3 - Composites

A composite area, volume, or line is one composed of a combination of simple parts. You can easily determine its centroid if you know the centroids of its parts.

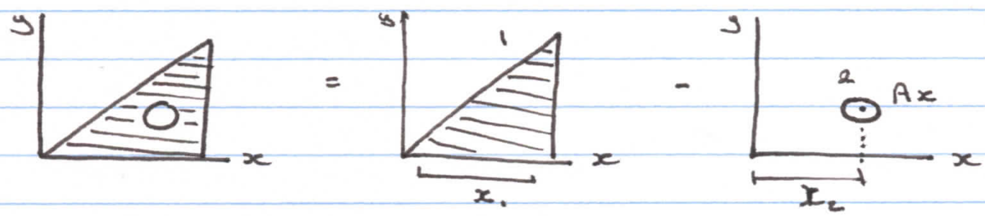
3.1 - Composite Areas

The x-coordinate of the centroid of a composite area is:

$$\bar{x} = \frac{\bar{x}_1 A_1 + \bar{x}_2 A_2 + \ldots \bar{x}_n A_n \ldots}{A_1 + A_2 + \ldots A_n} \Rightarrow \frac{\sum_i \bar{x}_i A_i}{\sum A_i}$$

and y: $$\bar{y} = \frac{\bar{y}_1 A_1 + \bar{y} A_2 + \ldots \bar{y} A_n}{A_1 + A_2 + \ldots A_n} \Rightarrow \frac{\sum_i \bar{y}_i A_i}{\sum A_i}$$

The terms corresponding to a cutout (hole) in these equations will be negative.



3.2 - Composite Volumes

$$\bar{x} = \frac{\sum_i \bar{x}_i V_i}{\sum_i V_i} \; ; \quad \bar{y} = \frac{\sum_i \bar{y}_i V_i}{\sum_i V_i} \; ; \quad \bar{z} = \frac{\sum_i \bar{z}_i V_i}{\sum_i V_i}$$

3.3 - Composite Lines

$$\bar{x} = \frac{\sum_i \bar{x}_i L_i}{\sum_i L_i} \; ; \quad \bar{y} = \frac{\sum_i \bar{y}_i L_i}{\sum_i L_i} \; ; \quad \bar{z} = \frac{\sum_i \bar{z}_i L_i}{\sum_i V_i}$$

3.4 Centers of Mass of Composite Objects

$$\bar{x} = \frac{\sum_i \bar{x}_i m_i}{\sum_i m_i} \; ; \quad \bar{y} \; \frac{\sum_i \bar{y}_i A_i}{\sum_i m_i} \; ; \quad \bar{z} = \frac{\sum_i z_i m_i}{\sum_i m_i}$$

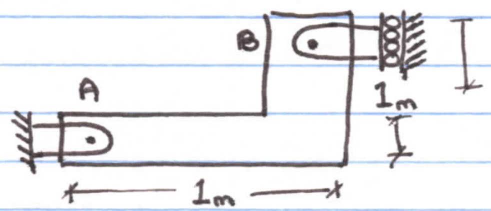
and

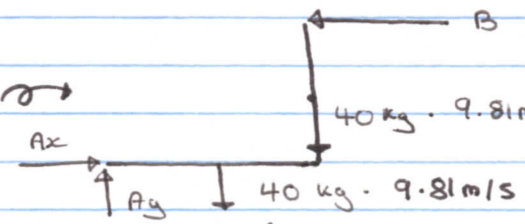
$$\bar{x} = \frac{\sum_i \bar{x}_i W_i}{\sum_i W_i} \; ; \quad \bar{y} = \frac{\sum_i y_i w_i}{\sum_i w_i} \; ; \quad \bar{z} = \frac{\sum_i z_i w_i}{\sum_i W_i}$$

When you know the masses or weights and the centers of mass or weights of the parts of a composite object, you can determine its center of mass

EXAMPLE:

The mass of a homogeneous slender bar is 80 kg, what are the reactions at A and B?





