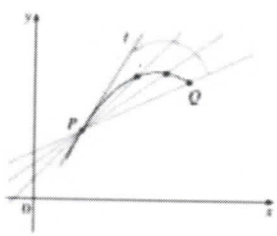
Nov. 16/16

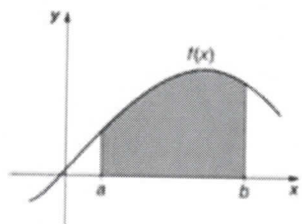
CHAPTER 5 INTEGRATION

RECALL: PREVIEW OF CALCULUS

Tangent Line Problem



Area Problem



ANTIDERIVATIVES

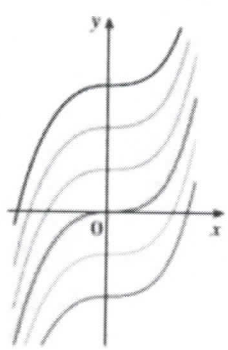
Definition A function F is called an antiderivative of f on an interval I if $F'(x) = f(x)$ for all x in I.

F.E. (For example:)

$f(x) = x^2$

a.d $= F(x) = \frac{1}{3}x^3$

anti-derivative.



— $y = \frac{x^3}{3} + 3$

— $y = \frac{x^3}{3} + 2$

— $y = \frac{x^3}{3} + 1$

— $y = \frac{x^3}{3}$

— $y = \frac{x^3}{3} - 1$

— $y = \frac{x^3}{3} - 2$

GENERAL ANTIDERIVATIVE

Theorem If F is an antiderivative of f on an interval I, then the most general antiderivative of f on I is

$$F(x) + C$$

where C is an arbitrary constant.

So in the previous slide, $\frac{1}{3}x^3 + C$ woud be the general antiderivative of x^2.

EXAMPLE 1

Find the general antiderivative of the following functions:

a) $f(x) = \sin x$ => $F(x) = -\cos(x) + C$

b) $g(x) = x^n, n \neq 0$ => $G(x) = \left(\frac{1}{n+1}\right) x^{n+1} + C$

c) $h(x) = \frac{1}{x}$ => x^{-1} (dont do that).

$H(x) = \ln|x| + C$

(JUST REMEMBER THIS)

INTEGRATION

The operation of finding all the general antiderivative is called **indefinite integration** and is denoted by an integral $\int$ sign. The general solution is denoted by

Variable of integration — Variable of integration

Constant of integration — (constant of integration)

$$y = \int f(x)\,dx = F(x) + C.$$

Integrand — (integrand)

An antiderivative of $f(x)$ — (an anti-derivative of $f(x)$)

INTEGRATION VS DIFFERENTIATION

These two operations are inverses of each other:

$\int F'(x)\,dx = F(x) + C.$ Integration is the "inverse" of differentiation.

$\frac{d}{dx}\left[\int f(x)\,dx\right] = f(x).$ Differentiation is the "inverse" of integration.

BASIC INTEGRATION RULES

Differentiation Formula	Integration Formula
$\frac{d}{dx}[C] = 0$	$\int 0\,dx = C$
$\frac{d}{dx}[kx] = k$	$\int k\,dx = kx + C$
$\frac{d}{dx}[kf(x)] = kf'(x)$	$\int kf(x)\,dx = k\int f(x)\,dx$
$\frac{d}{dx}[f(x) \pm g(x)] = f'(x) \pm g'(x)$	$\int [f(x) \pm g(x)]\,dx = \int f(x)\,dx \pm \int g(x)\,dx$
$\frac{d}{dx}[x^n] = nx^{n-1}$	$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C, \quad n \neq -1$
$\frac{d}{dx}[\sin x] = \cos x$	$\int \cos x\,dx = \sin x + C$
$\frac{d}{dx}[\cos x] = -\sin x$	$\int \sin x\,dx = -\cos x + C$

HUGE TABLE ON PAGE 282.

