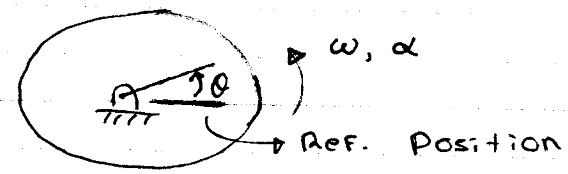
15.1C Equations Defining the Rotation of a Rigid Body about a Fixed Axis

A. Rotation of a single rigid body about a fixed axis.



The motion of the rigid body is represented by that of any radial line. Specifically, $\theta(t)$, $\omega(t)$, $\alpha(t)$

Mathematically:

	Rectilinear Motion	Rotation about Fixed axis
Position	$x(t)$	$\theta(t)$
Velocity	$v(t) = \dot{x}$	$\omega(t) = \dot{\theta}$
Acceleration	$a(t) = \dot{v} = \ddot{x}$	$\alpha(t) = \dot{\omega} = \ddot{\theta}$

∴ mathematically speaking, rotating about a fixed axis is treated the same way as the rectilinear motion of a particle.

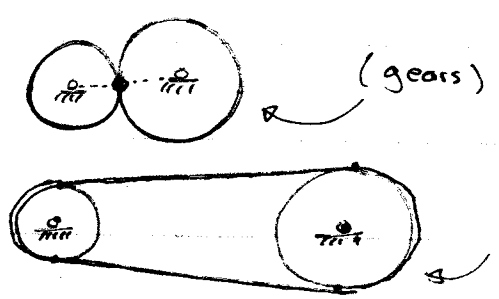
In particular,

11.1B
$$\begin{cases} a = a(t) \iff \alpha = \alpha(t) \\ a = a(v) \iff \alpha = \alpha(\omega) \\ a = a(x) \iff \alpha = \alpha(\theta) \end{cases}$$

11.2A
$$v = \text{const} \iff \omega = \text{const}$$

11.2B
$$\begin{cases} a = \text{const} \iff \alpha = \text{const} \\ v(t) = v_0 + a(t - t_0) \\ x(t) = x_0 + v_0(t - t_0) + a/2\,(t - t_0)^2 \\ v^2 - v_0^2 = 2a(x - x_0) \end{cases}$$

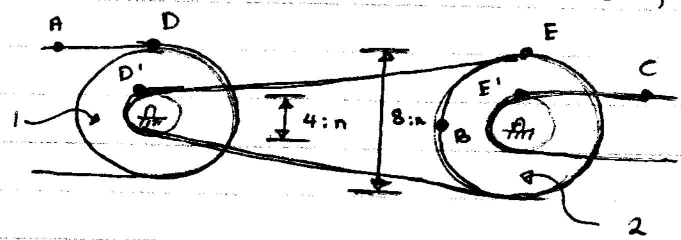
B. Rigid bodies in rotations about respective fixed axes.



how motion is transmitted from one rigid body to another, based on the "no slip" assumption at contact points then tangential components will be transmitted.

$\vec{v} = v\vec{e}_t$
$\vec{a} = \vec{a}_t + \vec{a}_n$ (only $\vec{a}_t$ will be transmitted.)

Problem 15.23 (From textbook)



Solution:

$v_A = 2\ ft/s \longrightarrow$
$a_A = 6\ ft/s^2 \longleftarrow$

$v_c,\ a_c,\ \vec{a}_B$

(1) D : Part of input belt, also part of pulley 1.
D on pulley 1

$\therefore\ v_D = r\omega = \underbrace{(4\,in)\,\omega_1} = v_A = \underbrace{24\ in/s}$

$\therefore\ \omega_1 = 6\ rad/s$

$(a_t)_D = r\alpha = (4\,in)\,\alpha_1 = 72\ in/s^2 = a_A$

$\therefore\ \alpha_1 = 18\ rad/s^2$

(2) D' on Pulley 1

$\therefore\ v_{D'} = (2\,in)(6\ rad/s) = 12\ in/s$

$(a_t)_{D'} = (2\,in)(18\ rad/s^2) = 36\ in/s^2$

(3) E : Part of the intermediate belt, also part of pulley 2.