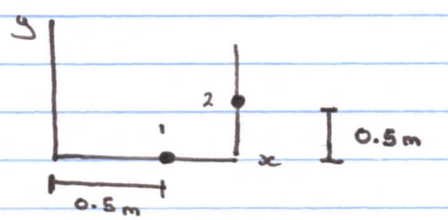
(essentially weight distribution by method of sections)

$$\xrightarrow{+} \Sigma F_x = A_x - B$$
$$+\uparrow \Sigma F_y = A_y - 2(40 \cdot 9.81) = 0$$
$$A_y = 784.8 \text{ N}$$

$$\circlearrowleft + \Sigma M_A = (1)B - (1)(40)(9.81) - (0.5)(40)(9.81)$$
$$B = 588.6 \text{ N} \; ; \; A_x = 588.6 \text{ N}$$

Second Method: We can treat the centreline of the bar as a composite line composed of two straight segments, the coordinate of the centroid of the composite line is:

Note: Figure 5.8A and B show Centroids of common shapes of areas.



$$\bar{x} = \frac{\bar{x}_1 L_1 + \bar{x}_2 L_2}{L_1 + L_2} = \frac{(0.5)(1) + (1)(1)}{(2)}$$

$$\bar{y} = \frac{\bar{y}_1 L_1 + \bar{y}_2 L_2}{L_1 + L_2} = \frac{(0)(1) + (0.5)(1)}{(2)}$$

$$\bar{x} = 0.75\text{m}$$
$$\bar{y} = 0.25 \text{ m}$$

In the free-body diagram we place the weight of the bar at its center of mass. From the equilibrium equations

$$\Sigma F_x = A_x - B = 0$$

$$\Sigma F_y = A_y - (80)(9.81) = 0$$

$$\Sigma M_A = (1)B - (0.75)(80)(9.81) = 0$$

Solving, we get $A_x = 588.6\ N$; $A_y = 784.8\ N$

$B = 588.6\ N$

Theorems of Pappus - Guldinus

Theorem I: The area of a surface of revolution is equal to the length of the generating curve times the distance traveled by the centroid of the curve while the surface is being generated.

Theorem II: The volume of a body of revolution is equal to the generating area times the distance traveled by the centroid of the area while the body is being generated.

Moments of Inertia

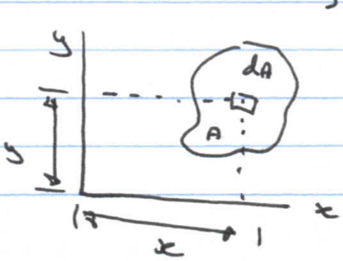
1. Definition

The moment of Inertia of an area are integrals similar in form to those used to determine the centroid of an area. We define four moments of inertia of an area A in the x-y plane:

1.1 - Moment of Inertia about the x-axis

$$I_x = \int_A y\, dA$$

where y is the y-coordinate of the differential element of area dA. This moment of inertia is sometimes expressed in terms of the radius of gyration about x-axis, K_x defined by $I_x = K_x A$



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Moments of Inertia

1.2 - Moment of Inertia about the y-axis

$$I_y = \int_A x^2 \, dA$$

$$I_x = \int_A y^2 \, dA$$

1.3 - Product of Inertia

$$I_{xy} = \int_A xy \, dA$$

1.4 - Polar moment of Inertia

$$J_O = \int_A r^2 \, dA$$

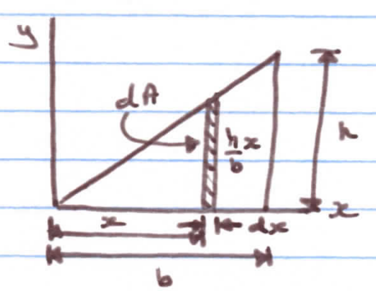
Where r is radial distance from the origin O to dA the radius of gyration about O, K_o is defined as: $J_o = K_o^2 A$

The polar moment of inertia is equal to the sum of the moment of inertia about the x and y axes.

$$J_o = \int_A r^2 dA = \int_A (x^2 + y^2) dA = I_x + I_y$$

and $K_o^2 = K_x^2 + K_y^2$

- Determine the moments of inertia and radius of gyration of the triangular area shown:



Let dA be the vertical strip

↳ its height is $\frac{h}{b}x$

$$dA = \frac{h}{b}x \cdot dA$$

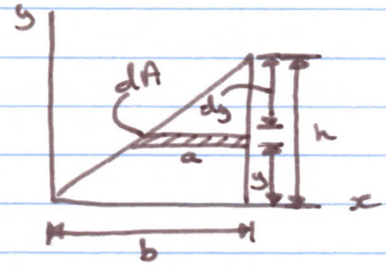
$$I_y = \int_A x^2 dA = \int_a^b x^2 \frac{h}{b} x dx$$

$$\Rightarrow \frac{h}{b}\left[\frac{x^4}{4}\right]_0^b = \frac{1}{4} hb^3$$

The radius of gyration K_y is $I_y = K_y^2 A$

$$K_y = \sqrt{\frac{I_y}{A}} = \sqrt{\frac{\frac{1}{4}hb^3}{\frac{1}{2}bh}} = \left(\frac{1}{\sqrt{2}} b\right)$$

- Determine the moment of inertia about the x-axis. Let dA be the horizontal strip.



Let dA be the horizontal strip

$$dA = (1 - y/h)\, b\, dy$$

$$I_x = \int_A y^2\, dA = \int_a^b y^2 (1 - y/h)\, b\, dy$$

$$\Rightarrow b\left[\frac{y^3}{3} - \frac{y^4}{4h}\right]_0^h$$

$$I_x = b\left(\frac{h^3}{3} - \frac{h^4}{4}\right) = \frac{1}{12} bh^3$$

and $$k_x = \sqrt{\frac{I_x}{A}} = \sqrt{\frac{(1/12)bh^3}{(1/2)bh}} = \frac{1}{\sqrt{6}} h$$

Product of Inertia

$$I_{xy} = \int_A xy\, dA = \int_A xy\,(\tfrac{1}{2}\, dx\, dy)$$

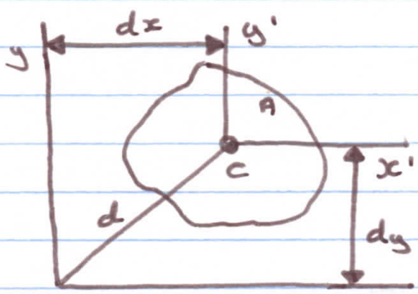
$$= \tfrac{1}{2}\int_a^b \int_0^h xy\, dx\, dy = \tfrac{1}{2}\left[\frac{x^2}{2}\right]_0^b \left[\frac{y^2}{2}\right]_0^h = \frac{b^2h^2}{8}$$

Polar moment of Inertia

$$J_o = I_x + I_y = \tfrac{1}{12} bh^3 + \tfrac{1}{4} hb^3$$

and $$k_o = \sqrt{k_x^2 + k_y^2} = \sqrt{\frac{1}{6}h^2 + \frac{1}{2}b^2}$$

2 - Parallel Axis Theorems



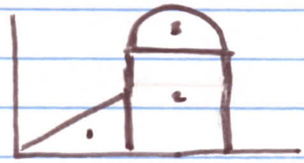
Centroid of an area A, and let x·y be a parallel coordinate system. The moments of inertia of A in terms of the two systems are related by the parallel axis theorem

$$I_x = I_{x'} + d_y^2 A$$

$$I_y = I_{y'} + d_x^2 A$$

$$I_{xy} = I_{x'y'} + d_x d_y A$$

$$J_o = J_o' + (d_x^2 + d_y^2) A = J_o' + d^2 A$$



Where d_x and d_y are the coordinates of the centroid of A in the xy - coordinate system, and where d is the distance from the origin of the x'y' - coordinate system to the origin of the xy - coordinate system.

3 - Moments of Inertia of Composite Areas

Determining a moment of inertia of a composite area in terms of a given coordinate system involves three steps:

1 - Choose the parts - Try to divide the composite area into parts whose moments of inertia you know or can easily determine.

