vectors

Sept. 19TH/16

$$A = \begin{pmatrix} a_{n} & \cdots & a_{1m} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}_{m \times n}$$

$AX = B$

Matrix equation

$$X = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \quad B = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$$

Find the inverse of

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}$$

IDENTITY MATRIX (2×2) OR I_2

$$\begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \Rightarrow \left(\begin{array}{cc|cc} 1 & 0 & 1 & -\frac{1}{2} \\ 0 & 1 & 0 & \frac{1}{2} \end{array}\right)$$

$$A^{-1} = \begin{pmatrix} 1 & -\frac{1}{2} \\ 0 & \frac{1}{2} \end{pmatrix}$$

This is through..

$$\left(\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0 & 2 & 0 & 1 \end{array}\right) \xrightarrow{\frac{1}{2}R_2} \left(\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & \frac{1}{2} \end{array}\right) \xrightarrow{-R_2+R_1} \left(\begin{array}{cc|cc} 1 & 0 & 1 & -\frac{1}{2} \\ 0 & 1 & 0 & \frac{1}{2} \end{array}\right)$$

Identity Matrix for $I_n = \begin{pmatrix} 1 & & \emptyset \\ & 1 & \\ \emptyset & & 1 \ldots \end{pmatrix}_{n \times n}$

Sept. 30TH - Quiz 1 (Friday ?)

(Systems of Linear Equations)

- Multiple choice
- One small calculation

Posted slides end.

- Homogeneous and Nonhomogeneous systems of linear equations.

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1 No solution if $p < q$

2 A unique solution if $p = q = n$

3 Infinitely many solutions if $p = q < n$

$s = n - p$ (number of free variables)

$$\begin{array}{c} \quad x \ y \ z \ w \\ \left(\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 0 \\ 1 & 2 & 0 & 2 & 0 \\ 0 & 2 & 6 & 3 & 0 \end{array}\right) \end{array} \xrightarrow{-R_1+R_2} \left(\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & \tfrac{1}{2} & 0 \\ 0 & 0 & 1 & \tfrac{1}{3} & 0 \end{array}\right) \qquad \begin{aligned} s &= n - p \\ &= 4 - 3 \\ &= 1 \end{aligned}$$

$$\Rightarrow \begin{array}{l} x + w \\ y + (\tfrac{1}{2}w) \\ z + (\tfrac{1}{3}w) \end{array} \longrightarrow \begin{array}{l} x = -w \\ y = -(\tfrac{1}{2})w \\ z = -(\tfrac{1}{3})w \end{array}$$

Solution: $(-w, -w/2, -w/3, w)$

$= w(-1, -1/2, -1/3, 1)$

where w is any real number

For example

$$\begin{array}{l} x_1 - 2x_2 + 3x_3 + 2x_4 + x_5 = \emptyset_1 \\ \qquad x_3 + 2x_5 = \emptyset_2 \\ \qquad x_4 - 4x_5 = \emptyset_3 \end{array} \qquad \left(\begin{array}{ccccc|c} 1 & -2 & 3 & 2 & 1 & \emptyset_1 \\ 0 & 0 & 1 & 0 & 2 & \emptyset_2 \\ 0 & 0 & 0 & 1 & -4 & \emptyset_3 \end{array}\right)$$

Must have a unique solution

Where

$Ax = \emptyset$	$Ax = B$, $B \neq \emptyset$
$m < n$	$m < n$
infinitely many solutions	A system has no solution or infinitely many solutions

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Write the system in the matrix form, solve it + write solution in vector form.

$$x + y - 2z + 4w = 5$$
$$2x + 2y - 3z + w = 3$$
$$3x + 3y - 4z - 2w = 1$$

$$\underbrace{\begin{pmatrix} 1 & 1 & -2 & 4 \\ 2 & 2 & -3 & 1 \\ 3 & 3 & -4 & -2 \end{pmatrix}_{3\times4}}_{A} \underbrace{\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}_{4\times1}}_{X} = \underbrace{\begin{pmatrix} 5 \\ 3 \\ 1 \end{pmatrix}}_{B}$$

$AX = B$

$(3\times4)(4\times1) = 3\times1$

Augmented matrix:

$$\left(\begin{array}{cccc|c} 1 & 1 & -2 & 4 & 5 \\ 2 & 2 & -3 & 1 & 3 \\ 3 & 3 & -4 & -2 & 1 \end{array}\right) \xrightarrow[-3R_1+R_3]{-2R_1+R_2} \left(\begin{array}{cccc|c} 1 & 1 & -2 & 4 & 5 \\ 0 & 0 & 1 & -7 & -7 \\ 0 & 0 & 2 & -14 & -14 \end{array}\right) \dots$$

augmented matrix

Free variables (corresponding columns do not contain ~~free variable~~ leading entries)

$$\dots \xrightarrow{-2R_2+R_3} \begin{array}{c} \begin{array}{ccccc} x & \boxed{y} & z & \boxed{w} & \end{array} \\ \left(\begin{array}{cccc|c} 1 & 1 & -2 & 4 & 5 \\ 0 & 0 & 1 & -7 & -7 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \end{array}$$

echelon form of the augmented matrix

$p = 2$

$n = 4$

$s = n - p$

$s = 2$

number of free variables.

