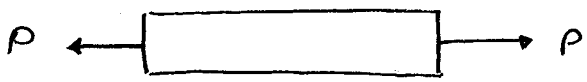
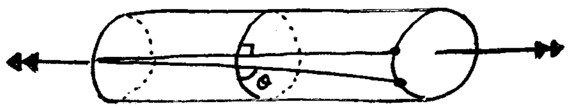
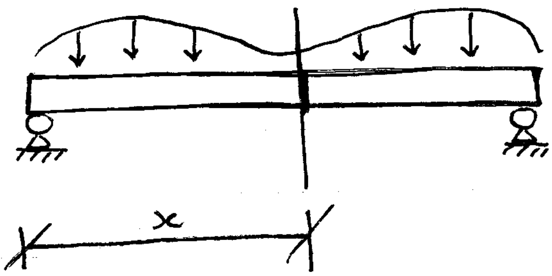
Ch. 12 Deflection of Beams and Shafts



$$\epsilon = \frac{\Delta L}{L}$$





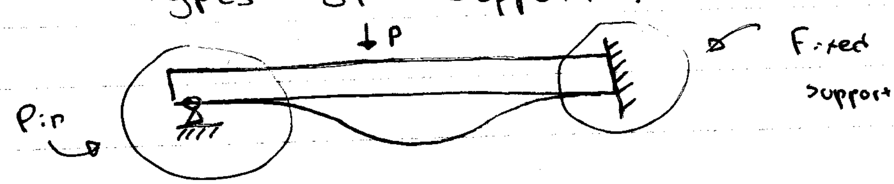
$$\sigma = -\frac{My}{I}$$

12.1 The Elastic Curve

The deflection diagram of the longitudinal axis that through the centroid of each cross-section area of the beam is called the elastic curve.

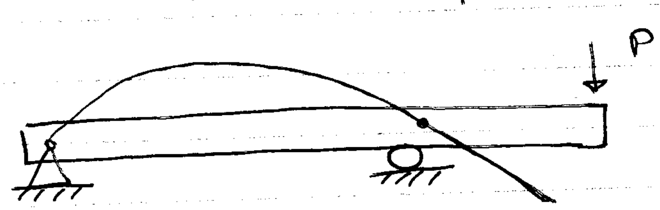
Sketch the elastic curve

1) The slope/disp variation at different types of supports.

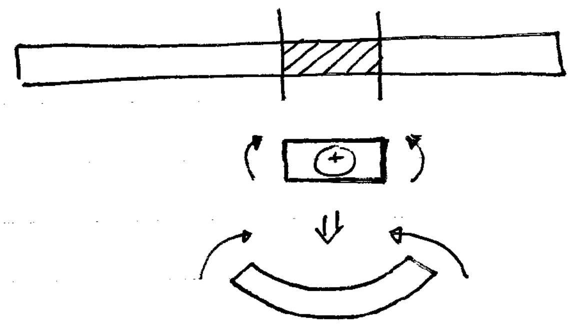


Pin : Resisting a Force, restrict disp.

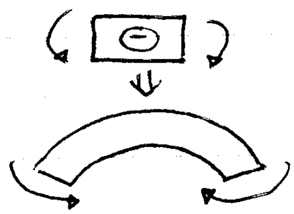
Fixed : Resisting a Force and a moment
Restrict displacement and rotation



2) The relationship between moments and elastic curve.