EXAMPLE 2

Integrate the following:

a) $\int(3x+2)\,dx$ → $\int 3x\,dx + \int 2\,dx$

$\Rightarrow 3\int x\,dx + \int 2\,dx$

$\Rightarrow 3(\frac{1}{2}x^2) + 2x + C$

$\boxed{\Rightarrow \frac{3}{2}x^2 + 2x + C}$

b) $\int\left(\frac{1}{x^3}\right)dx$

c) $\int[x \cdot \sin(x^2)]\,dx$

$-\cos x^2$

derivative: $\sin(x^2)2x$

$\therefore \boxed{\frac{-\cos(x^2)}{2} + C}$

$\int(\frac{1}{x^3})\,dx$

$\Rightarrow \int x^{-3}\,dx$

$\Rightarrow -(\frac{1}{2})x^{-2}$

$\boxed{= -\frac{1}{2x^2} + C}$

EXAMPLE 3

Find f if $f'(x) = x\sqrt{x}$ and $f(1) = 2$

$f'(x) = x\sqrt{x}$ → $x^{3/2}$

$\int f'(x)\,dx = \int x\sqrt{x}\,dx$

$f(x) = 2/5 \cdot x^{5/2} + C$

$f(1) = 2/5 \cdot (1)^{5/2} + C = 2$

$\therefore C = 8/5$

→ Particular solution:

$\therefore f(x) = 2/5\, x^{5/2} + 8/5$

SUMMATION NOTATION

This is a capital sigma (represents sum)

$$\sum_{k=1}^{n} a_k = a_1 + a_2 + a_3 + a_4 + \cdots + a_n$$

indices

(start @ 1, loops until n)

EXAMPLE 4

Find each sum.

(a) $\sum_{k=1}^{5} k^2$ (b) $\sum_{j=3}^{5} \frac{1}{j}$ (c) $\sum_{i=5}^{10} i$ (d) $\sum_{i=1}^{6} 2$

a) $\sum_{k=1}^{5} k^2 = (1)^2 + (2)^2 + (3)^2 + (4)^2 + (5)^2 \Rightarrow \boxed{55}$

b) $\sum_{j=3}^{5} \frac{1}{j} = \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} = \boxed{\frac{47}{60}}$

c) $\sum_{i=5}^{10} i = 5+6+7+8+9+10 = \boxed{45}$

d) $\sum_{i=1}^{6} 2 = \underset{(a1)}{2} + \underset{(a2)}{2} + \underset{(a3)}{2} + \underset{(a4)}{2} + \underset{(a5)}{2} + \underset{(a6)}{2} = \boxed{12}$

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IT SNOWED TODAY.

SUMMATION FORMULAS

1. $\sum_{i=1}^{n} c = cn$, c is a constant
2. $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$
3. $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$
4. $\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$

#2

② $\sum_{i=1}^{n} i = 1 + 2 + 3 + \dots + n$

$(+)\; n + (n-1) + (n-2) + \dots + 1$

$\underbrace{(n+1) + (n+1) + (n+1) + \dots + (n+1)}_{n(n+1)}$

$\Rightarrow \frac{n(n+1)}{2}$ ▩

EXAMPLE 5

Use the formulae from the previous slide to evaluate the following:

$$\sum_{i=0}^{15} \frac{3i^2 - 4}{5}$$

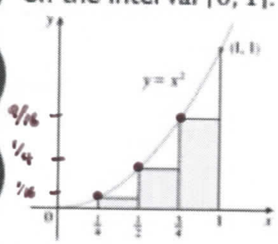
$$\sum_{i=0}^{15} \frac{3i^2-4}{5} = \frac{3}{5}\sum_{i=0}^{15} i^2 - \sum_{i=0}^{15} 4/5$$

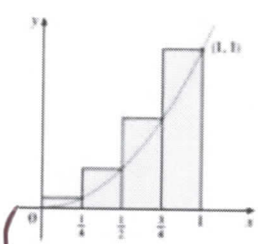
$$= 3/5\left[0^2 + \sum_{i=1}^{15} i^2\right] - \left[4/5 + \sum_{i=1}^{15} 4/5\right]$$

$$= 3/5\left[\frac{15(15+1)(2(15)+1)}{6}\right] - \left[4/5 + 15(4/5)\right] \Rightarrow \boxed{\frac{3656}{5}}$$

LET'S NOW TACKLE THE AREA PROBLEM...

Use rectangles to estimate the area under the parabola $y = x^2$ on the interval $[0, 1]$.





$0.21875 < A < 0.4$ ish.

GIVE IT A WHIRL.

