Sept. 14/16

P - Rank of a coefficient Matrix.

Q - Rank of an augmented Matrix

$n$ - number of unknowns

$m$ - number of equations

$n = m$?

when $p = q = n$ a system has a unique solution

$p < q$ a system has no solution

$\begin{cases} n > p = q & \text{a system has infinitely many solution} \\ S = n - p & \text{number of free variables.} \end{cases}$

e.g.

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & a \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & c \end{array}\right) \quad p = q = n$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right) \quad p < q$$

For example:

$$\left(\begin{array}{ccc|c} 1 & 2 & -4 & 6 \\ 0 & 1 & 2 & 5 \\ 0 & 0 & 1 & 3 \end{array}\right) \qquad \begin{array}{rl} x + 2y - 4z = 6 & \text{(eq. 1)} \\ + 1y + 2z = 5 & \text{(eq. 2)} \\ + z = 3 & \text{(eq. 3)} \end{array}$$

(the augmented matrix is in the echelon form)

$p = q = n$ ∴ system has unique solution.

Solution:

- Substitute (eq. 3) into (eq. 2) to solve for "y"
- Substitute "y" and "z" into (eq. 1) to solve for "x"

Sept. 14/16

For a matrix, the first number represents rows.
$(m \times n)$ - matrix $A$ = $m$ - rows, $n$ - columns

Sum of Matrices

$$A + B = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix} + \begin{pmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & & \vdots \\ b_{m1} & \cdots & b_{mn} \end{pmatrix}$$

(should be a)

$$cA = \begin{pmatrix} ca_{11} & \cdots & ca_{1n} \\ \vdots & & \vdots \\ ca_{m1} & \cdots & ca_{mn} \end{pmatrix}$$

Sept. 16/16

Let $A = \begin{pmatrix} -1 & 2 \\ 2 & -3 \\ \emptyset & 1 \\ 4 & -1 \end{pmatrix}_{4\times 2}$ Let $B = \begin{pmatrix} 1\emptyset & -1 & \emptyset \\ 1 & 2 & 1 \end{pmatrix}_{2\times 3}$

Find AB.

$$AB = (m\times r)(r\times n) = m\times n$$
$$= (4\times 2)(2\times 3) = (4\times 3)$$

COLUMN 1

so.. $= (-1)(10) + (2)(1) \Rightarrow -8$
$= (2)(10) + (-3)(1) \Rightarrow 17$
$= (\emptyset)(10) + (1)(1) \Rightarrow 1$
$= (4)(10) + (-1)(1) \Rightarrow 39$

Sept. 16 /16

Determine which matrices are in reduced echelon form and which are only in echelon form.

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$ reduced echelon form

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$ reduced echelon form

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$ not echelon form

$$\begin{pmatrix} 7 & 1 & 0 & 1 & 1 \\ 0 & 2 & 0 & 2 & 2 \\ 0 & 0 & 0 & 3 & 4 \\ 0 & 0 & 0 & 0 & 5 \end{pmatrix}$$ echelon form

$$\begin{aligned} x_1 - 2x_2 &= -1 \\ 3x_1 + 5x_2 &= 30 \\ 5x_1 + x_2 &= 28 \end{aligned}$$

$$\left(\begin{array}{cc|c} 1 & -2 & -1 \\ 3 & 5 & 30 \\ 5 & 1 & 28 \end{array}\right) \xrightarrow[-5R_1 + R_3]{-3R_1 + R_2}$$

Augmented Matrix

$$\left(\begin{array}{cc|c} 1 & -2 & -1 \\ 0 & 11 & 33 \\ 0 & 11 & 33 \end{array}\right) \xrightarrow{-R_2 + R_3} \left(\begin{array}{cc|c} 1 & -2 & -1 \\ 0 & 11 & 33 \\ 0 & 0 & 0 \end{array}\right)$$

echelon form

$$x_1 - 2x_2 = -1 \qquad\qquad x_1 - 2x_2 = -1$$
$$11x_2 = 33 \qquad\qquad x_1 = -1 + 2(3)$$
$$x_2 = 3 \qquad\qquad x_1 = 5$$

