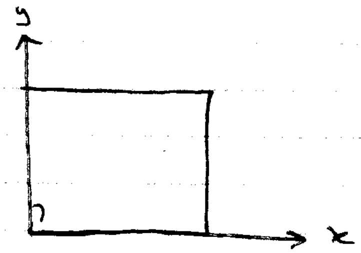
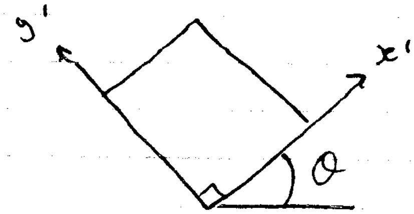


$\epsilon_x, \epsilon_y, \gamma_{xy}$



$\epsilon_{x'}, \epsilon_{y'}, \gamma_{x'y'}$

$$\begin{cases} \epsilon_{x'} = \epsilon_{AVG} + \dfrac{\epsilon_x - \epsilon_y}{2} \cos 2\theta + \dfrac{\gamma_{xy}}{2} \sin 2\theta \\ \epsilon_{y'} = \epsilon_{AVG} - \dfrac{\epsilon_x - \epsilon_y}{2} \cos 2\theta - \dfrac{\gamma_{xy}}{2} \sin 2\theta \\ \dfrac{\gamma_{x'y'}}{2} = \dfrac{\epsilon_x - \epsilon_y}{2} \sin 2\theta + \dfrac{\gamma_{xy}}{2} \cos 2\theta \end{cases}$$

Principal strains :

$$\tan 2\theta_p = \frac{(\gamma_{xy}/2)}{(\epsilon_x - \epsilon_y)/2} = \frac{\gamma_{xy}}{\epsilon_x - \epsilon_y}$$

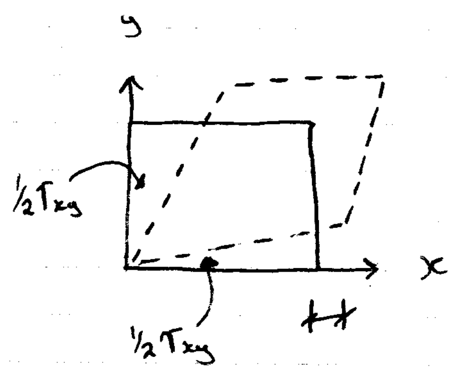
$$\epsilon_{1,2} = \epsilon_{AVG} \pm R$$

$$\epsilon_{AVG} = \frac{\epsilon_x + \epsilon_y}{2}, \quad R = \sqrt{\left(\frac{\epsilon_x - \epsilon_y}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2}$$

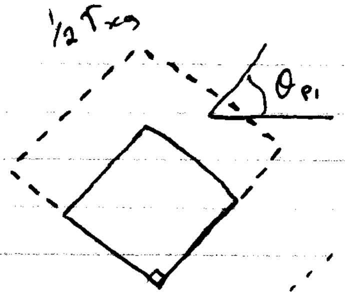
Max in-plane shear strain :

$$\tfrac{1}{2} \gamma_{\substack{max \\ in\text{-}plane}} = R = \sqrt{\left(\frac{\epsilon_x - \epsilon_y}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2}$$

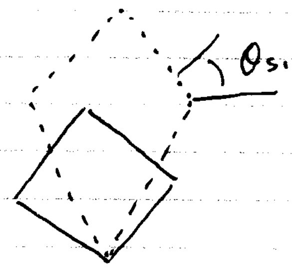
$$\tan 2\theta_s = -\frac{\epsilon_x - \epsilon_y}{\gamma_{xy}}$$



$\epsilon_x, \epsilon_y, \gamma_{xy}$



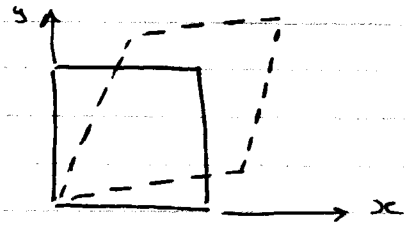
θ_{p1}, θ_{p2} , ϵ_1, ϵ_2



θ_{s1}, θ_{s2}

$\underbrace{\epsilon_{AVG}, \quad \epsilon_{AVG}}$ $\quad \gamma_{\substack{max \\ in\text{-}plane}}$