When $x = 1/4$

Area $= \frac{1}{4}f(\frac{1}{4}) + \frac{1}{4}f(\frac{1}{2}) \dots$

$\dots + \frac{1}{4}f(3/4)$

$= 1/4(1/16 + 1/4 + 9/16)$

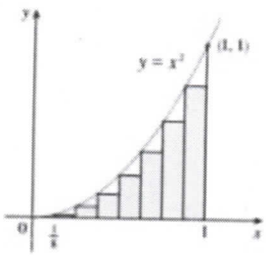
$\boxed{= 7/32}$ LOWER SUM

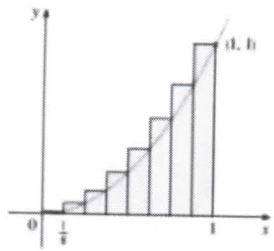
Area $= \frac{1}{4}f(\frac{1}{4}) + \frac{1}{4}f(\frac{1}{2}) + \frac{1}{4}f(3/4) + \frac{1}{4}f(1)$

$\Rightarrow \frac{1}{4}(\frac{1}{16} + \frac{1}{4} + 9/16 + 1)$

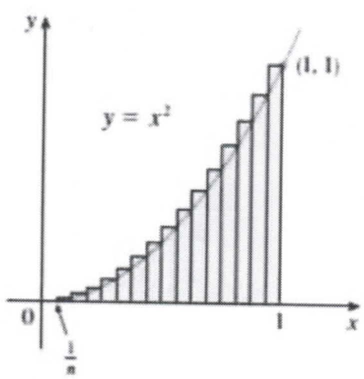
$\Rightarrow \boxed{15/32}$ UPPER SUM

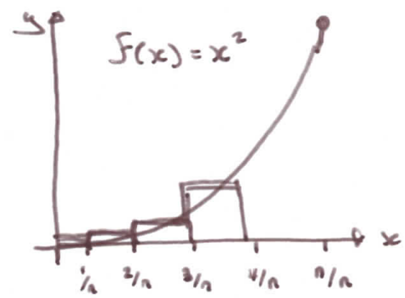
WE CAN EVEN ESTIMATE THE AREA MORE ACCURATELY...





INFINITE AMOUNT OF RECTANGLES





$$\text{Area} = \frac{1}{n}\left[f(1/n) + f(2/n) + \dots + f(n/n)\right]$$

$$= \frac{1}{n}\left[\left(\frac{1}{n}\right)^2 + \left(\frac{2}{n}\right)^2 + \dots + \left(\frac{n}{n}\right)^2\right]$$

$$= \frac{1}{n} \times \frac{1}{n^2}\left[1^2 + 2^2 + \dots + n^2\right]$$

USING FORMULA #3

$$= \frac{1}{n^{\cancel{3}2}}\left[\frac{\cancel{n}(n+1)(2n+1)}{6}\right]$$

$$= \frac{(n+1)(2n+1)}{6n^2}$$

$$= \frac{1}{6}\left[\frac{n+1}{n} \times \frac{2n+1}{n}\right]$$

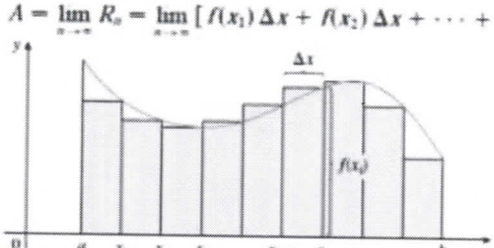
$$= \frac{1}{6}\left(1 + 1/n\right)\left(2 + 1/n\right)$$

as $n \to \infty$, Area $\to 2/6 = \boxed{1/3}$

AREA

Definition The area A of the region S that lies under the graph of the continuous function f is the limit of the sum of the areas of approximating rectangles:

$$A = \lim_{n\to\infty} R_n = \lim_{n\to\infty}\left[f(x_1)\,\Delta x + f(x_2)\,\Delta x + \cdots + f(x_n)\,\Delta x\right]$$



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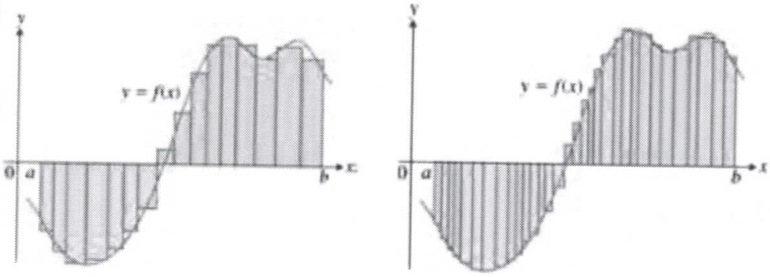
RIEMANN SUM