E on belt : $v_E = v_{D'} = 12\ in/s \longrightarrow$
$a_E = (a_t)_{D'} = 36\ in/s^2 \longleftarrow$

E on pulley 2 : $v_E = (4\,in)\,\omega_2 = 12\ in/s$

$\therefore\ \omega_2 = 3\ rad/s$

$(a_t)_E = (4\,in)\,\alpha_2 = 36\ in/s^2$

$\therefore\ \alpha_2 = 9\ rad/s^2$

(4) E' on pulley 2

$$v_{E'} = (2\text{ in})(3\text{ rad/s}) = 6\text{ in/s} \longrightarrow$$

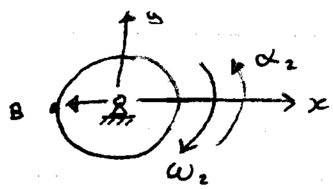
$$(a_t)_{E'} = (2\text{ in})(9\text{ rad/s}^2) = 18\text{ in/s}^2 \longleftarrow$$

E' on belt, and as a result, C on belt

$$v_c = 6\text{ in/s} \longrightarrow$$

$$a_c = 18\text{ in/s}^2 \longleftarrow$$

(5) B on pulley 2



$$\vec{r} = -4\vec{i} \text{ (in)}$$

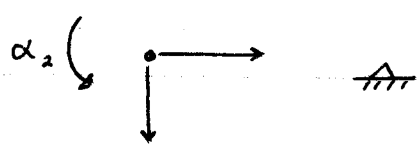
$$\vec{\omega}_2 = -3\vec{k} \text{ (rad/s)}$$

$$\vec{\alpha}_2 = 9\vec{k} \text{ (rad/s}^2\text{)}$$

$$\vec{v}_B = \vec{\omega}_2 \times \vec{r}$$

$$\vec{a}_B = \vec{\alpha}_2 \times \vec{r} + \vec{\omega}_2 \times \vec{v}_B$$

$$= 36\vec{i} - 36\vec{j} \quad (\text{in/s}^2)$$



SAMPLE PROBLEMS §15.1

15.1 $\alpha = \alpha(\theta)$

2 axis of rotation by $\vec{e}$

3 pulley + belt

4 constant α, two gears in contact

Sections dealing with general plane motion

§15.2 velocity, vector approach

3 velocity, semi-graphical approach

4 acceleration, vector approach

5 motion relative to rotating frame of ref.

§15.2 General Plane Motion : Velocity

15.2A Analyzing General Plane Motion

General Motion

= Translation with A

+ Rotation about A (as if it was fixed)

A is arbitrary, is known at the base point.

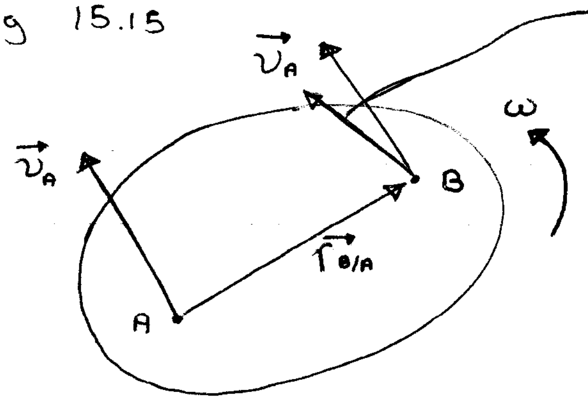
Fig. 15.12 General Motion = translation + rotation

Fig 15.13 Choice of base point has no effect on results

$\left.\begin{array}{l}\text{Translation}\\ \text{Rotation}\end{array}\right\}$ time-dependent.

15.2B Absolute and Relative Velocity in Plane Motion

Fig 15.15



$(\vec{v}_{B/A})$ $\vec{v}$ due to rotation

A, B, belong to the same rigid body.

$\vec{v}_B = \vec{v}_A + \vec{v}_{B/A}$

(Absolute $\vec{v}$) (Relative $\vec{v}$)

$\vec{v}_{B/A}$: Velocity of B rotating with respect to A as if A were fixed.

$\therefore \vec{v}_{B/A} = \vec{\omega} \times \vec{r}_{B/A}$

$\therefore \vec{v}_B = \vec{v}_A + \vec{\omega} \times \vec{r}_{B/A}$

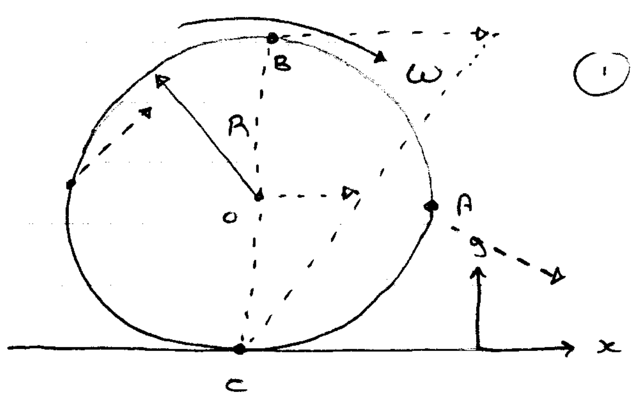
A : base point

For problem solving, it is typically a point whose velocity is known, or partially known.