Example



$\epsilon_x = 0$
$\epsilon_y = 0$
$\gamma_{xy} = 200(10^{-6})$

Find the state of strain on an element oriented at the point c.c.w. 45°

Solution : $\theta = 45°$

$$\epsilon_{x'} = \epsilon_{AVG} + \frac{\epsilon_x - \epsilon_y}{2}\cos 2\theta + \frac{\gamma_{xy}}{2}\sin 2\theta$$

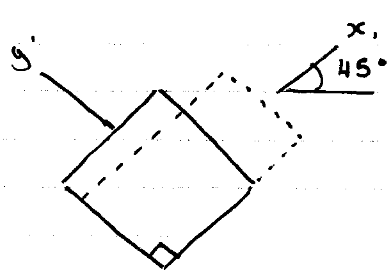
$$= 0 + 0 + \frac{200(\times 10^{-6})\sin 90°}{2}$$

$\epsilon_{x'} = 100(10^{-6})$ (T)

$\epsilon_{y'} = -100(10^{-6})$ (C)

$$\frac{1}{2}\gamma_{x'y'} = \frac{-\epsilon_x - \epsilon_y}{2}\sin 2\theta + \gamma_{xy}\cos 2\theta$$

$$= 0 + 0 \Rightarrow 0$$



Example

$\epsilon_x = -350\,(10^{-6})$
$\epsilon_y = 200\,(10^{-6})$
$\gamma_{xy} = 80\,(10^{-6})$

Determine:

a) The principle strains and associated orientations
b) The max in-plane shear strain and orientation

$\epsilon_{AVG} = -75\,(10^{-6})$

$R = 277.9\,(10^{-6})$

$\therefore\ \epsilon_1 = \epsilon_{AVG} + R = (-75 + 277.9)(10^{-6})$
$= 202.9\,(10^{-6})$

$\epsilon_2 = \epsilon_{AVG} - R$
$= -352.9\,(10^{-6})$

$$\tan 2\theta_p = \frac{\gamma_{xy}}{\epsilon_x - \epsilon_y} = \frac{80\,(10^{-6})}{(-350-200)(10^{-6})} = -0.1445$$

$2\theta_p = -8.28°$ and $-8.28° + 180°$
$\theta_p = -4.14°$ and $85.86°$

When $\theta = -4.14°$

$$\epsilon_{x'} = \epsilon_{AVG} + \frac{\epsilon_x - \epsilon_y}{2}\cos 2\theta + \frac{\gamma_{xy}}{2}\sin 2\theta$$

$$= \left[-75 + \frac{-350-200}{2}\cos(-8.28°) + \frac{80}{2}\sin(-8.28°)\right]\cdot 10^{-6}$$

$= -352.9\,(10^{-6}) = \epsilon_2$

$\therefore\ \theta_{p1} = 85.86°$ and $\theta_{p2} = -4.14°$

b) Max in-plane shear

$\frac{1}{2}\gamma_{\substack{max \\ in\text{-}plane}} = R = 277.9\,(10^{-6})$

$\gamma_{\substack{max \\ in\text{-}plane}} = 2R = 555.8\,(10^{-6})$

$$\tan 2\theta_s = -\frac{\epsilon_x - \epsilon_y}{\gamma_{xy}} = -\left(\frac{-350-200}{80}\right) = 6.875$$