RIEMANN SUM

Let f be defined on the closed interval $[a, b]$, and let Δ be a partition of $[a, b]$ given by

$$a = x_0 < x_1 < x_2 < \ldots < x_{n-1} < x_n = b$$

where Δx_i is the width of the ith subinterval

$[x_{i-1}, x_i]$. $\quad$ ith subinterval

If c_i is *any* point in the ith subinterval, then the sum

$$\sum_{i=1}^{n} f(c_i)\,\Delta x_i, \quad x_{i-1} \le c_i \le x_i$$

is called a **Riemann sum** of f for the partition Δ.

EXAMPLE 6

(a) Evaluate the Riemann sum for $f(x) = x^3 - 6x$, taking the sample points to be right endpoints and $a = 0$, $b = 3$, and $n = 6$.

(b) Evaluate $\int_0^3 (x^3 - 6x)\,dx$.

a) $\Delta x = \frac{b-a}{n} \Rightarrow \frac{1}{2}$

$0 < \frac{1}{2} < \frac{2}{2} < \frac{3}{2} < \frac{4}{2} < \frac{5}{2}, \frac{6}{2} = 3$

$x_0 \quad x_1 \quad x_2 \quad x_3 \quad x_4 \quad x_5 \quad x_6$

$\sum_{i=1}^{6} f(x_i)\Delta x \leftarrow$ all $= \frac{1}{2}$

$\Rightarrow \frac{1}{2}\left[f(x_1) + f(x_2) + f(x_3) + f(x_4) + f(x_5) + f(x_6)\right]$

$\Rightarrow \frac{1}{2}\left[f(\frac{1}{2}) + f(\frac{2}{2}) + \ldots + f(\frac{6}{2})\right]$

$= -63/16$ (negative answer means more area UNDER x-axis than above)

$\Delta x_i = \frac{b-a}{n}$

$\Delta x_i = 3/n$

b) $\int_0^3 (x^3 - 6x)\,dx$

$= \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x_i$

$= \lim_{n\to\infty} \frac{3}{n} \sum_{i=1}^{n} f\left(\frac{3i}{n}\right)$

$= \lim_{n\to\infty} \frac{3}{n} \sum_{i=1}^{n} \left[\left(\frac{3i}{n}\right)^3 - 6\left(\frac{3i}{n}\right)\right]$

$= \lim_{n\to\infty} \frac{3}{n} \sum_{i=1}^{n} \left(\frac{27^3}{n^3} - \frac{18i}{n}\right)$

$= \lim_{n\to\infty} \left[\frac{81}{n^4} \sum_{i=1}^{n} i^3 - \frac{54}{n^2} \sum_{i=1}^{n}\right]$

NORM

The width of the largest subinterval of a partition Δ is the **norm** of the partition and is denoted by $\|\Delta\|$. If every subinterval is of equal width, then the partition is **regular** and the norm is denoted by

$$\|\Delta\| = \Delta x = \frac{b-a}{n}. \qquad \text{Regular partition}$$

For a general partition, the norm is related to the number of subintervals of $[a, b]$ in the following way.

$$\frac{b-a}{\|\Delta\|} \leq n \qquad \text{General partition}$$

$$\|\Delta\| \to 0 \text{ implies that } n \to \infty.$$

$$\lim_{n \to \infty} \left[\frac{8}{n^4} \cdot \frac{(n^2)(n+1)^2}{4} - \ldots \right.$$

$$\left. \ldots \frac{54}{n^2} \cdot \left(\frac{n(n+1)}{2} \right) \right]$$

$$\lim_{n \to \infty} \left[\frac{81}{4}\left(1 + \cancel{\tfrac{1}{n}}^{\,0}\right)^2 - 27\left(1 + \cancel{\tfrac{1}{n}}^{\,0}\right) \right]$$

$$= \frac{81}{4} - 27 \Rightarrow \boxed{-\frac{27}{4}}$$

DEFINITE INTEGRAL

If f is defined on the closed interval $[a, b]$ and the limit of Riemann sums over partitions Δ

$$\lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} f(c_i)\,\Delta x_i$$

exists (as described above), then f is said to be **integrable** on $[a, b]$ and the limit is denoted by

$$\lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} f(c_i)\,\Delta x_i = \int_a^b f(x)\,dx.$$

The limit is called the **definite integral** of f from a to b. The number a is the **lower limit** of integration, and the number b is the **upper limit** of integration.

