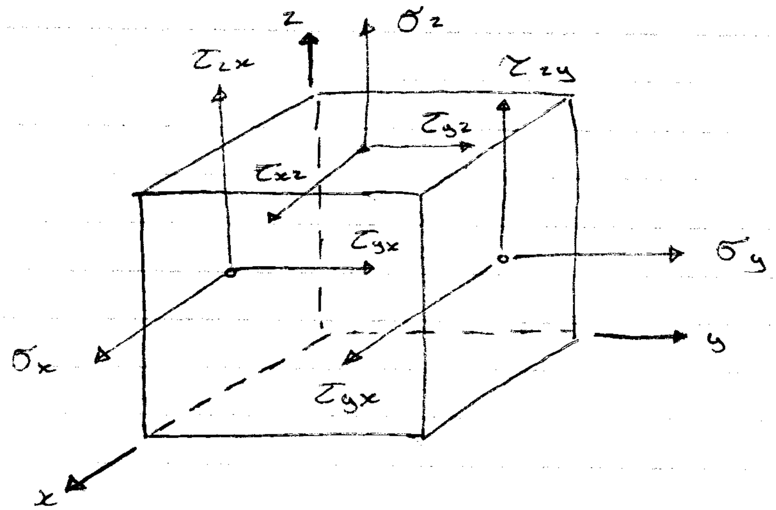
Chapter 9 - Stress Transformation

9.1 - Plane stress Transformation

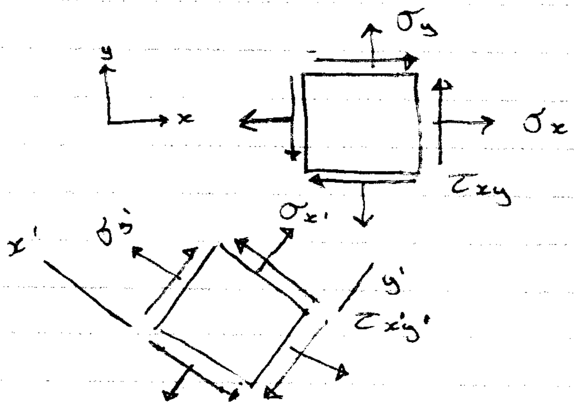
State of Stress



$\tau_{xy} = \tau_{yx}$
$\tau_{xz} = \tau_{zx}$
$\tau_{yz} = \tau_{zy}$

$$\begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{bmatrix}$$ 3×3 Symmetric Matrix.

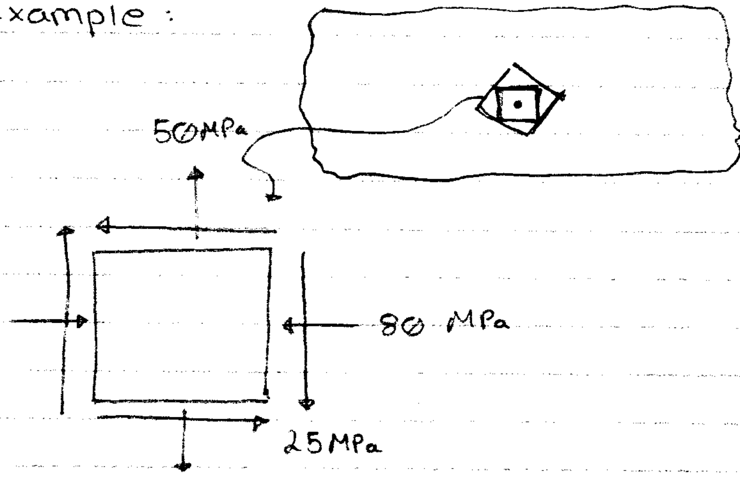
$\sigma_z = \tau_{zx} = \tau_{zy} = 0$ Plane stress