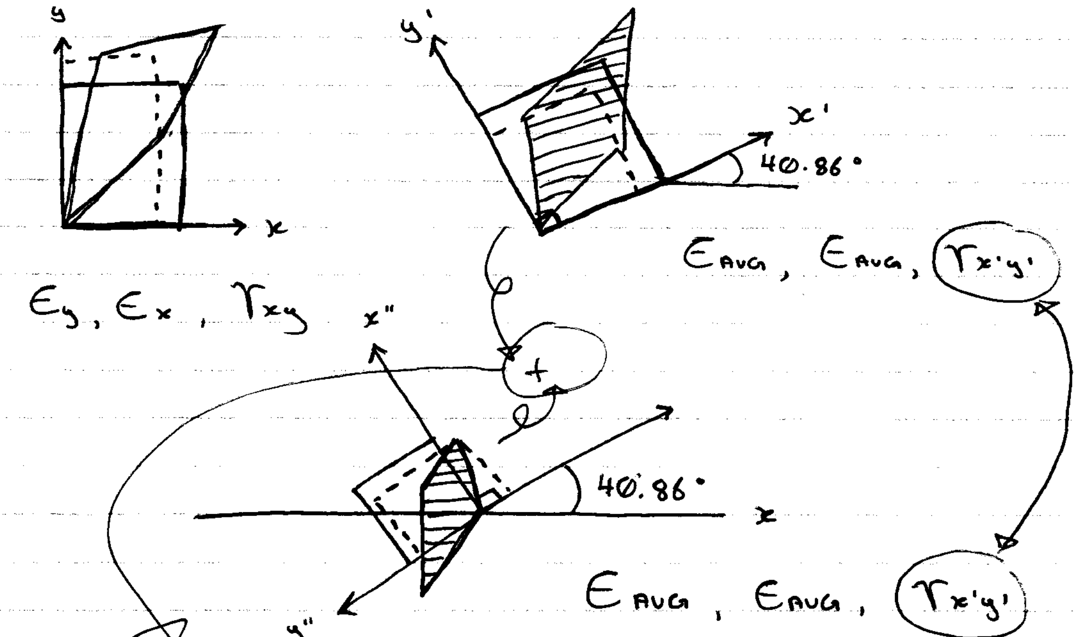
$2\theta_s = 81.72°$ and $81.72° + 180°$

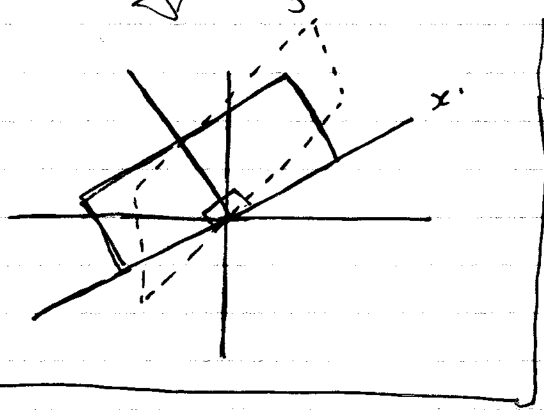
$\theta_s = 40.86°$ and $130.86°$

When $\theta = 40.86°$

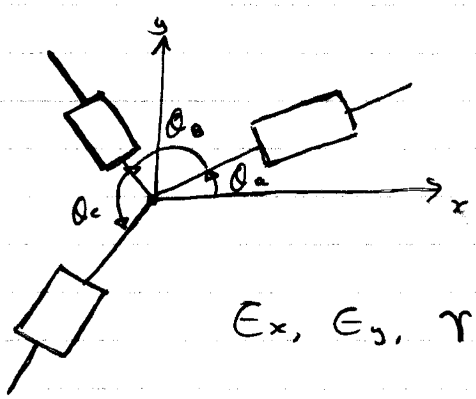
(2?) $\gamma_{x'y'} = 277.9\,(10^{-6})$

$\theta_{s1} = 40.86°$ $\qquad$ $\theta_{s2} = 130.86°$





Strain Rosettes

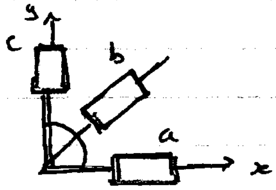


$\epsilon_x, \epsilon_y, \gamma_{xy}$

Transformation

$$\begin{cases} \epsilon_a = \epsilon_x \cos^2\theta_a + \epsilon_y \sin^2\theta_a + \gamma_{xy} \sin\theta_a \cos\theta_a \\ \epsilon_b = \epsilon_x \cos^2\theta_b + \epsilon_y \sin^2\theta_b + \gamma_{xy} \sin\theta_b \cos\theta_b \\ \epsilon_c = \epsilon_x \cos^2\theta_c + \epsilon_y \sin^2\theta_c + \gamma_{xy} \sin\theta_c \cos\theta_c \end{cases}$$

Case 1:



$\theta_A = 0°$

$\theta_b = 45°$

$\theta_c = 90°$

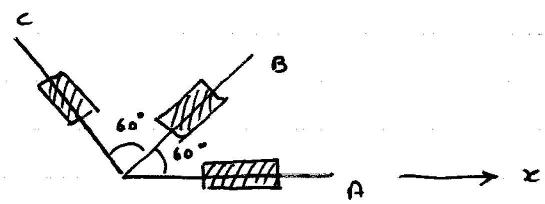
$$\epsilon_x = \epsilon_a \qquad \epsilon_y = \epsilon_c$$

and

$$\begin{aligned}
\epsilon_b &= \epsilon_x \cos^2\theta_b + \epsilon_y \sin^2\theta_b + \gamma_{xy} \sin\theta_b \cos\theta_b \\
&= \epsilon_a \cos^2 45^\circ + \epsilon_c \sin^2 45^\circ + \gamma_{xy} \sin 45^\circ \cos 45^\circ \\
&= \tfrac{1}{2}\epsilon_a + \tfrac{1}{2}\epsilon_c + \tfrac{1}{2}\gamma_{xy}
\end{aligned}$$

$$\gamma_{xy} = 2\epsilon_b - \epsilon_a - \epsilon_c$$

Case 2:



$$\theta_A = 0^\circ$$
$$\theta_B = 60^\circ$$
$$\theta_C = 120^\circ$$

$$\therefore \epsilon_x = \epsilon_a$$

$$\begin{cases} \epsilon_b = \epsilon_x \cos^2 60^\circ + \epsilon_y \sin^2 60^\circ + \gamma_{xy} \sin 60^\circ \cos 60^\circ \\ \epsilon_c = \epsilon_x \cos^2 120^\circ + \epsilon_y \sin^2 120 + \gamma_{xy} \sin 120^\circ \cos 120^\circ \end{cases}$$

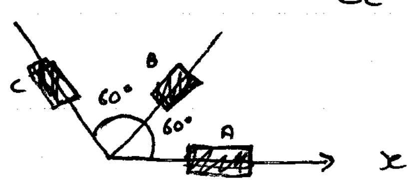
$$\epsilon_y = \tfrac{1}{3}\left(2\epsilon_b + 2\epsilon_c - \epsilon_a\right)$$

$$\gamma_{xy} = \frac{2}{\sqrt{3}}\left(\epsilon_b - \epsilon_c\right)$$

Example:

$$\left.\begin{aligned} \epsilon_a &= 60\,(10^{-6}) \\ \epsilon_b &= 135\,(10^{-6}) \\ \epsilon_c &= 264\,(10^{-6}) \end{aligned}\right\}$$

Determine principle strains and orientations.



From previous lecture -

Determine the principal strains and orientations.

Solution :
$$\begin{cases} \epsilon_x = \epsilon_a \\ \epsilon_y = \frac{1}{3}(2\epsilon_b + 2\epsilon_c - \epsilon_a) \\ \gamma_{xy} = \frac{2}{\sqrt{3}}(\epsilon_b - \epsilon_c) \end{cases}$$

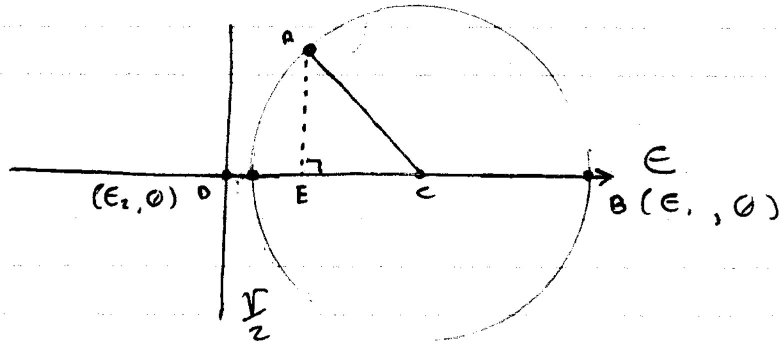
$$\Rightarrow \epsilon_x = 60(10^{-6})$$
$$\epsilon_y = 246(10^{-6})$$
$$\gamma_{xy} = -149(10^{-6})$$

Mohr's circle :
$$\epsilon_{AVG} = \frac{\epsilon_x + \epsilon_y}{2} = \frac{60 + 246}{2}(10^{-6})$$
$$= 153(10^{-6})$$

$\therefore$ centre $C(\epsilon_{AVG}, 0) = C(153(10^{-6}), 0)$

Reference Point $A(\epsilon_x, \frac{1}{2}\gamma_{xy})$

$= (60 \times 10^{-6}, -74.5 \times 10^{-6})$



radius
$$R = \sqrt{\left(\frac{\epsilon_x - \epsilon_y}{2}\right)^2} \ \ldots \text{ etc. use formula.}$$

$$R = 119.2(10^{-6})$$
$$\therefore \epsilon_1 = \epsilon_{AVG} + R = 153(10^{-6}) + 119.20(10^{-6})$$
$$= 272.2(10^{-6})$$
$$\epsilon_2 = \epsilon_{AVG} - R = 153(10^{-6}) - 119.20(10^{-6})$$
$$= 33.8(10^{-6})$$