THEOREM: CONTINUITY IMPLIES INTEGRABILITY

If a function f is continuous on the closed interval $[a, b]$, then f is integrable on $[a, b]$. That is, $\int_a^b f(x)\,dx$ exists.

EXAMPLE 7

Compute the following definite integral:

$$\int_{-1}^{3} (2 - 3x)\, dx$$

$= \int_{-1}^{3} (2-3x)\, dx$ $\qquad \Delta x_i = \frac{b-a}{n} = \frac{3-(-1)}{n} = 4/n$

$= \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i)\Delta x_i$

$= \lim_{n \to \infty} \frac{4}{n} \sum_{i=1}^{n} \left[2 - 3\left(-1 + \frac{4i}{n}\right)\right]$

$= \lim_{n \to \infty} \frac{4}{n} \sum_{i=1}^{n} \left(5 - \frac{12i}{n}\right) \Rightarrow \lim_{n \to \infty} \left[\frac{4}{n} \times 5n - \frac{48}{n^2} \sum_{i=1}^{n} i\right]$

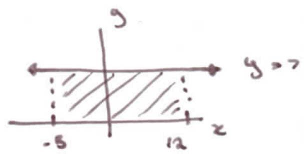
$\lim_{n \to \infty} \left[20 - 24\left(1 + \frac{1}{n}\right)\right]$

$= 20 - 24$

$\boxed{= -4}$

AREAS OF COMMON GEOMETRIC FIGURES

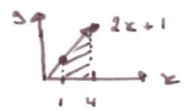
a) $\int_{-5}^{12} 7\, dx \Rightarrow y = 7 \Rightarrow$



$\Rightarrow A = bh$

$A = 17 \cdot 7 \Rightarrow \boxed{119}$

b) $\int_{1}^{4} (2x + 1)\, dx \Rightarrow y = 2x + 1$

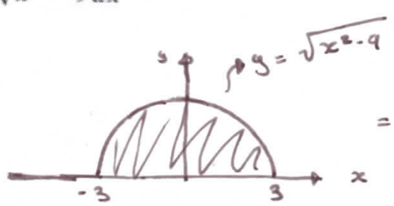


$(4-1)(3) + \frac{1}{2}(4-1)(6)$

$\Rightarrow 9 + 9$

$\boxed{= 18}$

c) $\int_{-3}^{3} \sqrt{x^2 - 9}\, dx$



$\Rightarrow \frac{\pi r^2}{2}$

$\boxed{\Rightarrow \frac{9\pi}{2}}$

DEFINITIONS OF TWO SPECIAL DEFINITE INTEGRALS

1. If f is defined at $x = a$, then $\int_a^a f(x)\, dx = 0$.

2. If f is integrable on $[a, b]$, then $\int_b^a f(x)\, dx = -\int_a^b f(x)\, dx$.

ADDITIVE INTERVAL PROPERTY

If f is integrable on the three closed intervals determined by a, b, and c, then

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

PROPERTIES OF DEFINITE INTEGRALS

If f and g are integrable on $[a, b]$ and k is a constant, then the functions kf and $f \pm g$ are integrable on $[a, b]$, and

1. $\displaystyle\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$

2. $\displaystyle\int_a^b [f(x) \pm g(x)]\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx.$

EXAMPLE 8

Use the properties of integrals to evaluate $\int_0^1 (4 + 3x^2)\,dx$.

$\int_0^1 4\,dx + 3\int_0^1 x^2\,dx$

$= 4(1-0) + 3(1/3)$

$= 4 + 1$

$= 5$

EXAMPLE 9

(*) "WOULD MAKE A GOOD M.C. QUESTION"

If it is known that $\int_0^{10} f(x)\,dx = 17$ and $\int_0^{8} f(x)\,dx = 12$, find $\int_8^{10} f(x)\,dx$.

