EXAMPLE 22

Nov. 9/16

Let $f(x) = \dfrac{2x^2}{x^2 - 1}$

Intercepts:

y-int: $x \Rightarrow \varnothing$

$f(0) \Rightarrow \varnothing$

$\boxed{(\varnothing, \varnothing) \text{ is a y-int}}$

x-int: $y \Rightarrow \varnothing$

$\dfrac{2x^2}{x^2 - 1} = \varnothing$

$x \Rightarrow \varnothing$

$\boxed{(\varnothing, \varnothing) \text{ is a x-int}}$

V.A.: $x^2 - 1 = \varnothing \Rightarrow x \pm 1$

(Test by plugging in values.)

$2(\pm 1)^2 \neq \varnothing \Rightarrow \boxed{x = 1}$ & $\boxed{x = -1}$ are $\boxed{\text{V.A.}}$

H.A.: $\lim\limits_{x \to \infty} \dfrac{2x^2}{x^2 - 1} \cdot \dfrac{1/x^2}{1/x^2}$

$\lim\limits_{x \to \infty} \dfrac{2}{1 - \cancel{1/x^2}^{\,\varnothing}} \Rightarrow \dfrac{2}{1}$ $\therefore$ $\boxed{y = 2 \text{ is a H.A.}}$

$\lim\limits_{x \to -\infty} \dfrac{2x^2}{x^2 - 1} \cdot \dfrac{1/x^2}{1/x^2} \Rightarrow 2$

C.V.'s: $f'(x) = \dfrac{(4x)(x^2 - 1) - (2x^2)(2x)}{(x^2 - 1)^2} \Rightarrow \dfrac{-4x}{(x^2 - 1)^2}$

$f'(x) = \varnothing \Rightarrow x = \varnothing$ is a CV

$f'(x)$ DNE $\Rightarrow x^2 - 1 = \varnothing \Rightarrow x = \pm 1$ (however, these are not in the domain - they are vertical asymptotes)

	-2	-1	-0.5	0	0.5	1	2
f'	+	NOT CV	+	LOCAL MAX	−	NOT CV	−

$\therefore f$ is inc on $(-\infty, -1) \cup (-1, 0)$ and dec on $(0, 1) \cup (1, \infty)$.

f has a local max at $(0, \varnothing)$

Concavity $f''(x) = \dfrac{-4(x^2-1)^2 + 4x(2)(x^2-1)(2x)}{(x^2-1)^4}$

$= \dfrac{12x^2+4}{(x^2-1)^3}$

$f''(x) = 0 \Rightarrow 12x^2+4 = 0$ X (not possible, $12x^2$ cant equal -4)

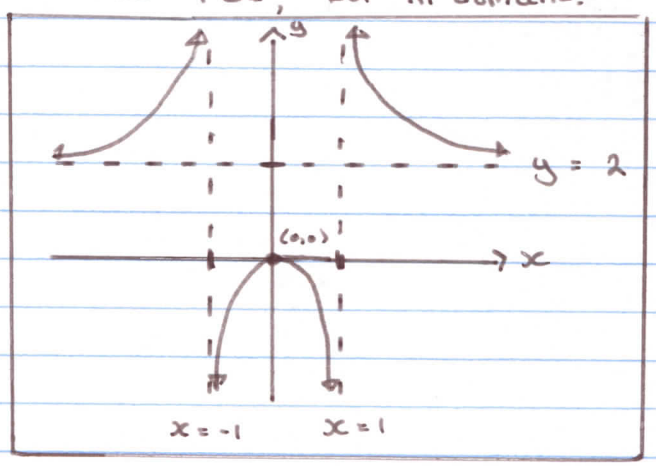
$f''(x)$ DNE $\Rightarrow x = \pm 1$

	-2	-1	0	1	2
f''	$+$		$-$		$+$
	CU		CD		CU

not POI, not in domain.

$\therefore f$ is CU on $(-\infty,-1)(1,\infty)$ and CD on $(-1,1)$

f has no POIs



Domain: $(-\infty,-1) \cup (-1,1) \cup (1,\infty)$

Range: $(-\infty,0] \cup (2,\infty)$

EXAMPLE 23

Let $f(x) = \dfrac{x^2}{\sqrt{x+1}}$

Intercepts: y-int: $x=0 \Rightarrow f(0) \Rightarrow 0$

$(0,0)$ is a y-int

x-int: $y=0 \Rightarrow x=0$

$(0,0)$ is a x-int

Asymptote: V.A. $x+1=0 \Rightarrow x=-1$

$(-1)^2 \neq 0$ $x=-1$ is a V.A

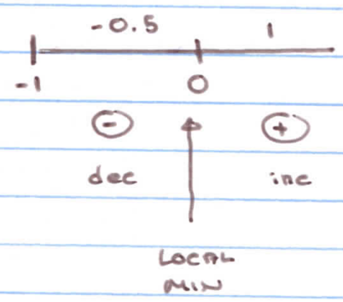
H.A. $\lim\limits_{x\to\infty} \dfrac{x^2}{\sqrt{x+1}} = \infty$, $\lim\limits_{x\to-\infty} \dfrac{x^2}{\sqrt{x+1}}$ DNE

$\therefore$ No horizontal asymptotes.