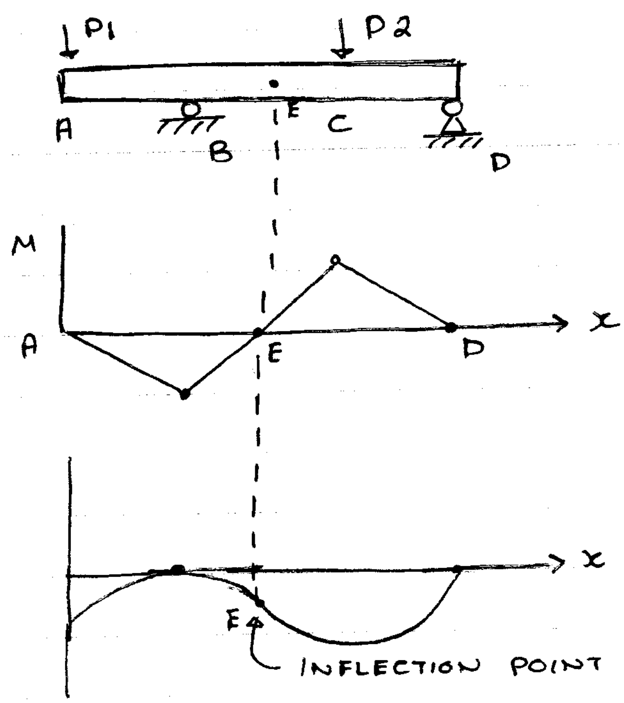


↳ concave upwards



↳ concave downwards

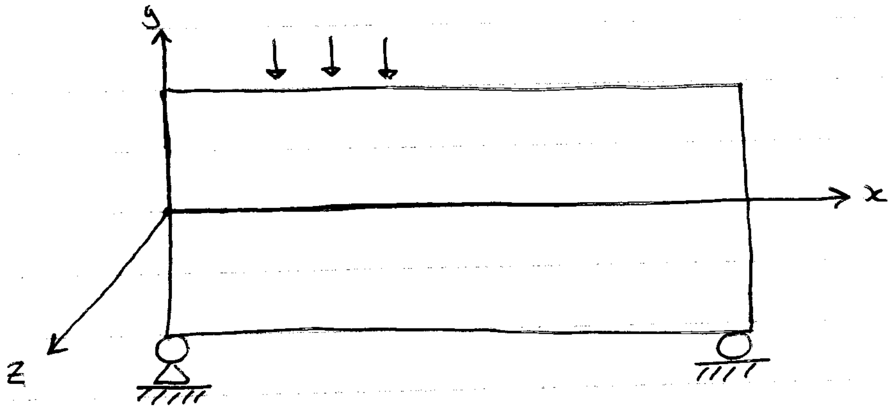
Example:



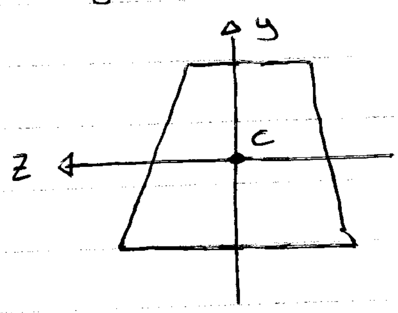
Bending Deformation:

#1 : The longitudinal axis (x) which lies within the neutral surface does not experience any change in length

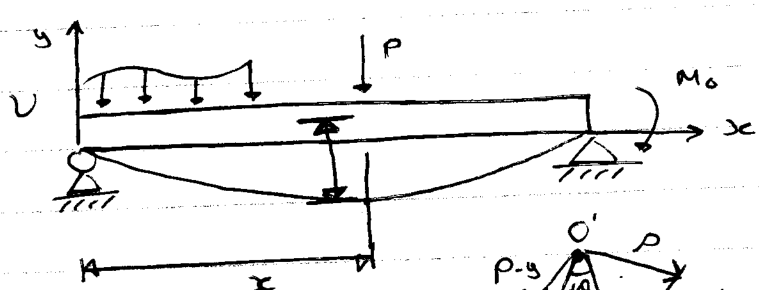
#2 : All cross-sections of the beam remain plane and perpendicular to the longitudinal axis during the deformation.