$\therefore$ the system has a unique solution $(p = q = n)$

$p = 2 \quad q = 2$

$n = 2$

Example 2

$$6x_1 - 5x_2 - 3x_3 = 55$$
$$x_1 - x_2 - x_3 = 8$$
$$7x_1 - 6x_2 - 4x_3 = 63$$

$$\left(\begin{array}{ccc|c} 6 & -5 & -3 & 55 \\ 1 & -1 & -1 & 8 \\ 7 & -6 & -4 & 63 \end{array}\right) \xrightarrow[e]{\text{reorganize}} \left(\begin{array}{ccc|c} 1 & -1 & -1 & 8 \\ 6 & -5 & -3 & 55 \\ 7 & -6 & -4 & 63 \end{array}\right) \cdots$$

$$\cdots \xrightarrow[-7R_1 + R_3]{-6R_1 + R_2} \left(\begin{array}{ccc|c} 1 & -1 & -1 & 8 \\ 0 & 1 & 3 & 7 \\ 0 & 1 & 3 & 7 \end{array}\right) \xrightarrow{-R_2 + R_3}$$

$$\longrightarrow \left(\begin{array}{ccc|c} 1 & -1 & -1 & 8 \\ 0 & 1 & 3 & 7 \\ 0 & 0 & 0 & 0 \end{array}\right) \qquad p = 2,\quad q = 2,\quad n = 3$$

echelon form

$$x_1 - x_2 - x_3 = 8 \quad = \quad x_1 - (7 - 3)x_3) - x_3 = 8$$
$$x_2 + 3x_3 = 7 \quad \Rightarrow \quad x_2 = 7 - 3x_3$$

... where $x_3$ is any real number.

$x_1 = 7 - 3x_3$

$x_2 = 7 - 3x_3$

Infinitely many solutions.

$$\begin{aligned} x_1 + 3x_2 + x_3 &= 0 \\ 4x_1 + 2x_2 - x_3 &= 0 \\ -x_1 + x_2 + x_3 &= 1 \end{aligned} \Rightarrow \left(\begin{array}{ccc|c} 1 & 3 & 1 & 0 \\ 4 & 2 & -1 & 0 \\ -1 & 1 & 1 & 1 \end{array}\right) \ldots$$

$$\ldots \xrightarrow[R_1 + R_3]{-4R_1 + R_2} \left(\begin{array}{ccc|c} 1 & 3 & 1 & 0 \\ 0 & -10 & -5 & 0 \\ 0 & 4 & 2 & 1 \end{array}\right) \xrightarrow{-(1/5)R_2} \left(\begin{array}{ccc|c} 1 & 3 & 1 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 4 & 2 & 1 \end{array}\right) \ldots$$

$$\ldots \xrightarrow{-2R_2 + R_3} \left(\begin{array}{ccc|c} 1 & 3 & 1 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right) \qquad \begin{aligned} p &= 2 \\ q &= 3 \end{aligned} \qquad p < q$$

$$0x_1 + 0x_2 + 0x_3 = 1$$

& $\therefore$ the system has no solution.

Compute the Matrix $3A + 4B$ where $A = \begin{pmatrix} 3 & -5 \\ 2 & 7 \end{pmatrix}$

and $B = \begin{pmatrix} -1 & 0 \\ 3 & -4 \end{pmatrix}$

$$3A = \begin{pmatrix} 9 & -15 \\ 6 & 21 \end{pmatrix}_{2\times 2} \qquad 4B = \begin{pmatrix} -4 & 0 \\ 12 & -16 \end{pmatrix}_{2\times 2}$$

$$3A + 4B = \begin{pmatrix} 9 & -15 \\ 6 & 21 \end{pmatrix} + \begin{pmatrix} -4 & 0 \\ 12 & -16 \end{pmatrix} \Rightarrow \begin{pmatrix} 9-4 & -15-0 \\ 6+12 & 21+(-16) \end{pmatrix}$$

$$= \cancel{\begin{pmatrix} 13 & -15 \\ -6 & 5 \end{pmatrix}}$$

$$= \begin{pmatrix} 5 & -15 \\ 18 & 5 \end{pmatrix}$$

Calculate AB and BA

$A = (1\ 2\ 3)_{1\times 3}$

$B = \begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix}_{3\times 1}$

AB

$(1\times 3)(3\times 1) = (1\times 1)$

$= (1\ \ 2\ \ 3)\begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix} = (1(3) + 2(4) + 3(5)) = (26)_{1\times 1}$

BA

$(3\times 1)(1\times 3) = (3\times 3)$

$= \begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix}(1\ \ 2\ \ 3) = \begin{pmatrix} 3(1) & 3(2) & 3(3) \\ 4(1) & 4(2) & 4(3) \\ 5(1) & 5(2) & 5(3) \end{pmatrix}$

$= \begin{pmatrix} 3 & 6 & 9 \\ 4 & 8 & 12 \\ 5 & 10 & 15 \end{pmatrix}_{3\times 3}$

$A = \begin{pmatrix} 8 \\ -2 \end{pmatrix}_{2\times 1}$ $B = \begin{pmatrix} 0 & -2 \\ 3 & 1 \\ -4 & 5 \end{pmatrix}_{3\times 2}$ (row × column)

For AB

$(2\times 1)(3\times 2)$

The product is not defined.