2 - Determine the moment of inertia of the parts. Determine the moment of inertia of each part in terms of a parallel coordinate system with its origin at the centroid of the part, then use the parallel axis theorem to determine the moment of inertia in terms of the given coordinate system.

3 - Sum the moments

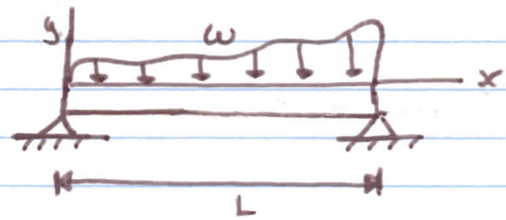
Sum the moments of inertia of the parts (or subtract in the case of cutout) to obtain the moment of inertia of the composite area.

Beams

- Loads distributed along a line

Let's assume that the function ω describing a particular distributed load is known. The graph of ω is called the loading curve. The force acting on an element dx of the line is ωdx. The total force F is:

$$F = \int_L \omega dx$$



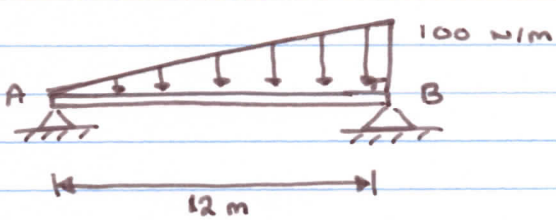
The moment about the origin due to the force exerted on the element dx is $x\omega dx$ so the total moment about the origin due to the distributed load is:

$$M = \int_L x\omega dx \quad \text{or} \quad M = \bar{x}F = \int_L x\omega dx$$

where F is the equivalent load placed at the position

$$\bar{x} = \frac{\int_L x\omega dx}{\int_L \omega dx}$$

Q - The beam is subjected to a triangular distributed load whose value at B is 100 N/m. Determine the reactions at A and B.



First Method:

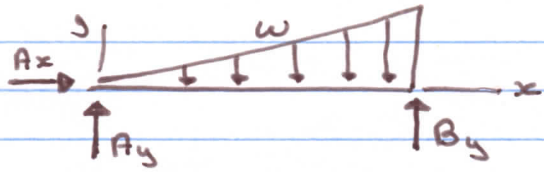
$$\omega = \frac{100x}{12} \text{ N/m}$$

The total load is:

$$F = \int_L \omega dx = \int_0^{12} \frac{100}{12} x dx \Rightarrow 600 \text{ N}$$

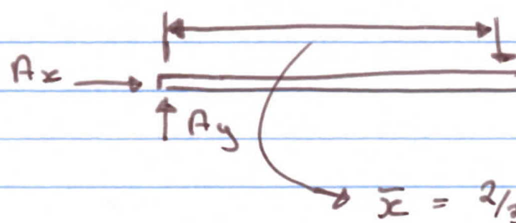
The clockwise moment about A due to the load is:

$$M_A = \int_L x w \, dx = \int_0^{12} \frac{100}{12} x \, dx = 4800 \text{ N/m}$$



$$\Sigma F_x = A_x = 0$$
$$\Sigma F_y = A_y + B_y - 600\text{N} = 0$$
$$\Sigma M_A = 12(B_y) - 4800 = 0$$
$$A_x = 0$$
$$A_y = 200\text{N}$$
$$B_y = 400\text{N}$$

Second Method:



$$F = \tfrac{1}{2}(12\text{m})(100)\text{ N/m} = 600\text{N}$$

$$\bar{x} = \tfrac{2}{3}(12\text{m}) = 8 \text{ m}$$

$$F = \tfrac{1}{2} \cdot 12\text{m} \times 100 \text{ N/m} = 600\text{N}$$

$$\bar{x} = \tfrac{2}{3} \cdot 12\text{m} = 8 \text{ m}$$

$$\Sigma F_x = 0 = A_x$$
$$\Sigma F_y = A_y + B_y - 600 = 0$$
$$\Sigma M_A = 12 B_y - 8 \cdot 600 = 0$$

$$\therefore A_x = 0$$
$$A_y = 200\text{N}$$
$$B_y = 400\text{N}$$