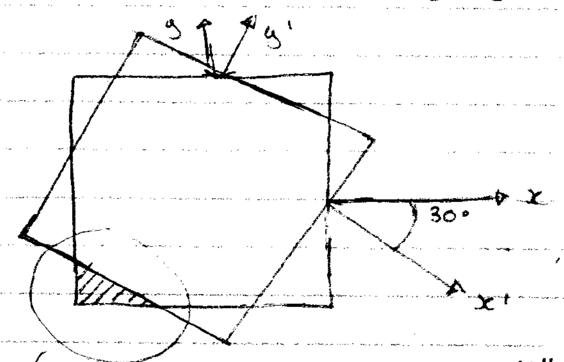
σ_x, σ_y, τ_{xy} : positive

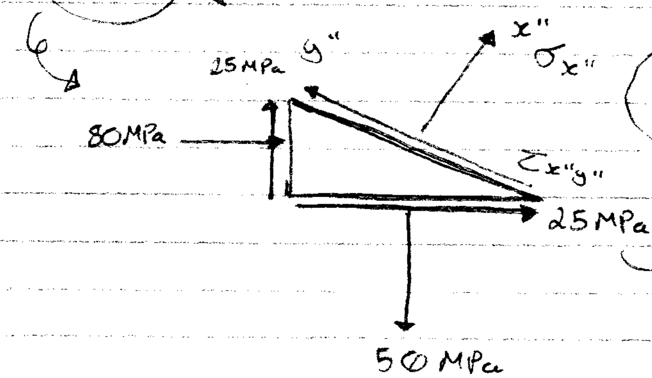
$\sigma_{x'}$, $\sigma_{y'}$, $\tau_{x'y'}$: positive

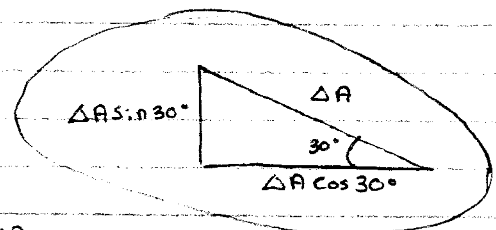
Example :



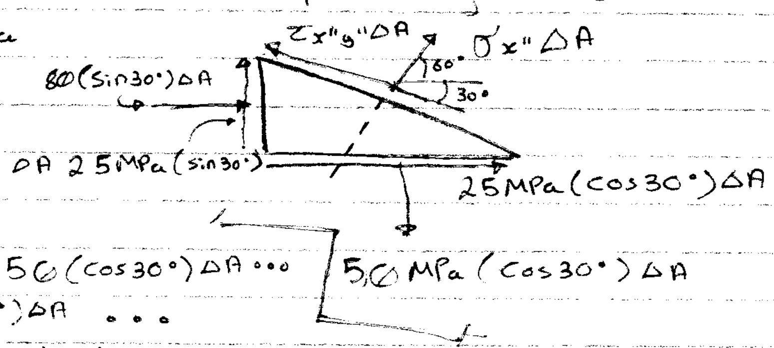
Find the state of stress at the point on an element oriented 30° c.w. from this position,