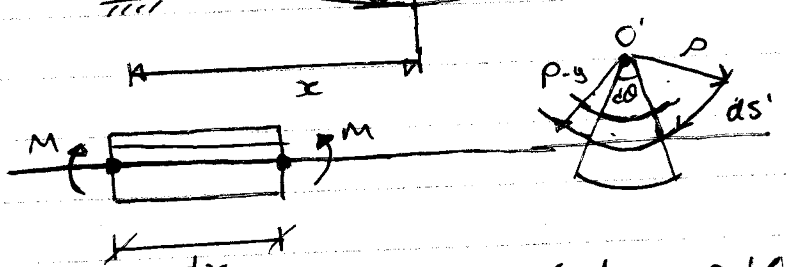


xy: plane of symmetry



Moment - curvature relationship





$$\begin{cases} dx = \rho\, d\theta \\ ds' = (\rho - y)\, d\theta \end{cases} = \frac{(\rho - y)d\theta - \rho\, d\theta}{\rho\, d\theta}$$

For Normal Strain

$$\epsilon = \frac{ds' - dx}{dx} = \frac{(\rho y)d\theta - \rho d\theta}{\rho d\theta}$$

$$\boxed{\epsilon = -\frac{y}{\rho}}$$

Hooke's Law:

$$\sigma = E\epsilon = \frac{-Ey}{\rho} \quad *$$

The Flexure Formula:

$$\Rightarrow \frac{-Ey}{\rho} = \frac{-My}{I} \quad *$$

$$\boxed{\Rightarrow \frac{1}{\rho} = \frac{M}{EI}}$$

→ Flexural Rigidity: EI

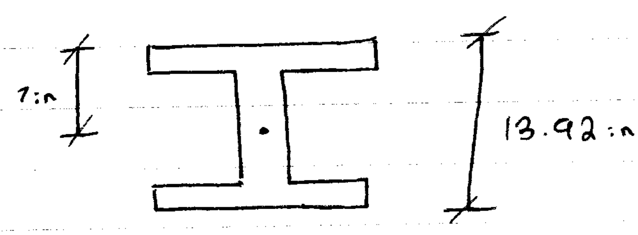
Special Case: M = const (pure bending)

ρ will be a constant as well.

→ circular arc.

A beam, A36 steel, W14×53

$$\begin{cases} E_{ST} = 29(10^3) \text{ ksi} \\ \sigma_y = 36 \text{ ksi} \end{cases}$$

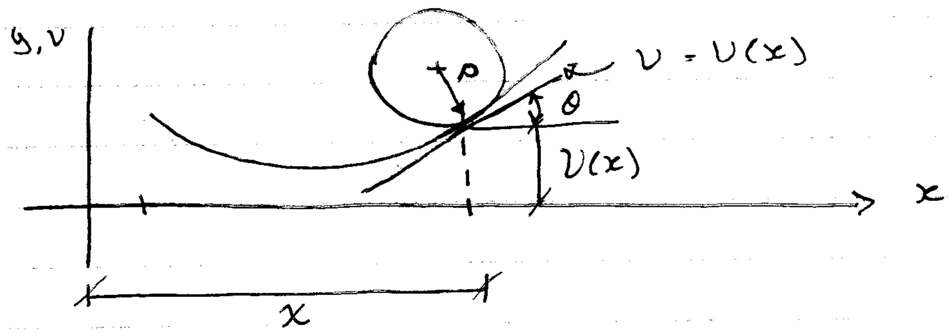


If $\sigma_{max} = \sigma_y$

$$1/\rho = -\epsilon/y = -\frac{\sigma_{max}}{Ey}$$

$$|\rho| = \left|\frac{Ey}{\sigma_{max}}\right| = \frac{(29\times10^3)(7)}{36} = 5639 \text{ in}$$

12.2 Slope and Displacement by Integration

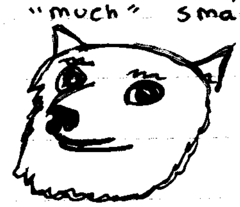


$$\boxed{\frac{1}{\rho} = \frac{M}{EI}}$$

$$\frac{1}{\rho} = \frac{d^2v/dx^2}{\left[1 + (dv/dx)^2\right]^{3/2}}$$

Small Deformation:

$v(x)$ is small, $\left|\frac{dv}{dx}\right| \ll 1$ notation represents "much" small



$$\frac{dv}{dx} = \tan\theta \approx \theta$$

$$\boxed{\theta = \frac{dv}{dx}}$$

* slope angle

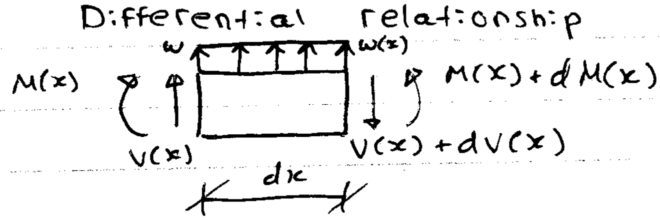
* rotation of the cross-section

Small Deformation

$$1/\rho = \frac{d^2v}{dx^2}$$

$$\Rightarrow \boxed{\frac{d^2v}{dx^2} = \frac{M}{EI}}$$

Differential relationship



$\Sigma F = 0 \quad : \quad V(x) + w(x)\,dx - V(x) - dV(x) = 0$

$$\boxed{\frac{dV}{dx} = w}$$

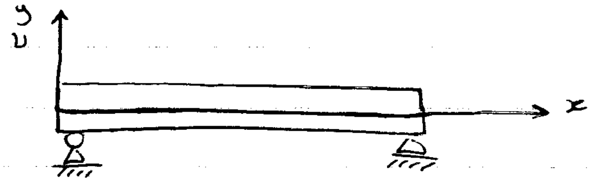
$\Sigma M = 0 \quad : \quad \boxed{\frac{dM}{dx} = V}$

Since $M(x) = EI \dfrac{d^2 v}{dx^2}$

$$\Rightarrow \quad \boxed{\frac{d}{dx}\left(EI \frac{d^2 v}{dx^2} \right) = V}$$

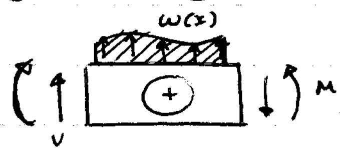
$$\Rightarrow \quad \boxed{\frac{d^2}{dx^2}\left(EI \frac{d^2 v}{dx^2} \right) = w}$$

12.2 Slope and displacement by integration



$$(1) \quad E = \frac{d^2 v(x)}{dx^2} = M(x)$$

Sign convention:

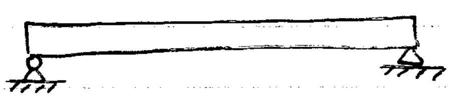


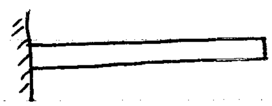
$$(2) \quad \frac{d}{dx}\left(EI \frac{d^2 v(x)}{dx^2} \right) = V(x)$$