$\triangle ACE$:

$AE = 74.5(10^{-6})$

$CE = 153(10^{-6}) - 60(10^{-6})$

$= 93(10^{-6})$

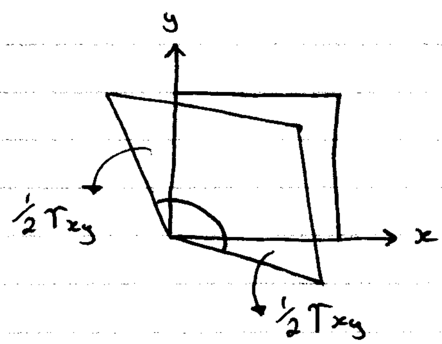
$\therefore \tan \angle ACE = \frac{AE}{CE} = \frac{74.5(10^{-6})}{93(10^{-6})}$

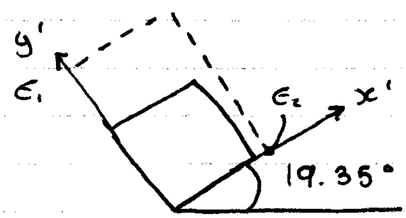
$\angle ACE = 38.7°$

$\therefore 2\theta_{P2} = \angle ACE = 38.7°$

$\theta_{P2} = 19.35°$

and $\theta_{P1} = 90° + 19.35° = 109.35°$





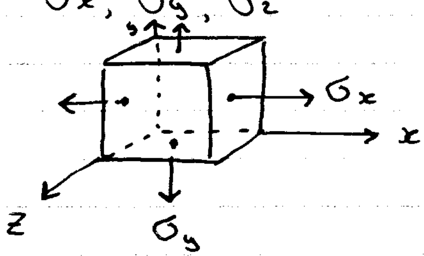
10.6 Material Property Relationships

General Hookes Law

↳ A 3D state of stress

Normal stress and normal strains

$\sigma_x, \sigma_y, \sigma_z$ $\qquad \epsilon_x, \epsilon_y, \epsilon_z$



$\epsilon_x = \frac{\sigma_x}{E}$

$\epsilon_y = -\nu\epsilon_x = -\frac{\nu\sigma_x}{E}$

$\epsilon_z = -\nu\epsilon_x = -\frac{\nu\sigma_x}{E}$

$\epsilon_x = -\nu\epsilon_y = -\frac{\nu\sigma_y}{E}$

$\epsilon_y = \frac{\sigma_y}{E}$

$\epsilon_z = -\nu\epsilon_y = -\frac{\nu\sigma_y}{E}$

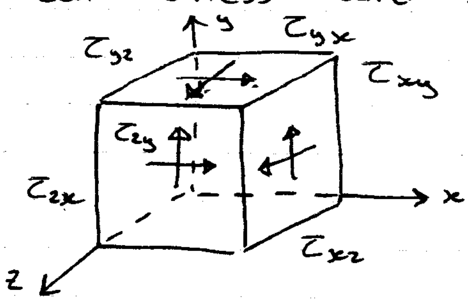
$\epsilon_y = -\nu\epsilon_z = \frac{-\nu\sigma_z}{E}$

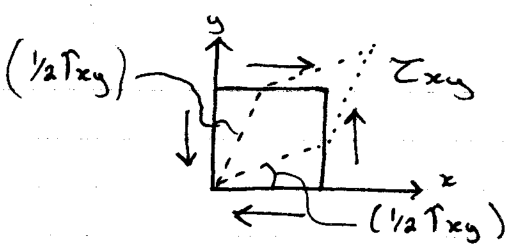
$\epsilon_x = -\nu\epsilon_z - \frac{-\nu\sigma_z}{E}$