→ not a force, needs to be multiplied by area



$\Sigma F_{x''} = 0$

$\sigma_{x''}\Delta A + 80(\sin 30°)\Delta A - 50(\cos 30°)\Delta A \ldots$

$\ldots + 25(\sin 30°)(\cos 30°)\Delta A \ldots$

$\ldots + 25(\cos 30°)(\cos 60°)\Delta A = 0$

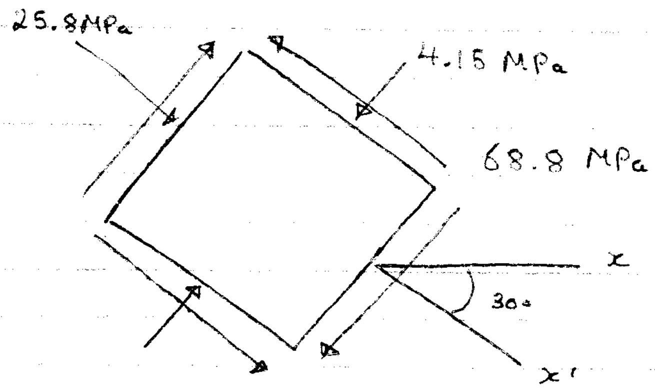
→ ΔA cancels out

∴ Area does not matter,

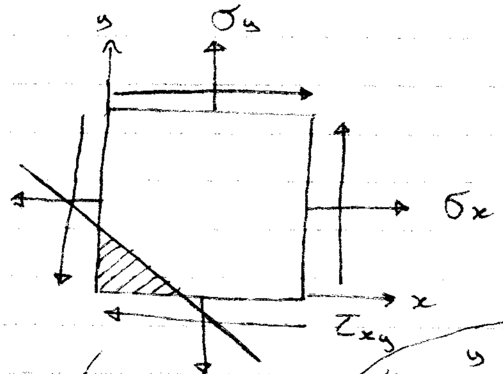
$\sigma_{x''} = -4.15$ MPa

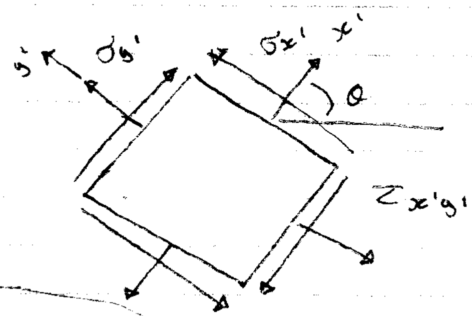
$\Sigma F_{y''} = 0$

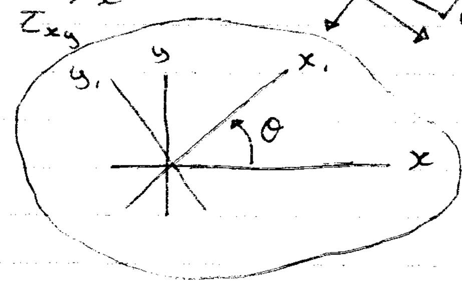
$\tau_{x''y''} = 68.8$ MPa

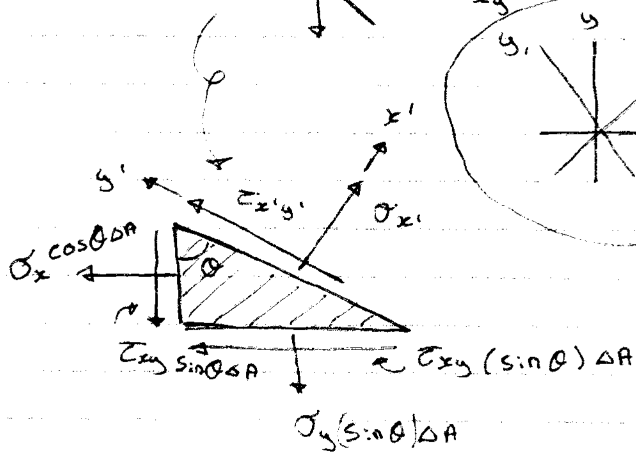


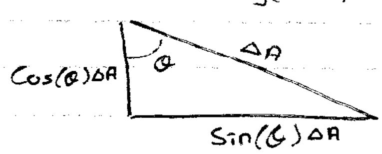
9.2 General Equations of Plane Stress Transformation.











$$\Sigma F_{x'} = 0$$

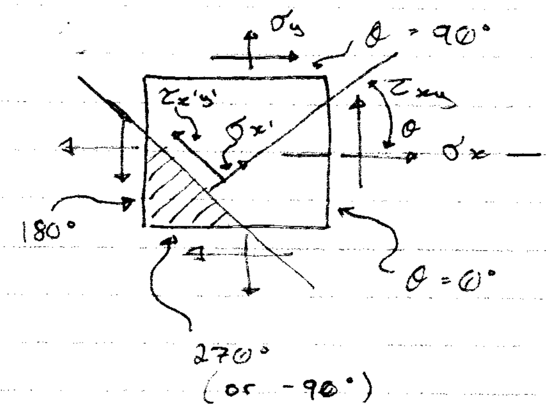
$$\sigma_{x'} \Delta A - \sigma_x \Delta A \cos\theta \cdot \cos\theta - \tau_{xy} \Delta A \sin\theta \cdot \cos\theta - \sigma_y \Delta A \sin\theta \cdot \sin\theta - \tau_{xy} \Delta A \cos\theta \cdot \sin\theta = 0$$

$$\sigma_{x'} = \sigma_x \cos^2\theta + \sigma_y \sin^2\theta + 2\tau_{xy} \sin\theta \cos\theta$$

