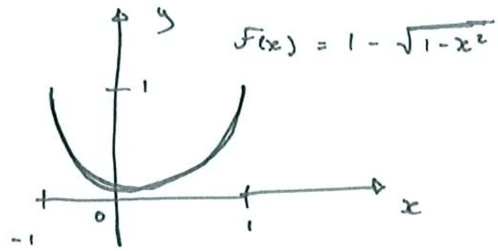
and

$$\lim_{x \to b^-} f(x) = f(b).$$



EXAMPLE 11

Discuss the continuity of $f(x) = 1 - \sqrt{1 - x^2}$ on the interval $[-1, 1]$



$f(x)$ is continuous on $(-1, 1)$

$\lim_{x \to 1^-} f(x) = f(1)$

$\lim_{x \to 1^+} f(x) = f(-1)$

Thus, f is continuous on $[-1, 1]$

PROPERTIES OF CONTINUITY

If b is a real number and f and g are continuous at $x = c$, then the functions listed below are also continuous at c.

1. Scalar multiple: bf
2. Sum or difference: $f \pm g$
3. Product: fg
4. Quotient: $\frac{f}{g}$, $g(c) \neq 0$

EXAMPLES OF CONTINUOUS FUNCTIONS

1. Polynomial: $p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$ → always continuous
2. Rational: $r(x) = \frac{p(x)}{q(x)}$, $q(x) \neq 0$ → continuous if denom $\neq \emptyset$
3. Radical: $f(x) = \sqrt[n]{x}$ → continuous domains
4. Trigonometric: $\sin x$, $\cos x$, $\tan x$, $\cot x$, $\sec x$, $\csc x$
5. Exponential and logarithmic: $f(x) = a^x$, $f(x) = e^x$, $f(x) = \ln x$ → continuous where $x > 0$

The above functions are continuous everywhere in their domains

RESULT.....

The following are all continuous functions in their respective domains:

$$f(x) = x + e^x, \quad f(x) = 3\tan x, \quad f(x) = \frac{x^2+1}{\cos x}$$

as long as $\cos x \neq 0$

COMPOSITION OF CONTINUOUS FUNCTIONS

If g is continuous at c and f is continuous at $g(c)$, then the composite function given by $(f \circ g)(x) = f(g(x))$ is continuous at c.

For example,

$g(x) = \ln x$ and $f(x) = x^2 + 3x - 4$

are continuous every where.

Thus

$(g \cdot f)(x) = \ln(x^2 + 3x - 4)$

is continuous on their domains.

EXAMPLE 12

Describe the interval(s) in which the following function ~~is~~ is continuous:

$$f(x) = \frac{1}{\sqrt{x^2+7} - 4}$$

Find the domain of f.

$x^2 + 7 \geq 0$ (always true because x^2)

$\rightarrow$ not an issue

How about denom. $= 0$

$\sqrt{x^2+7} - 4 = 0$

$\sqrt{x^2+7} = 4$

$x^2 + 7 = 16$

$x^2 = 9 \Rightarrow x = \pm 3$

$\therefore$ dom f = $x \in (-\infty, -3) \cup (-3, 3) \cup (3, \infty)$

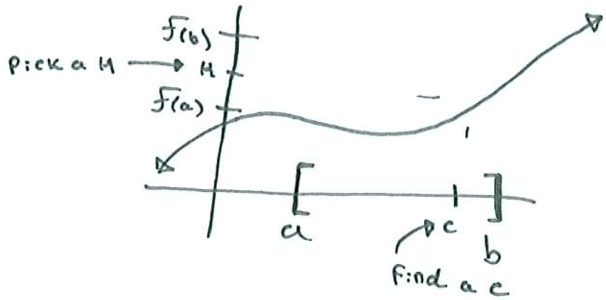
The f is continuous on

$x \in (-\infty, -3) \cup (-3, 3) \cup (3, \infty)$

INTERMEDIATE VALUE THEOREM

If f is continuous on the closed interval $[a, b]$, $f(a) \neq f(b)$, and k is any number between $f(a)$ and $f(b)$, then there is at least one number c in $[a, b]$ such that

$f(c) = k$.