C.V.'s : $f'(x) = \dfrac{x(3x+4)}{2(x+1)^{3/2}}$

$f'(x) = 0 \Rightarrow x = 0$ or $x = -4/3$ (not in domain)

$f'(x)$ DNE $\Rightarrow x+1 = 0 \Rightarrow x = -1$



$\therefore f$ is decreasing on $(-1, 0)$

and increasing on $(0, \infty)$

f has a local min at $(0, 0)$

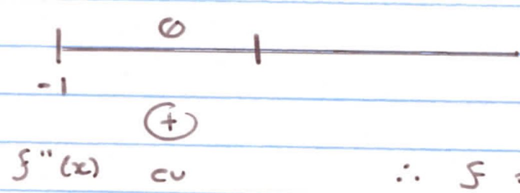
Concavity : $f''(x) = \dfrac{3x^2 + 8x + 8}{4(x+1)^{5/2}}$

$f''(x) = 0 \Rightarrow x$

$\sqrt{8^2 - 4(3)(8)}$ DNE

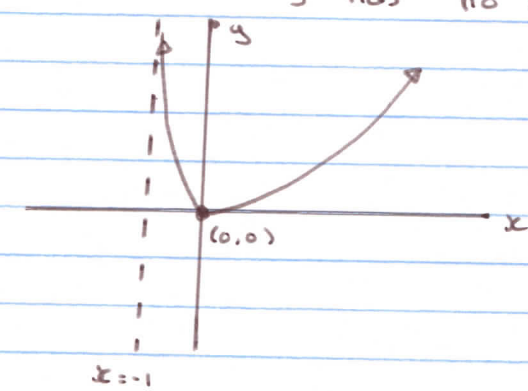
$(b^2 - 4ac)$

$f''(x)$ DNE $\Rightarrow x = -1$



$\therefore f$ is CU on $(-1, \infty)$

f has no POIs



Domain : $(-1, \infty)$

Range : $[0, \infty)$

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Nov. 11 /16

FINAL EXAM INFO

~~Short Answer~~

Kind of questions: (short answer)
- related rates
- optimization problem
- graphing
- 1 proof
- 1 matching questions (graph to functions)

→ Q1 will not be on final, but question about it will be on the exam

For graphing, do work on back of other page → fill in chart.

(*) EXAMPLE 24

$[0, 2\pi]$ First

x-int: $y = 0 \Rightarrow \cos x = 0$

$\Rightarrow x = \pi/2,\ 3\pi/2$

$(\pi/2, 0),\ (3\pi/2, 0)$

y-int: $x = 0 \Rightarrow y = 1/2$ $(0, 1/2)$ is the y-int

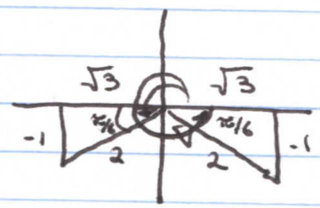
Asymptotes: V.A. $2 + \sin x \neq 0$ $(-1 \leq \sin x \leq 1)$ – No V.A.'s

H.A. $\lim_{x \to \pm\infty} \frac{\cos x}{2 + \sin x}$ DNE ($\sin x$ & $\cos x$ oscillate)

– No H.A.'s

C.V.'s $f'(x) = \frac{-\sin x(2 + \sin x) - \cos x(\cos x)}{(2 + \sin x)^2}$

$= -\frac{2\sin x + 1}{(2 + \sin x)^2}$



$f'(x) = 0 \Rightarrow \sin x = -1/2$

$\Rightarrow x = 7\pi/6,\ 11\pi/6$

$f'(x)$ exists everywhere $\Rightarrow$ no other CVs

	$\pi/2$		$3\pi/2$		$15\pi/8$	
0		$\pi/6$		$11\pi/6$		2π
$f'(x)$	$-$		$+$		$\boxed{-}$	
	dec	local min	inc	local max	dec	