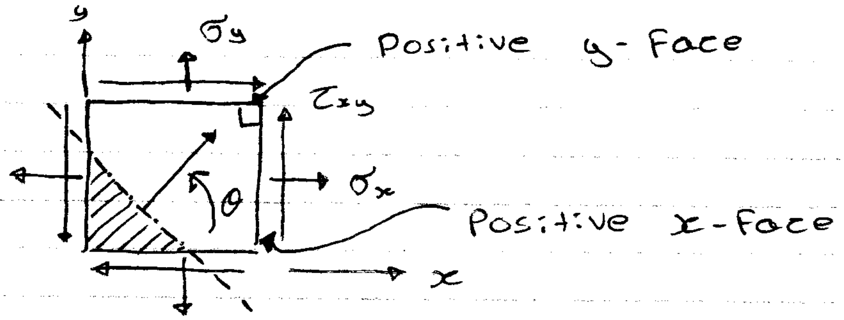
$$\Sigma F_{y'} = 0$$

$$\tau_{x'y'} \Delta A + \sigma_x \Delta A \cos\theta \sin\theta + \tau_{xy} \Delta A \sin\theta \sin\theta - \sigma_y \Delta A \sin\theta \cos\theta - \tau_{xy} \Delta A \cos\theta \cos\theta$$

$$\tau_{x'y'} = -\sigma_x \cos\theta \sin\theta + \sigma_y \sin\theta \cos\theta + \tau_{xy} (\cos^2\theta - \sin^2\theta)$$

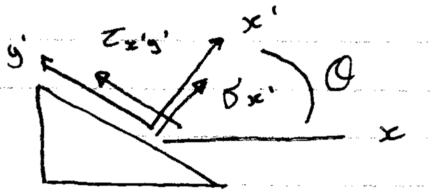


σ_y
$\theta = 90°$
$\tau_{x'y'}$
τ_{xy}
$\sigma_{x'}$
θ
σ_x
180°
$\theta = 0°$
270°
(or -90°)



Positive x-Face, $\theta = 0°$

Positive y-Face, $\theta = 90°$



$$\sigma_{x'} = \sigma_x \cos^2\theta + \sigma_y \sin^2\theta + 2\tau_{xy}\sin\theta\cos\theta$$

$$\tau_{x'y'} = -(\sigma_x - \sigma_y)\sin\theta\cos\theta + \tau_{xy}(\cos^2\theta - \sin^2\theta)$$

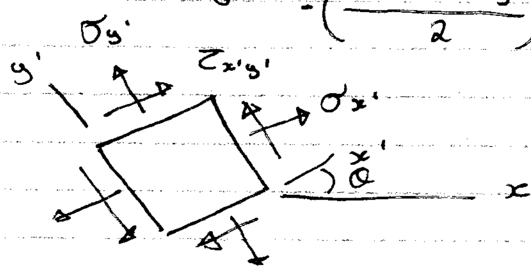
$$\theta = 0°, \quad \cos\theta = 1, \quad \sin\theta = 0$$

$$\begin{cases} \sigma_{x'} = \sigma_x \\ \tau_{x'y'} = \tau_{xy} \end{cases}$$

$$\Rightarrow \cos^2\theta = \frac{1+\cos 2\theta}{2} \qquad \sin^2\theta = \frac{1-\sin 2\theta}{2}$$

$$\sin\theta\cos\theta = \frac{1}{2}\sin 2\theta$$

$$\Rightarrow \begin{cases} \sigma_{x'} = \dfrac{\sigma_x + \sigma_y}{2} + \dfrac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta \\ \tau_{x'y'} = -\left(\dfrac{\sigma_x - \sigma_y}{2}\right)\sin 2\theta + \tau_{xy}\cos 2\theta \end{cases}$$



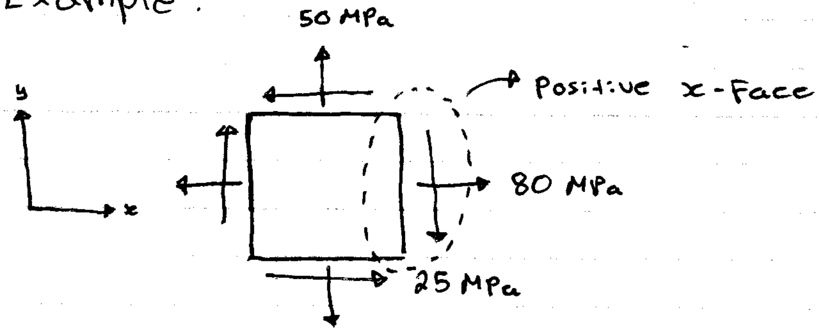
Positive y' Face : $\theta + 90°$

$$\sigma_{y'} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos(2\theta + 180°) + \tau_{xy}\sin 2\theta$$

$$\boxed{\sigma_{x'} + \sigma_{y'} = \sigma_x + \sigma_y}$$

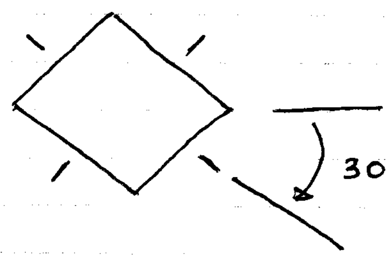
$$\boxed{\sigma_{x'}\sigma_{y'} - \tau_{x'y'}^2 = \sigma_x\sigma_y - \tau_{xy}^2}$$

Example:



Determine the state of stress at the point on another element oriented 30° cw from the position shown.