EXAMPLE 13

Show that there is a root of the equation:

$$4x^3 - 6x^2 + 3x - 2 = 0$$

between 1 and 2.

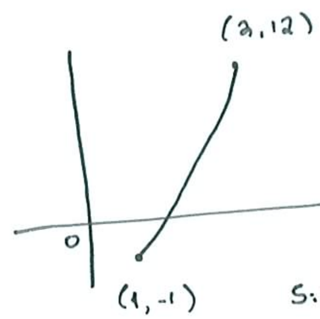
Let $f(x) = 4x^2 - 6x^2 + 3x - 2$ $[1,2]$

$f(1) = 4(1)^3 - 6(1)^2 + 3(1) - 2$

$\Rightarrow -1$

$f(2) = 4(2)^3 - 6(2)^2 + 3(2) - 2$

$\Rightarrow 12$



Since f is continuous on $[1,2]$ and $f(1) < 0$, $f(2) > 0$

then $\exists\, c \in [1,2]$ s.

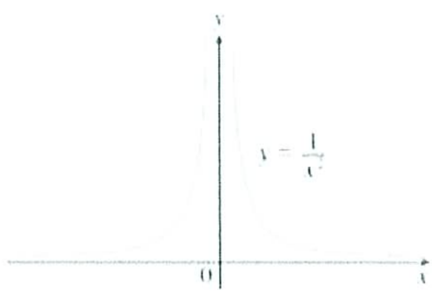
(there exists) $f(c) = 0$

(by IVT)

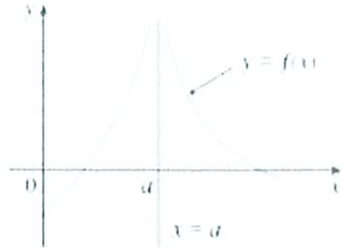
INFINITE LIMITS

Find $\lim_{x \to 0} \frac{1}{x^2}$ if it exists.

x	$\frac{1}{x^2}$
± 1	1
± 0.5	4
± 0.2	25
± 0.1	100
± 0.05	400
± 0.01	10 000
± 0.001	1 000 000



DEFINITION



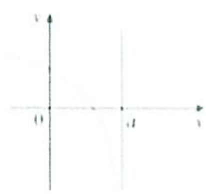
Let f be a function defined on both sides of a, except possibly at a itself. Then

$$\lim_{x \to a} f(x) = \infty$$

means that the values of $f(x)$ can be made arbitrarily large (as large as we please) by taking x sufficiently close to a, but not equal to a

(a) $\lim f(x) = \infty$

(b) $\lim f(x) = \infty$



(c) $\lim f(x) = -\infty$

(d) lim

EXAMPLE 14

Determine the following limits for $f(x)$ if

$$f(x) = \frac{x+3}{4-x}$$

a) $\lim_{x \to 4^+} f(x)$

b) $\lim_{x \to 4^-} f(x)$

c) $\lim_{x \to 4} f(x)$

VERTICAL ASYMPTOTES

Let f and g be continuous on an open interval containing c. If $f(c) \neq 0$, $g(c) = 0$, and there exists an open interval containing c such that $g(x) \neq 0$ for all $x \neq c$ in the interval, then the graph of the function

$$h(x) = \frac{f(x)}{g(x)}$$

has a vertical asymptote at $x = c$.

EXAMPLE 14

Determine any vertical asymptotes for the following functions:

a) $f(x) = \dfrac{2x}{x-3}$

b) $g(x) = \dfrac{x^2 - 2x - 3}{x^2 + 2x - 15}$

PROPERTIES OF INFINITE LIMITS

Let c and L be real numbers, and let f and g be functions such that

$$\lim_{x \to c} f(x) = \infty \quad \text{and} \quad \lim_{x \to c} g(x) = L.$$

1. Sum or difference: $\lim_{x \to c} [f(x) \pm g(x)] = \infty$
2. Product: $\lim_{x \to c} [f(x)g(x)] = \infty, \quad L > 0$

   $\lim_{x \to c} [f(x)g(x)] = -\infty, \quad L < 0$
3. Quotient: $\lim_{x \to c} \dfrac{g(x)}{f(x)} = 0$