$\sin 15\pi/8 > -1/2$

$0 > \sin 15\pi/8 > -1/2$

$0 > 2\sin 15\pi/8 > -1$

$1 > 2\sin \frac{15\pi}{8} + 1 > 0$

$\therefore f$ is decreasing on $(0, 7\pi/6) \cup (11\pi/6, 2\pi)$

f is increasing on $(7\pi/6, 11\pi/6)$

f has a local min at $(7\pi/6, -\sqrt{3}/3)$

f has a local max at $(11\pi/6, \sqrt{3}/3)$

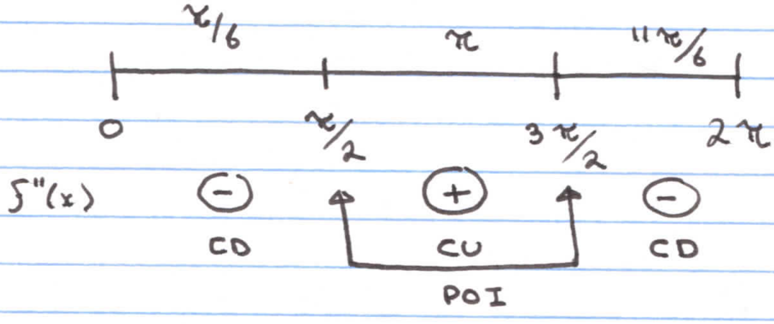
Concavity: $f''(x) = -\dfrac{2\cos x\,(1-\sin x)}{(2+\sin x)^3} = 0$

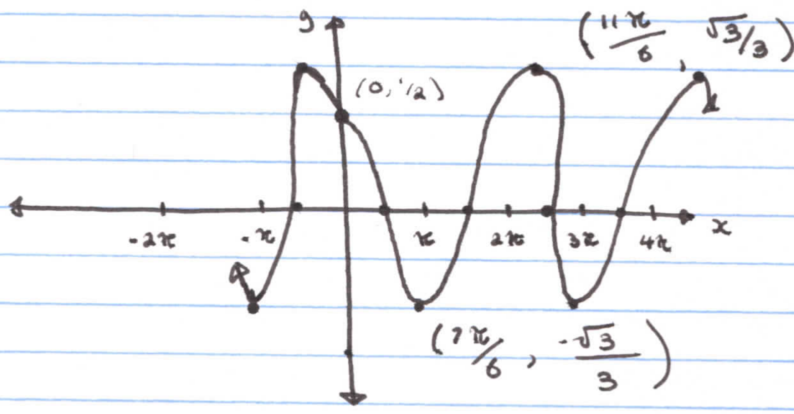
$\Rightarrow \cos x = 0$

$x = \pi/2,\ 3\pi/2$

$\sin x = 1$

$x = \pi/2$





(*) EXAMPLE 25

$f(x) = \dfrac{x^3}{x^2+1}$

Intercepts:

y-int: $x = 0$ $\Rightarrow y = 0$

x-int: $y = 0$ $\Rightarrow x = 0$

$(0, 0)$ is the x and y-intercept.

Asymptotes: V.A. $x^2 + 1 \neq 0$, $\Rightarrow$ No V.A.

$\lim_{x \to +\infty} f(x) = +\infty$ $\qquad \lim_{x \to -\infty} f(x) = -\infty$

$\Rightarrow$ No H.A.

[POLYNOMIAL DIVISION]

$$
\begin{array}{r|l}
 & x \\
\hline
x^2 + 0x + 1 & x^3 + 0x^2 + 0x + 0 \\
 & -\ x^3 + 0x^2 + x \\
\hline
 & -x + 0
\end{array}
$$

$x^3 = x(x^2+1) - x$ $\Rightarrow \dfrac{x(x^2+1) - x}{x^2+1}$

$\Rightarrow \dfrac{x(x^2+1)}{(x^2+1)} - \dfrac{(x)}{x^2+1}$

$\Rightarrow x - \dfrac{x}{x^2+1}$

So as $x \to \infty$, $\dfrac{x}{x^2+1} \Rightarrow 0$

Then as $x \to \infty$, $f(x) \to x$

i.e. as x gets very large, $f(x)$ "behaves" like the line $\boxed{y = x}$

$\boxed{y = x}$ is called an oblique asymptote (slant asymptote)

(Exercise)

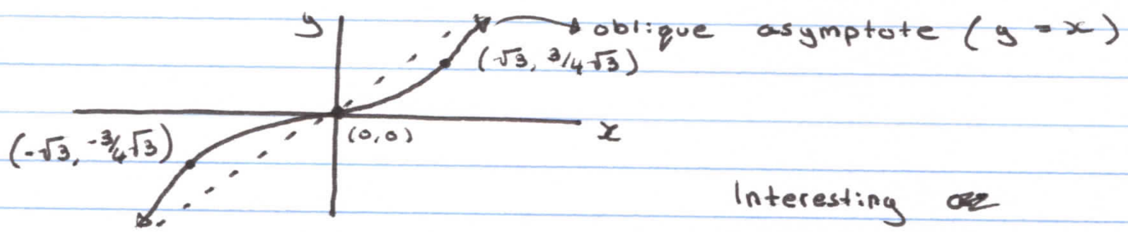
$\therefore f$ is increasing on $(-\infty, 0) \cup (0, \infty)$

[no local max or min]

f is concave up (CU) on $(-\infty, -\sqrt{3}) \cup (0, \sqrt{3})$

CD on $(-\sqrt{3}, 0) \cup (\sqrt{3}, \infty)$

POIs $(-\sqrt{3}, -3/4\sqrt{3})$, $(0, 0)$, $(\sqrt{3}, 3/4\sqrt{3})$



Interesting example