Solution:



$\sigma_x = -80$ MPa

$\sigma_y = +50$ MPa   (from positive x-Face)

$\tau_{xy} = -25$ MPa

$\theta = -30°$

$$\Rightarrow \sigma_{x'} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$

$$= \frac{-80+50}{2} + \frac{-80-50}{2}\cos(-60°) - 25\sin(-60°)$$

$$= 25.8 \text{ MPa}$$

$$\Rightarrow \tau_{x'y'} = \frac{\sigma_x - \sigma_y}{2}\sin 2\theta + \tau_{xy}\cos 2\theta$$

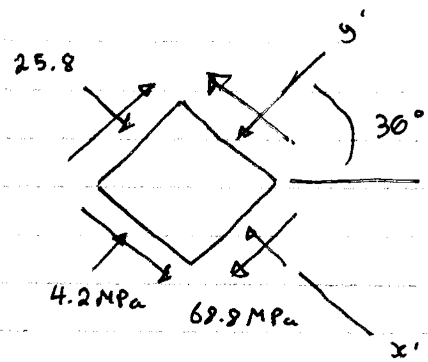
$$= -\left(\frac{-80-50}{2}\right)\sin(-60°) - 25\cos(-60°)$$

$$= -68.8 \text{ MPa}$$

$$\Rightarrow \sigma_{y'} = \sigma_x + \sigma_y - \sigma_{x'}$$

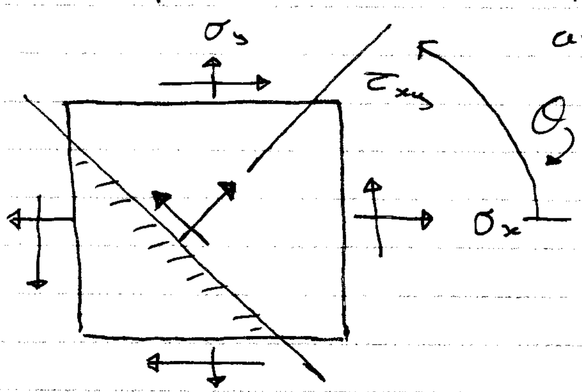
$$= -80 + 50 - (-25.8)$$

$$= 4.2 \text{ MPa}$$



9.3 Principal stress and maximum in-plane shear stress

Principal stresses: the max/min normal stress at a point



The max/min normal stress occurs when

$$\frac{d\sigma_{x'}}{d\theta} = 0$$

$$\Rightarrow \frac{\sigma_x - \sigma_y}{2}\left(-\cancel{2}\sin 2\theta\right) + \tau_{xy}\left(\cancel{2}\cos 2\theta\right) = 0$$

$$\Rightarrow -\frac{\sigma_x - \sigma_y}{2}\sin 2\theta + \tau_{xy}\cos 2\theta$$

this equation says no shear stress

$$\boxed{\Rightarrow \tan 2\theta_p = \frac{\tau_{xy}}{(\sigma_x - \sigma_y)/2}}$$

subscript
↳ principal stress

$$\tan 2\theta_p = \frac{\tau_{xy}}{(\theta_x - \theta_y)/2}$$

$$\theta_{p1} = \sigma_{p1} = \sigma_{max}$$

$$\theta_{p2} = \sigma_{p2} = \sigma_{min}$$

$$|\theta_{p1} - \theta_{p2}| = 90^\circ$$

$$\sin 2\theta_p = \pm\frac{\tau_{xy}}{R}$$

$$\cos 2\theta_p = \pm\frac{(\sigma_x - \sigma_y)/2}{R}$$

Here, $R = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$

↳

$$\therefore \quad \sigma = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2} \cdot \left(\pm \frac{(\sigma_x - \sigma_y)/2}{R} \right) + \tau_{xy} \left(\pm \frac{\tau_{xy}}{R} \right)$$

$$\Rightarrow \quad \frac{\sigma_x + \sigma_y}{2} + \frac{1}{R}\left(\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2 \right)$$