$\int_0^{10} f(x)\,dx = \int_0^{8} f(x)\,dx + \int_8^{10} f(x)\,dx$

$17 = 12 + \int_8^{10} f(x)\,dx$

$\int_8^{10} f(x)\,dx = \boxed{5}$

PRESERVATION OF INEQUALITY

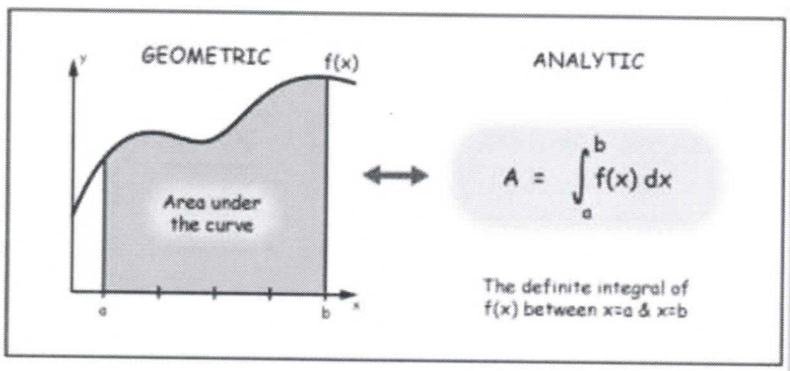
1. If f is integrable and nonnegative on the closed interval $[a, b]$, then

$$0 \le \int_a^b f(x)\,dx.$$

2. If f and g are integrable on the closed interval $[a, b]$ and $f(x) \le g(x)$ for every x in $[a, b]$, then

$$\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx.$$

FUNDAMENTAL THEOREM OF CALCULUS



ANTIDIFFERENTIATION AND DEFINITE INTEGRATION

Throughout this chapter, you have been using the integral sign to denote an antiderivative (a family of functions) and a definite integral (a number).

Antidifferentiation: $\int f(x)\,dx$ Definite integration: $\int_a^b f(x)\,dx$

FUNDAMENTAL THEOREM OF CALCULUS

(FTOC)

If a function f is continuous on the closed interval $[a, b]$ and F is an antiderivative of f on the interval $[a, b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

PROOF:

Let Δ be some portion of $[a, b]$

i.e. $a = x_0 < x_1 < x_2 \dots < x_{n-1} < x_n = b$

$\Rightarrow F(b) - F(a) = F(x_n) - F(x_0)$

$\Rightarrow F(x_n) - F(x_{n-1}) + F(x_{n-1}) \dots$

$\dots - F(x_{n-2}) + F(x_{n-2})$

By MVT, $\exists c_i \in (x_i, x_{i-1})$ such that

$$F'(c_i) = \frac{F(x_i) - F(x_{i-1})}{x_i - x_{i-1}}$$

Since $F'(c_i) = f(c_i)$, and if we let $\Delta x_i = x_i - x_{i-1}$

$$f(c_i) = \frac{F(x_i) - F(x_{i-1})}{\Delta x_i}$$

$$F(x_i) - F(x_{i-1}) = f(c_i) \cdot \Delta x_i$$

$$\underbrace{\sum_{i=1}^{n} \left[F(x_i) - F(x_{i-1})\right]}_{\text{everything is going to cancel except } F(x_n) \text{ \& } F(x_0)} = \sum_{i=1}^{n} f(c_i)\Delta x_i$$

$$F(x_n) - F(x_0) = \sum_{i=1}^{n} f(c_i)\Delta x_i$$

$$\lim_{n\to\infty} \left[F(b) - F(a)\right] = \lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\Delta x_i \leftarrow \text{Riemann Sum}$$

$$F(b) - F(a) = \int_a^b f(x)\,dx$$

EXAMPLE 10

Evaluate the integral $\int_{-2}^{1} x^3\,dx$.

what is the area under curve between -2 and 1?

$\Rightarrow \left(\tfrac{1}{4}\right)x^4 \Big|_{-2}^{1}$

$\Rightarrow \left(\tfrac{1}{4}\right)(1)^4 - \left(\tfrac{1}{4}\right)(-2)^4$

$\Rightarrow \left(\tfrac{1}{4}\right) - 4 \Rightarrow \boxed{-3\,\tfrac{3}{4}}$

(in definite integration the C will cancel - there's endpoints)

EXAMPLE 11

Find the area under the parabola $y = x^2$ from 0 to 1.

$$\int_0^1 x^2\,dx$$

$$\Rightarrow \left(\tfrac{1}{3}\right) x^3 \Big|_0^1$$

$$\Rightarrow \left(\tfrac{1}{3}\right)(1)^3 - \left(\tfrac{1}{3}\right)(0)^3$$

$$\boxed{= \tfrac{1}{3}}$$

EXAMPLE 12

Find the area under the cosine curve from 0 to b, where $0 \leqslant b \leqslant \pi/2$.