BA

$(3\times 2)(2\times 1) = (3\times 1) \Rightarrow \begin{pmatrix} 0 & -2 \\ 3 & 1 \\ -4 & 5 \end{pmatrix}\begin{pmatrix} 3 \\ -2 \end{pmatrix} \Rightarrow$

$$\begin{pmatrix} 0(3) + (-2)(-2) \\ 3(3) + 1(-2) \\ -4(3) + 5(-2) \end{pmatrix} \Rightarrow \begin{pmatrix} 0 + (+4) \\ 9 + (-2) \\ -12 + (-10) \end{pmatrix} \Rightarrow \begin{pmatrix} 4 \\ 7 \\ -22 \end{pmatrix}_{3\times 1}$$

Identity Matrix $I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$$AI = \begin{pmatrix} 2 & 3 & 4 \\ 5 & 6 & 7 \\ 8 & 9 & 3 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$IA = A$

Example:

$-x_1 - 2x_2 + x_3 = 5$
$-2x_1 + 3x_2 + x_3 = 1$
$-x + 3x_2 + 2x_3 = 2$

$AX = B$
A - coefficient matrix
X - column vector of unknowns
B - column vector of constants

$$\left(\begin{array}{ccc|c} -1 & -2 & 1 & 5 \\ -2 & 3 & 1 & 1 \\ -1 & 3 & 2 & 2 \end{array}\right)\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

|— A —|— B —|— x —|

$$\left(\begin{array}{ccc|c} -1 & -2 & 1 & 5 \\ -2 & 3 & 1 & 1 \\ -1 & 3 & 2 & 2 \end{array}\right) \longrightarrow ?$$

answer on board resolves ...

$$\left(\begin{array}{ccc|c} 1 & -2 & 1 & 5 \\ -2 & 3 & 1 & 1 \\ 1 & 3 & 2 & 2 \end{array}\right)$$ diff. Matrix. ?

$x + y - 2z + 4w = 5$
$2x + 2y - 3z + w = 3$
$3x + 3y - 4z - 2w = 1$

$$\underset{A \;\; 3\times4}{\left(\begin{array}{cccc|c} 1 & 1 & -2 & 4 & 5 \\ 2 & 2 & -3 & 1 & 3 \\ 3 & 3 & -4 & -2 & 1 \end{array}\right)}^{\;Ax}_{\;3\times1} \underset{x \;\; 4\times1}{\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}}$$

$\therefore Ax = 3 \times 1$

$$\left(\begin{array}{cccc|c} 1 & 1 & -2 & 4 & 5 \\ 2 & 2 & -3 & 1 & 3 \\ 3 & 3 & -4 & -2 & 1 \end{array}\right) \xrightarrow[-3R_1 + R_2]{-2R_1 + R_2} \left(\begin{array}{cccc|c} 1 & 1 & -2 & 4 & 5 \\ 0 & 0 & 1 & -7 & -7 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$x + y - 2z + 4w = 5$
$z - 7w = -7$

$$\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} -9 & -y & +10w \\ -7 & y & +7w \\ 0 & 0 & w \end{pmatrix}$$

$z = -7 + 7w$
$x + y - 2(-7 + 7w) + 4w = 5$
$x = -y + 2(-7 + 7w) - 4w + 5$
$x = -y - 14 + 14w - 4w + 5$
$\boxed{x = -9 - y + 10w}$

$$\begin{pmatrix} -9 \\ 0 \\ -7 \\ 0 \end{pmatrix} + \begin{pmatrix} -y \\ y \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 10w \\ 0 \\ 7w \\ w \end{pmatrix}$$

$$x = x_0 + yx_1 + wx_2$$

$$\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} -9 \\ 0 \\ -7 \\ 0 \end{pmatrix} + y\begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + w\begin{pmatrix} 10 \\ 0 \\ 7 \\ 1 \end{pmatrix}$$

$x = x_0 + yx_1 + wx_2$
where ~~x~~, $y, w$ are any real number.