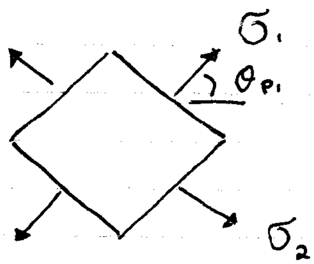
$$\Rightarrow \quad \sigma_{AVG} \pm R$$

$$\sigma_{AVG} = \frac{\sigma_x + \sigma_y}{2}$$

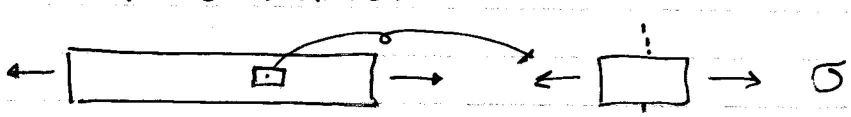
$$\therefore \quad \sigma_1 = \sigma_{AVG} + R$$

$$\sigma_2 = \sigma_{AVG} - R$$

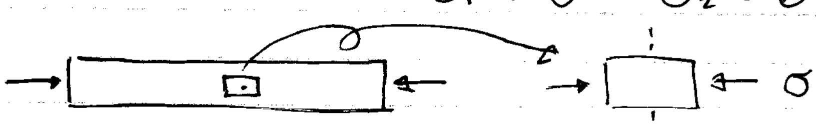
$$\boxed{\sigma_{AVG} = \frac{\sigma_x + \sigma_y}{2}} \qquad \boxed{R = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 - \tau_{xy}^2}}$$



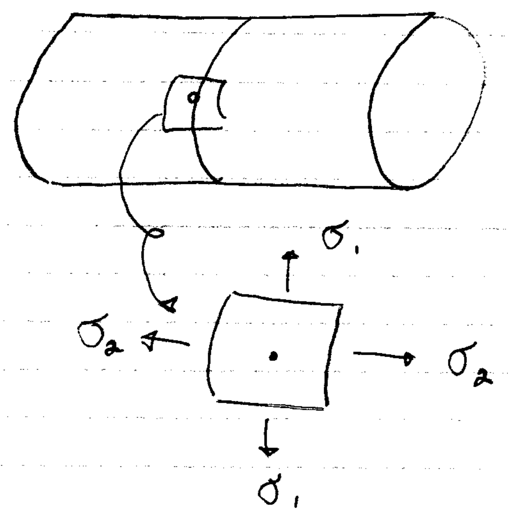
Axial force member:



$$\sigma_1 = \sigma \qquad \sigma_2 = 0$$



$$\sigma_1 = 0 \qquad \sigma_2 = -\sigma$$



Max in-plane shear stress

$$\tau_{x'y'} = -\frac{\sigma_x - \sigma_y}{2}\sin 2\theta + \tau_{xy}\cos 2\theta$$

$$\frac{d\tau_{x'y'}}{d\theta} = -\frac{\sigma_x - \sigma_y}{2}(2\cos\theta) + \tau_{xy}(-\sin 2\theta) = 0$$

$$\Rightarrow \tan 2\theta_s = -\frac{(\sigma_x - \sigma_y)/2}{\tau_{xy}}$$

$$\begin{cases} \theta = \theta_{s1} \quad \text{and} \quad \theta = \theta_{s2} \\ \tau_{\substack{max \\ \text{in-plane}}} = R \end{cases}$$

Max in-plane shear

$$\sigma_{x1} = \frac{\sigma_x + \sigma_y}{2} = \sigma_{y1}$$

$$\begin{cases} \tan 2\theta_p = \dfrac{\tau_{xy}}{(\theta_x - \theta_y)/2} \\ \tan 2\theta_s = \dfrac{(\theta_x - \theta_y)/2}{\tau_{xy}} \end{cases}$$

$$\Rightarrow \tan 2\theta_p \cdot \tan 2\theta_s = -1$$

$$\Rightarrow \cos 2\theta_p \cos 2\theta_s + \sin 2\theta_p \sin 2\theta_s = 0$$

$$\Rightarrow \cos 2(\theta_p - \theta_s) = 0$$

$$\Rightarrow 2(\theta_p - \theta_s) = \pm 90°$$

$$\Rightarrow \theta_p - \theta_s = \pm 45°$$

principle angle

max in-plane shear angle