$$\int_0^b \cos x\,dx = \sin x\,dx \Big|_0^b$$

$$\Rightarrow \sin b - \sin(0)$$

$$\boxed{\Rightarrow \sin b}$$

EXAMPLE 13

Evaluate:

$$\int_0^1 (u + 2)(u - 3)\,du$$

$$\int_0^1 (u+2)(u-3)\,du$$

$$\Rightarrow \int_0^1 (u^2 - u - 6)\,du$$

$$\Rightarrow \left(\tfrac{1}{3}\right)u^3 - \left(\tfrac{1}{2}\right)u^2 - 6u \Big|_0^1$$

$$\Rightarrow \left[\left(\tfrac{1}{3}\right)(1)^3 - \left(\tfrac{1}{2}\right)(1)^2 - 6(1)\right] - \left[\left(\tfrac{1}{3}\right)(0)^3 - \left(\tfrac{1}{2}\right)(0)^2 - 6(0)\right]$$

$$\boxed{= \frac{-37}{6}}$$

EXAMPLE 14

Evaluate: $\displaystyle\int_1^9 \frac{x-1}{\sqrt{x}}\,dx$

$\Rightarrow (x-1)(x^{-1/2})\,dx$

$\Rightarrow \left(x^{1/2} - x^{-1/2}\right)dx$

$\Rightarrow \left(\frac{2}{3}\right)x^{3/2} - 2x^{1/2}\Big|_1^9$

$= \left[(2/3)(9)^{3/2} - 2(9)^{1/2}\right] - \left[(2/3)(1)^{3/2} - 2(1)^{1/2}\right]$

$\boxed{\Rightarrow \frac{40}{3}}$

EXAMPLE 15

Evaluate:

$$\int_{-3}^{5} |x^2 - 4|\,dx$$

$|x^2 - 4|$

$\Rightarrow x^2 - 4 \quad x \le -2,\ x \ge 2$

$\Rightarrow -(x^2-4) \quad -2 \le x \le 2$

$\Rightarrow \int_{-3}^{-2} (x^2-4)\,dx + \int_{-2}^{2} -(x^2-4)\,dx \ \ldots$

$\ldots + \int_{2}^{5} (x^2-4)\,dx$

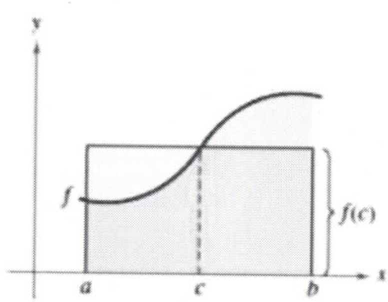
$\Rightarrow \left[\frac{1}{3}\cdot x^3 - 4x\right]\Big|_{-3}^{-2} + \left[-\frac{1}{3}x^3 + 4x\right]\Big|_{-2}^{2} + \left[\frac{1}{3}x^3 - 4x\right]\Big|_{2}^{5}$

$\boxed{\Rightarrow 40}$

MEAN VALUE THEOREM FOR INTEGRALS

If f is continuous on the closed interval $[a, b]$, then there exists a number c in the closed interval $[a, b]$ such that

$$\int_a^b f(x)\,dx = f(c)(b-a).$$



THE SUBSTITUTION RULE

Today, we are going to discuss how to evaluate integral of the following form:

$$\int 2x\sqrt{1+x^2}\,dx$$

THE SUBSTITUTION RULE

If $u = g(x)$ is a differentiable function whose range is an interval I and f is continuous on I, then

$$\int f(g(x))\,g'(x)\,dx = \int f(u)\,du$$

EXAMPLE 16

$$\int 2x\sqrt{1+x^2}\,dx$$

$\int 2x\sqrt{1+x^2}\,dx$ Let $u = 1 + x^2$

$\frac{du}{dx} = 2x$ $\int \underbrace{\sqrt{1+x^2}}_{u}\,\underbrace{2x\,dx}_{du}$

$du = 2x\,dx$

$\Rightarrow \int \sqrt{u}\cdot du$

$\Rightarrow \int u^{1/2}\,du = \frac{2}{3}u^{3/2} + C$

$\Rightarrow \frac{2}{3}(1+x^2)^{3/2} + C$

Checking:

$\frac{2}{3}(1+x^2)^{3/2} + C$

$\Rightarrow \frac{2}{3}\cdot\frac{3}{2}(1+x^2)^{1/2}(2x)$

$\Rightarrow 2x\cdot\sqrt{1+x^2}$

EXAMPLE 17

$$\int x^3 \cos(\underbrace{x^4 + 2}_{u})\, dx.$$

let $u = x^4 + 2$

$\frac{du}{dx} = 4x^3 \Rightarrow \left(\frac{1}{4}\right) du = x^3\, dx$

$\Rightarrow \int \cos u \left(\frac{1}{4}\right) du$

$\Rightarrow \sin(u)\left(\frac{1}{4}\right) + C$

$\Rightarrow \boxed{\frac{1}{4}\sin(x^2+2) + C}$

EXAMPLE 18

$$\int \sqrt{\underbrace{2x + 1}_{du}}\, dx.$$

let $u = 2x + 1$

$\frac{du}{dx} = 2 \Rightarrow du = 2\,dx \Rightarrow \frac{1}{2}du = dx$

$\Rightarrow \int \sqrt{u}\, dx$

$\Rightarrow \int \sqrt{u} \cdot \frac{1}{2} du \Rightarrow \left(\frac{1}{2}\right) \int \sqrt{u}\, du$

$\Rightarrow \left(\frac{1}{2}\right)\left(\frac{2}{3}\right)(u)^{3/2} + C$

$\hookrightarrow \boxed{\frac{1}{3}(2x+1)^{3/2} + C}$

EXAMPLE 19

$$\int \frac{x}{\sqrt{\underbrace{1 - 4x^2}}}\, dx.$$

Let $u = 1 - 4x^2$

$\frac{du}{dx} = -8x \Rightarrow du = -8x\, dx$

$\Rightarrow -\frac{1}{8} du = x\, dx$

$\Rightarrow \int \frac{x}{\sqrt{u}}\, dx \Rightarrow \int -\frac{1}{8} du \cdot \frac{1}{\sqrt{u}}$

$\Rightarrow -\frac{1}{8} \int u^{-1/2} du \Rightarrow -\frac{1}{8} \cdot 2 \cdot u^{1/2} + C$

$\Rightarrow \boxed{\frac{1}{4}(1 - 4x^2)^{1/2} + C}$

EXAMPLE 20

$$\int \sqrt{1 + x^2}\, x^5\, dx.$$

$\Rightarrow \underbrace{\sqrt{1+x^2}}_{u} \cdot x^4 \cdot \underbrace{x\,dx}_{\frac{1}{2}du}$

$\Rightarrow \int u^{1/2} \cdot (u^2 - 2u + 1) \cdot \frac{1}{2} du$

$\Rightarrow \frac{1}{2} \int (u^{5/2} - 2u^{3/2} + u^{1/2})\, du$

let $u = 1 + x^2 \Rightarrow (x^2)^2 = (u-1)^2$

$x^4 = u^2 - 2u + 1$

$\frac{du}{dx} = 2x$

$\frac{1}{2} du = x\,dx$

$\frac{1}{2}\left[\frac{2}{7} u^{7/2} - 2 \cdot \frac{2}{5} u^{5/2} + \ldots\right.$

$\left.\ldots + \frac{2}{3} u^{3/2}\right] + C$

$\Rightarrow \frac{1}{7}(1+x^2)^{7/2} - \frac{2}{5}(1+x^2)^{5/2} + \ldots$

$\ldots + \frac{1}{3}(1+x^2)^{3/2} + C$

EXAMPLE 21

$$\int_1^2 \frac{dx}{(3 - 5x)^2}$$

1st, Evaluate $\int \frac{dx}{(3-5x)^2}$

Let $u = 3 - 5x$

$\frac{du}{dx} = -5 \Rightarrow (-\frac{1}{5}) du = dx$

$\Rightarrow \int \frac{1}{u^2} \cdot (-\frac{1}{5}) du$

$\Rightarrow (-\frac{1}{5}) \int u^{-2} du$

$\Rightarrow -\frac{1}{5}(-1) u^{-1} + C$

$\Rightarrow \frac{1}{5(3-5x)} + C$

$\int_1^2 \frac{dx}{(3-5x)^2} = \frac{1}{5(3-5x)}\Big|_1^2$

$= \frac{1}{5(-7)} - \frac{1}{5(-2)}$

$= \frac{1}{14}$