$\epsilon_z = \frac{\sigma_z}{E}$

$$\begin{cases} \epsilon_x = \dfrac{\sigma_x}{E} - \dfrac{\nu\sigma_y}{E} - \dfrac{\nu\sigma_z}{E} = \dfrac{1}{E}\left[\sigma_x - \nu(\sigma_y + \sigma_z)\right] \\ \epsilon_y = \dfrac{-\nu\sigma_x}{E} + \dfrac{\sigma_y}{E} - \dfrac{2\sigma_z}{E} = \dfrac{1}{E}\left[\sigma_y - \nu(\sigma_z + \sigma_x)\right] \\ \epsilon_z = \dfrac{-\nu\sigma_x}{E} - \dfrac{\nu\sigma_y}{E} + \dfrac{\sigma_z}{E} = \dfrac{1}{E}\left[\sigma_z - \nu(\sigma_x + \sigma_y)\right] \end{cases}$$

Shear stress and shear strain :





$$\begin{cases} \gamma_{xy} = \dfrac{\tau_{xy}}{G} \\ \gamma_{yz} = \dfrac{\tau_{yz}}{G} \\ \gamma_{zx} = \dfrac{\tau_{zx}}{G} \end{cases}$$

Principal Stress and principal strain :

$xyz \longleftrightarrow$ principal stress directions

$\longleftrightarrow \tau_{xy} = \tau_{yz} = \tau_{zx} = 0$

Hooke's Law

$\longleftrightarrow \gamma_{xy} = \gamma_{yz} = \gamma_{zx} = 0$

$\longleftrightarrow$ principal stress directions

Furthermore,

$\sigma_1, \sigma_2, \sigma_3$; $\epsilon_1, \epsilon_2, \epsilon_3$

$$\Leftrightarrow \epsilon_1 = \frac{1}{E}\left[\sigma_1 - \nu(\sigma_2 + \sigma_3)\right]$$

$$\epsilon_2 = \frac{1}{E}\left[\sigma_2 - \nu(\sigma_3 + \sigma_1)\right]$$

$$\epsilon_3 = \frac{1}{E}\left[\sigma_3 - \nu(\sigma_2 + \sigma_3)\right]$$

Plane stress and plane strain

Plane stress: $\sigma_z = \tau_{xz} = \tau_{yz} = 0$

Hooke's Law:

$$\epsilon_z = \frac{1}{E}\left[\sigma_z - \nu(\sigma_x + \sigma_y)\right] = \frac{\nu}{E}(\sigma_x + \sigma_y)$$

$$\neq 0$$

$$\gamma_{xz} = 0 \quad , \quad \gamma_{yz} = 0$$

Not a plane strain.

Plane strain $\epsilon_z = 0 \quad \gamma_{xz} = \gamma_{yz} = 0$

Hooke's Law: $\tau_{xz} = \tau_{yz} = 0$

Since $\epsilon_z = \frac{1}{E}\left[\sigma_z - \nu(\sigma_x + \sigma_y)\right] = 0$

$$\Rightarrow \sigma_z = \nu(\sigma_x + \sigma_y) \neq 0$$

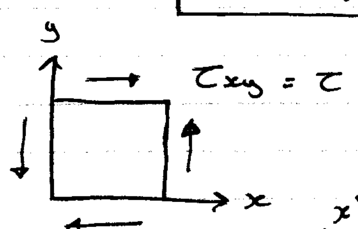
(Poisson's ratio)

E, ν, G

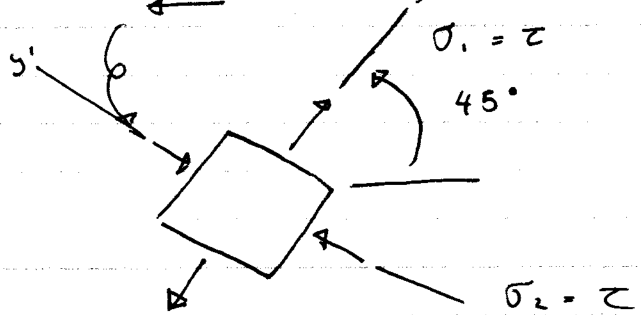
(shear modulus)