$$(3) \quad \frac{d^2}{dx^2}\left(EI \frac{d^2 v(x)}{dx^2} \right) = w(x)$$

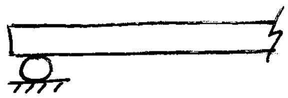
Domain: $0 \le x \le L$

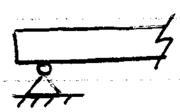
Boundary Conditions

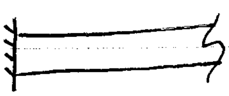


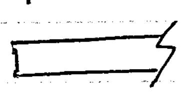


Deflection

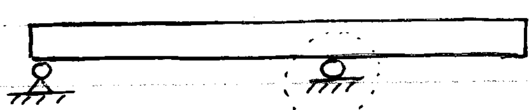
Roller:  $v = 0$, $M = 0$

Pin:  $v = 0$, $M = 0$

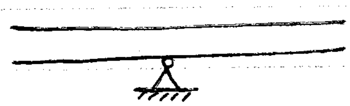
Fixed:  $v = 0$, $M = 0$

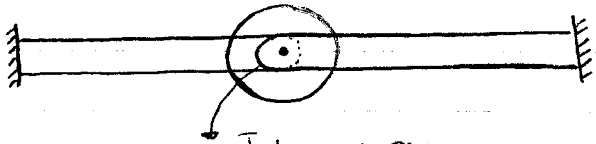
Free:  $V = 0$, $M = 0$

Shear force

 $v = 0$

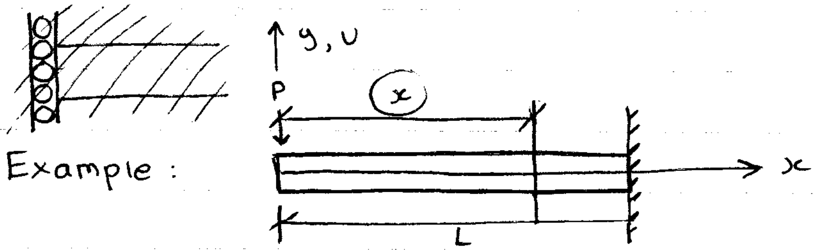
Roller

Pin  $v = 0$



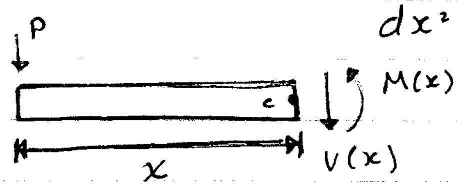
$M = 0$

Example :



Determine the elastic curve, $EI = const$

Solution : $EI \dfrac{d^2 v}{dx^2} = M$



$\Sigma M_c = 0$:

$M(x) + Px = 0$

$M(x) = -Px$

Elastic Curve

$$EI \frac{d^2 v}{dx^2} = -Px$$

$$EI \frac{dv}{dx} = -\frac{P}{2}x^2 + C_1$$

$$* \quad EIv = \frac{-Px^3}{6} + C_1 x + C_2$$

Boundary conditions

$$\begin{cases} x = L, & v = 0 \\ x = L & \theta = \dfrac{dv}{dx} = 0 \end{cases}$$

$$\hookrightarrow \quad -\frac{P}{6}L^3 + C_1 L + C_2 = 0$$

$$\Rightarrow \quad \frac{-P}{2}L^2 + C_1 = 0$$

$$\Rightarrow \quad C_1 = \frac{P}{2}L^2, \qquad C_2 = -\frac{1}{3}PL^3$$

$$EIv = -\frac{P}{6}x^3 + \frac{P}{2}L^2 x - \frac{1}{3}PL^3$$

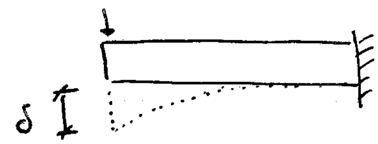
$$\Rightarrow \quad v = \frac{-P}{6EI}\left(x^3 - 3L^2x + 2L^3\right) \quad *$$

and $\quad \theta = \dfrac{dv}{dx} = \dfrac{-P}{2EI}\left(x^2 - L^2\right)$

max deflection

it occurs at $x = 0$

$$v_{max} = \frac{-PL^3}{3EI}$$

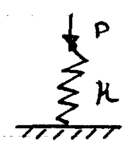


and the rotation at $x = 0$ is

$$\theta\big|_{x=0} = \frac{PL^2}{2EI}$$

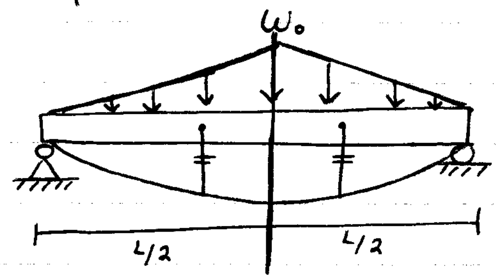
At the Free end

$$\delta = \frac{PL^3}{3EI} \Rightarrow P = \frac{3EI}{L^3}\cdot\delta$$



$P = k\delta$

Example:

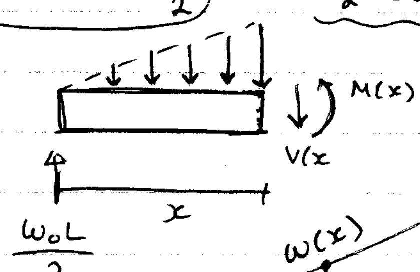


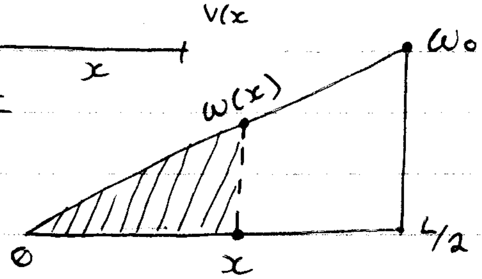
EI = const

Determine the max deflection of the beam.

Solution : $\quad EI\dfrac{d^2v}{dx^2} = M(x)$

$0 \le x \le \frac{L}{2}$: $\quad \frac{1}{2} w_0 \times L/2 \times 2 \times \frac{1}{2} = \dfrac{w_0 L}{4}$

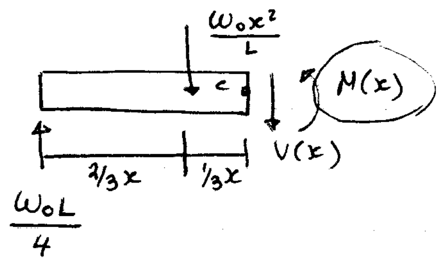




$$\frac{w(x)}{w_0} = \frac{x}{L/2}$$

$$w(x) = \frac{2w_0 x}{L}$$

$$\frac{1}{2}x \cdot w(x) = \frac{1}{2}x \cdot \frac{2w_o x}{L} = \frac{w_o x^2}{L}$$



$\Sigma M_c = 0$:

$$0 = M(x) + \frac{w_o x^2}{L} \cdot \frac{1}{3}x - \frac{w_o L \cdot x}{4}$$

$$\Rightarrow M(x) = -\frac{w_o}{3L}x^3 + \frac{w_o L}{4}x$$

Elastic Curve

$$EI\frac{d^2 v}{dx^2} = -\frac{w_o}{3L}x^3 + \frac{w_o L}{4}x$$

$$EI\frac{dv}{dx} = -\frac{w_o}{12L}x^4 + \frac{w_o L}{8}x^2 + C_1$$

$$EI\,v = -\frac{w_o x^5}{60L} + \frac{w_o L x^3}{24} + C_1 x + C_2$$

Boundary Conditions

At $x = 0$, $v = 0$

$$0 = 0 + C_2 \Rightarrow C_2 = 0$$

At $x = L/2$, $\theta = 0$

$$0 = -\frac{w_o}{12L}\left(\frac{L}{2}\right)^4 + \frac{w_o L}{8}\left(\frac{L}{2}\right)^2 + C_1$$

$$\Rightarrow C_1 = -\frac{5w_o L^3}{192}$$

Deflection

$$EI\,v = -\frac{w_o}{60L}x^5 + \frac{w_o L}{24}x^3 - \frac{5w_o L}{192}x$$

The max deflection occurs when $x = L/2$

$$v_{max} = \frac{1}{EI}\left[-\frac{w_o}{60L}\left(\frac{L}{2}\right)^5 + \frac{w_o L}{24}\left(\frac{L}{2}\right)^3 - \frac{5w_o L^3}{192}\left(\frac{L}{2}\right)\right]$$

$$= -\frac{w_o L^4}{120EI}$$