$$x + y - 2z + 4w = 5$$
$$z - 7w = -7$$

y and w are free variables

$z - 7w = -7$

$z = -7 + 7w$

$x + y - 2z + 4w = 5$

$x = 5 - y + 2z - 4w$

$x = 5 - y + 2(-7 + 7w) - 4w$

$x = -9 - y + 10w$

$\therefore$ $z = -7 + 7w$

$x = -9 - y + 10w$

where y and w are any real numbers.

System has infinitely many solutions.

$$\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} -9 - y + 10w \\ y \\ -7 \quad + 7w \\ w \end{pmatrix} = \begin{pmatrix} -9 \\ 0 \\ -7 \\ 0 \end{pmatrix} + \begin{pmatrix} -y \\ y \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 10w \\ 0 \\ 7w \\ w \end{pmatrix} = \dots$$

$$\dots \underbrace{\begin{pmatrix} -9 \\ 0 \\ -7 \\ 0 \end{pmatrix}}_{x_0} + \underset{t_1}{\underset{\updownarrow}{y}} \underbrace{\begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}}_{x_1} + \underset{t_2}{\underset{\updownarrow}{w}} \underbrace{\begin{pmatrix} 10 \\ 0 \\ 7 \\ 1 \end{pmatrix}}_{x_2}$$

$$x = \underbrace{x_0}_{\text{particular solution to the given system}} + \underbrace{t_1 x_1 + t_2 x_2}_{\text{general solution to the associated homogeneous system}}$$

where t_1, t_2 are any real number.

$S = n - p$

$x = x_0 + \sum_{i=1}^{s} t_i x_i : s = n - p$

(homogeneous ~~solution~~ system)

or $(x, y, z, w) = (-9, 0, -7, 0) + t_1(-1, 1, 0, 0) + t_2(0, 0, 7, 1)$

y, w are any real number

Write the system in the matrix Form, solve it and write the solution in the vector form.

$$\begin{aligned} 3x + 6y \qquad\qquad - w + 4v &= 10 \\ 2x + 4y + z \qquad\quad - 10v &= 19 \\ -x - 2y \qquad\quad + w + 2v &= 2 \\ -4x - 8y - z \qquad\quad - 16v &= -31 \end{aligned}$$

$$\underbrace{\begin{pmatrix} 3 & 6 & 0 & -1 & 4 \\ 2 & 4 & 1 & 0 & -10 \\ -1 & -2 & 0 & 1 & 2 \\ -4 & -8 & -1 & 0 & -16 \end{pmatrix}_{4\times 5}}_{A} \underbrace{\begin{pmatrix} x \\ y \\ z \\ w \\ v \end{pmatrix}_{5\times 1}}_{x} = \underbrace{\begin{pmatrix} 10 \\ 19 \\ 2 \\ -31 \end{pmatrix}_{4\times 1}}_{B}$$

cont'd. ...

$$\left(\begin{array}{ccccc|c} 3 & 6 & 0 & -1 & 4 & 10 \\ 2 & 4 & 1 & 0 & -10 & 19 \\ -1 & -2 & 0 & 1 & 2 & 2 \\ -4 & -8 & -1 & 0 & -16 & -31 \end{array}\right) \longrightarrow \begin{array}{c} \begin{array}{ccccc} x & \boxed{y} & z & w & \boxed{v} \end{array} \\ \left(\begin{array}{ccccc|c} 1 & 2 & 0 & 0 & 3 & 6 \\ 0 & 0 & 1 & 0 & 4 & 7 \\ 0 & 0 & 0 & 1 & 5 & 8 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) \end{array}$$

y and V are Free-variables

x, z, w are basic (leading) Variables

$$\begin{aligned} x + 2y \quad\quad\quad\quad + 3v &= 6 \\ + z \quad\quad + 4v &= 7 \\ + w + 5v &= 8 \end{aligned}$$

$r = 5 \quad p = 3 \quad s = r - p \Rightarrow 5 - 3 \Rightarrow 2$

$w = 8 - 5v$

$z = 7 - 4v$

$x = 6 - 2y - 3v$

$$\begin{pmatrix} x \\ y \\ z \\ w \\ v \end{pmatrix} = \begin{pmatrix} 6 - 2y & -3v \\ y & \\ 7 & -4v \\ 8 & -5v \\ & v \end{pmatrix} = \underbrace{\begin{pmatrix} 6 \\ 0 \\ 7 \\ 8 \\ 0 \end{pmatrix}}_{x_0} + y \underbrace{\begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}}_{x_1} + v \underbrace{\begin{pmatrix} -3 \\ 0 \\ -4 \\ -5 \\ 1 \end{pmatrix}}_{x_2}$$

The System has infinitely many solutions.