(modulus of elasticity)

$$\Rightarrow \boxed{G = \frac{E}{2(1+\nu)}}$$



$$\gamma_{xy} = \frac{\tau}{G}$$



$$\epsilon_{x'} = \epsilon_x \cos^2 45° + \epsilon_y \sin^2 45° + \gamma_{xy} \sin 45° \cos 45°$$

$$= \tfrac{1}{2}\gamma_{xy}$$

$$\epsilon_{y'} = -\tfrac{1}{2}\gamma_{xy}$$

In $x'y'$: Hooke's Law

$$\epsilon_{x'} = \frac{1}{E}(\sigma_{x'} - \nu\sigma_{y'})$$

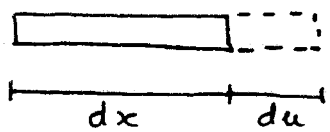
Since $\sigma_{x'} = \tau$, $\sigma_{y'} = -\tau$

$$\Rightarrow \frac{1}{2}\gamma_{xy} = \frac{1}{E}(\tau + \nu\tau) = \frac{1+\nu}{E}\tau$$

$$\Rightarrow \frac{1}{2}\frac{\cancel{\tau}}{G} = \frac{1+\nu}{E}\cdot\frac{1}{\cancel{E}}$$

$$\boxed{\Rightarrow G = \frac{E}{2(1+\nu)}}$$

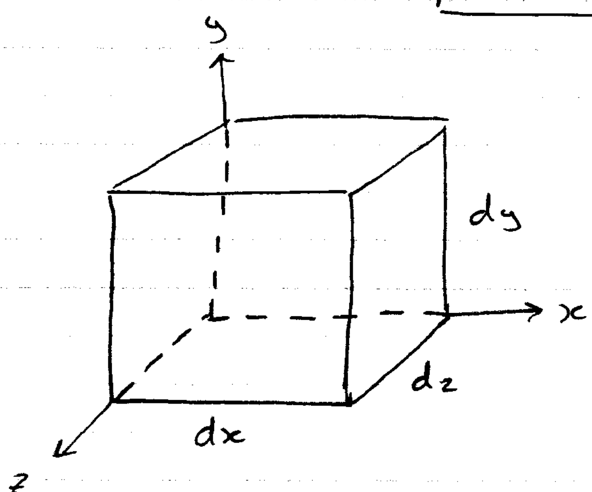
Dilation and Bulk Modules



ϵ_x : $\epsilon_x = \dfrac{du}{dx}$

$du = \epsilon_x dx$

$$\boxed{dx' = du + dx = (1+\epsilon_x)dx}$$



$\epsilon_x, \epsilon_y, \epsilon_z$

$dx \rightarrow dx' = (1+\epsilon_x)dx$
$dy \rightarrow dy' = (1+\epsilon_y)dy$
$dz \rightarrow dz' = (1+\epsilon_z)dz$

Volume old $dv = dx\,dy\,dz$
new $dv' = dx'dy'dz'$

Volume change

$$\begin{aligned}\delta v &= dv' - dv \\ &= (1+\epsilon_x)(1+\epsilon_y)(1+\epsilon_z)\,dx\,dy\,dz - dv \\ &= (1 + \epsilon_x + \epsilon_y + \epsilon_z + \epsilon_x\epsilon_y + \epsilon_y\epsilon_z + \epsilon_z\epsilon_x \\ &\quad + \epsilon_x\epsilon_y\epsilon_z)\,dv - dv \\ &= (\epsilon_x + \epsilon_y + \epsilon_z)\,dv\end{aligned}$$

Volumetric Strain $\longleftrightarrow$ dilation

$$e = \frac{\delta v}{dv} = \epsilon_x + \epsilon_y + \epsilon_z$$

Hooke's Law

$$\epsilon_x = \frac{1}{E}(\sigma_x - \nu(\sigma_y + \sigma_z))$$

$$\epsilon_y = \frac{1}{E}(\sigma_y - \nu(\sigma_z + \sigma_x))$$

$$\epsilon_z = \frac{1}{E}(\sigma_z - \nu(\sigma_x + \sigma_y))$$

$$\epsilon_x + \epsilon_y + \epsilon_z = \frac{1-2\nu}{E}(\sigma_x + \sigma_y + \sigma_z)$$

Bulk modulus:

IF $\sigma_x = \sigma_y = \sigma_z = -p$

$$e = -\frac{3(1-2\nu)}{E} p$$

$$\frac{p}{e} = -\frac{E}{3(1-2\nu)}$$

Define

$$k = \frac{E}{3(1-2\nu)}$$

bulk modulus

$$\boxed{\Rightarrow p = -ke}$$

No volume change

$$1 - 2\nu = 0$$

$$\nu = 1/2$$