Spaces of a Matrix

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}_{m \times n}$$

$R_1 = (a_{11}, \dots, a_{1n})$

$Row(A) = \{(x : x = c_1 R_1, \dots, c_m R_m\}$

$C_1 = (C_{11}, \dots, C_{m1})$

$Col(A) = \{y \cdot y = b_1 C_1 + \dots b_n C_n\}$

$\dim Row(A)$ = rank of a row of space

= $\dim(c)$ = rank of a column space = rank A

1) Let A be given

$$A = \begin{pmatrix} 1 & -1 & 3 \\ \emptyset & 1 & 1 \\ 2 & -1 & 1 \end{pmatrix}$$

Describe the row and column subspaces of A and final bases for them, row and column ranks.

$Col(A) = \{Y : Y = b_1(1, \emptyset, 2) + b_2(-1, 1, -1) + b_3(3, 1, 1)\}$

$Row(A) = \{X : X = c_1(1, -1, 3) + C_2(\emptyset, 1, 1) + C_3(2, -1, 1)\}$

Basis for Col(A)

$$A = \begin{pmatrix} 1 & -1 & 3 \\ \emptyset & 1 & 1 \\ 2 & -1 & 1 \end{pmatrix} \xrightarrow{R_3 - 2R_1} \begin{pmatrix} 1 & -1 & 3 \\ \emptyset & 1 & 1 \\ \emptyset & 1 & -5 \end{pmatrix} \xrightarrow{R_3 - R_2} \begin{pmatrix} 1 & -1 & 3 \\ \emptyset & 1 & 1 \\ \emptyset & \emptyset & 6 \end{pmatrix}$$

echelon form

The basis for $Col(A) = \{(1, \emptyset, 2), (-1, 1, 1), (3, 1, 1)\}$

$\dim Col(A) = 3$ = rank of Col(A)

$$A^T = \begin{pmatrix} 1 & \emptyset & 2 \\ -1 & 1 & -1 \\ 3 & 1 & 1 \end{pmatrix} \xrightarrow[R_3 - 3R_1]{R_1 + R_2} \begin{pmatrix} 1 & \emptyset & 2 \\ \emptyset & 1 & 1 \\ \emptyset & 1 & -5 \end{pmatrix} \xrightarrow{R_3 - R_2} \begin{pmatrix} 1 & \emptyset & 2 \\ \emptyset & 1 & 1 \\ \emptyset & \emptyset & -6 \end{pmatrix}$$

The basis for $Row(A) = \{(1, -1, 3), (\emptyset, 1, 1), (2, -1, 1)\}$

$\dim Row(A) = 3$

Find the basis for the null space of A

$NS(A)$

$AX = 0$

$$\begin{pmatrix} -1 & -1 & 3 \\ 0 & 1 & 1 \\ 2 & -1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$x_1 \pm x_2 = x_3 = 0$ $\qquad NS(A) = \{0\}$

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right)$$

2) Let A be given

$$A = \begin{pmatrix} 3 & 4 & 0 & 7 \\ 1 & -5 & 2 & -2 \\ -1 & 4 & 0 & 3 \\ 1 & -1 & 2 & 2 \end{pmatrix} \longrightarrow \begin{array}{c} \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & \cancel{0}1 \\ 0 & 1 & 0 & \cancel{0}1 \\ 0 & 0 & 1 & 1\,\cancel{0} \\ 0 & 0 & 0 & \cancel{0} \end{pmatrix} \end{array}$$

row-reduced echelon form

Describe the row and column subspaces of A and find basis for them, row and column ranks.

$Col(A) = \{y : y = b_1(3, 1, -1, 1) + b_2(4, -5, 4, -1) + b_3$ to erase.

$Col(A) = \{\cancel{x} : y = b_1(3, 1, -1, 1) + b_2(4, -5, 4, 1) + b_3(0, 2, 0, 2) + b_4(7, -2, 3, 2)$

$Row(A) = \{x : x = c_1(3, 4, 0, 7) + c_2(1, -5, 2, -2) + c_3(-1, 4, 0, 3) + c_4(1, -1, 2, 2)$

$$A^T = \begin{pmatrix} 3 & 1 & -1 & 1 \\ 4 & -5 & 4 & -1 \\ 0 & 2 & 0 & 2 \\ 7 & -2 & 3 & 2 \end{pmatrix} \longrightarrow \text{echelon form}$$

basis for $Col(A) = \{(3, 1, -1, 1), (4, -5, 4, -1), (0, 2, 0, 2)\}$

dim $Col(A) = 3$

~~dim~~ $Row(A) = \{(1, 0, 0, 1), (0, 1, 0, 1), (0, 0, 1, 1)\}$

dim $Row(A) = 3$

3) Find the basis of the space ω generated by

$S = \{(0,1,2,0), (0,1,0,0), (0,1,1,0\}$

$\omega = (x : x = c_1 x_1 + c_2 x_2 + c_3 x_3)$

1st way $\Rightarrow$ $\begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \\ 2 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 1/2 \\ 0 & 1 & 1/2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$S_1 = \{(0,1,2,0), (0,1,0,0)\}$

$\dim \omega = 2$

2nd way $\Rightarrow$ $A^T = \begin{pmatrix} 0 & 1 & 2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

"row way"

$\underline{A}$ $\qquad\qquad$ $\underline{B}$

The basis for ω is $S_1 = \{(0,1,2,0), (0,0,1,0)\}$

$\dim \omega = 2$

4) $\omega = \text{span}\{(-1,2,1), (3,3,6), (-2,1,-3)\}$ Find the basis for ω.

(a) the "weeding out" procedure

$\begin{pmatrix} -1 & 3 & -2 \\ 2 & 3 & 1 \\ 1 & 6 & -3 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1/3 \\ 0 & 0 & 0 \end{pmatrix}$

$S = \{(-1,2,1), (3,3,6)\}$

$\dim \omega = 2$

$\boxed{\text{Rank } A = 2}$

(b) the "row form" procedure

$\begin{pmatrix} -1 & 2 & 1 \\ 3 & 3 & 6 \\ -2 & 1 & -3 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & -2 & 1 \\ 0 & 3 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

echelon form

$S = \{(1,-2,1), (0,3,1)\}$

$\dim \omega = 2$