Find A^{-1}, then use A^{-1} to solve $AX = B$

Where $A = \begin{pmatrix} 6 & 7 \\ 5 & 6 \end{pmatrix} \quad B = \begin{pmatrix} 2 \\ -3 \end{pmatrix} \quad X = A^{-1}B$

$$(A|I) = \left(\begin{array}{cc|cc} 6 & 7 & 1 & 0 \\ 5 & 6 & 0 & 1 \end{array}\right) \xrightarrow{(1/6)R_1} \left(\begin{array}{cc|cc} 1 & 7/6 & 1/6 & 0 \\ 5 & 6 & 0 & 1 \end{array}\right) \ldots \text{cont'd.}$$

$$\downarrow$$

$$(I|A^{-1}) \approx \left(\begin{array}{cc|cc} 1 & 0 & a & b \\ 0 & 1 & c & d \end{array}\right)$$

$$\xrightarrow{-5R_1 + R_2} \left(\begin{array}{cc|cc} 1 & 7/6 & 1/6 & 0 \\ 0 & 1/6 & -5/6 & 1 \end{array}\right) \xrightarrow{6R_2} \left(\begin{array}{cc|cc} 1 & 7/6 & 1/6 & 0 \\ 0 & 1 & -5 & 6 \end{array}\right) \ldots$$

$$\ldots \xrightarrow{-7/6 R_2 + R_1} \left(\begin{array}{cc|cc} 1 & 0 & 6 & -7 \\ 0 & 1 & -5 & 6 \end{array}\right)$$

$Ax = B$

$6x_1 + 7x_2 = 2$
$5x_1 + 6x_2 = -3$

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \underset{A}{\begin{pmatrix} 6 & -7 \\ -5 & 6 \end{pmatrix}_{2\times 2}} \underset{B}{\begin{pmatrix} 2 \\ -3 \end{pmatrix}_{2\times 1}} = \begin{pmatrix} a \\ b \end{pmatrix}_{2\times 1} \Rightarrow \begin{pmatrix} 6(2) + (-7)(-3) \\ (-5)2 + (6)(-3) \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 33 \\ -28 \end{pmatrix}_{2\times 1}$$

$x_1 = 33$
$x_2 = -28$

(upper triangular)?

Find the 'uppetriangular' matrix A such that $A^3 = \begin{pmatrix} 8 & -57 \\ 0 & 27 \end{pmatrix}$

$$A = \begin{pmatrix} a_{11} & a_{12} \cdots \\ \emptyset & \ddots a_{mn} \end{pmatrix} \qquad A = \begin{pmatrix} x & y \\ 0 & z \end{pmatrix}_{2\times 2}$$

$$A^2 = AA = \begin{pmatrix} x & y \\ 0 & z \end{pmatrix}\begin{pmatrix} x & y \\ 0 & z \end{pmatrix} = \begin{pmatrix} x^2 + 0y + xy + y^2 \\ 0x + 0z + 0y + z^2 \end{pmatrix} \ldots$$

$$\Rightarrow \begin{pmatrix} x^2 & xy & y^2 \\ 0 & & z^2 \end{pmatrix} \ldots ?$$

impossible to read example on board...

The orthogonal matrix A is a matrix for which

$AA^T = I$

$A^{-1}(AA^T) = A^{-1}I \longrightarrow \underbrace{(A^{-1}A)}_{I} A^T = A^{-1}I = IA^T = A^{-1}I \rightarrow A^T = A^{-1}$

$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is the orthogonal matrix

Show that $a^2 + b^2 = 1$

$a^2 + b^2 = 1$
$c^2 + d^2 = 1$
$ac + bd = 0$

$AA^T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} a & c \\ b & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} a & c \\ b & d \end{pmatrix} = \begin{pmatrix} a^2 + b^2 & + ac + bd \\ ac + bd & + c^2 + d^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Evaluate the determinant of each matrix

$I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $A = \begin{pmatrix} 2 & 1 & 1 \\ 3 & 0 & -1 \\ 4 & 5 & 2 \end{pmatrix}$

$\det A = \begin{vmatrix} 2 & 1 & 1 \\ 3 & 0 & -1 \\ 4 & 5 & 2 \end{vmatrix} = (-1)^{2+1} 3 \begin{vmatrix} 1 & 1 \\ 5 & 2 \end{vmatrix} + (-1)^{2+3} \begin{vmatrix} 2 & 1 \\ 4 & 5 \end{vmatrix} \ldots$

$\ldots = (-3) \underbrace{[1(2) - 5(1)]}_{9} + \underbrace{[2(5) - 1(4)]}_{6} = 9 + 6 = 15$

$\ldots ?$