Types of Problems :

$\begin{cases} \omega \text{ is known} \\ \omega \text{ is to be determined} \end{cases}$

4 sample Problems :



(1) The disk rolls without slip at C.

Find : $\vec{v}_c$, $\vec{v}_o$, $\vec{v}_A$ and $\vec{v}_B$ in terms of R and ω

Solution : $\vec{\omega} = -\omega \vec{k}$

$\therefore$ no slip at C

$\therefore$ $\vec{v}_c = \vec{0}$

O : $\vec{v}_o = \vec{v}_c + \vec{\omega} \times \vec{r}_{o/c}$

$= \vec{0} + (-\omega \vec{k}) \times (R \vec{j})$

$= R\omega \vec{i}$

A : $\vec{v}_A = \vec{v}_c + \vec{\omega} \times \vec{r}_{A/c}$

$= R\omega \vec{i} - R\omega \vec{j}$

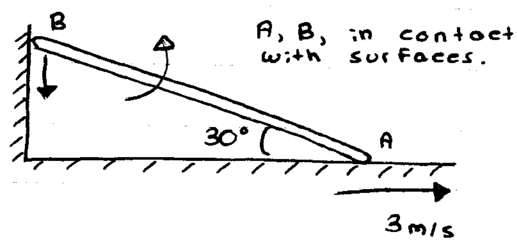
B : $\vec{v}_B = \vec{v}_c + \vec{\omega} \times \vec{r}_{B/c}$

$= 2R\omega \vec{i}$

Example 2 :

Bar AB : 1.5 m in length

Find ω_{AB} , $\vec{v}_B$



Solution :

A : base point

assume : ccw ω_{AB}

$\vec{\omega}_{AB} = \omega_{AB} \vec{k}$

next : $\vec{v}_B \downarrow$ $\qquad \vec{v}_B = -v_B \vec{j}$

$\vec{v}_B = \vec{v}_A + \vec{\omega}_{AB} \times \vec{r}_{B/A}$

$-\vec{v}_B \vec{j} = 3\vec{i} + (\vec{\omega}_{AB} \vec{k}) \times$

$\times (-1.5 \cos 30° \vec{i} + 1.5 \sin 30° \vec{j})$

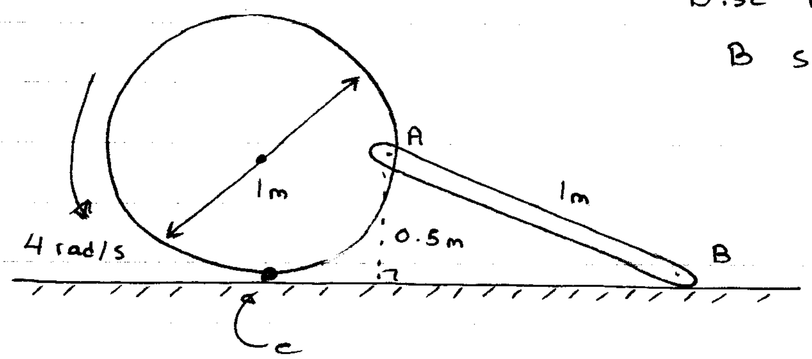
$= 3\vec{i} - 0.75\, \omega_{AB} \vec{i} - 1.299\, \omega_{AB} \vec{j}$

$$\begin{cases} i : & 0 = 3 - 0.75\, \omega_{AB} \\ j : & -\vec{v}_B = -1.299\, \omega_{AB} \end{cases}$$

Solving leads to $\omega_{AB} = 4$ rad/s

$v_B = 5.196$ m/s

Example 3:



Disc rolls without slip

B slides on surface

$\omega_{AB} = ?$

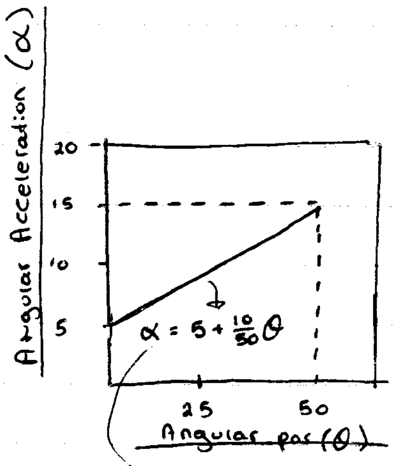
$v_B = ?$

$\omega_{AB} = 2.309$ rad/s

$v_B = 3.155$ m/s ←

$\vec{v}_A = -2\vec{i} + 2\vec{j}$

1



$\alpha = 5 + 0.2\theta$

$$\omega = \omega(\theta)$$
$$\alpha = \alpha(\theta)$$
$$a = a(x)$$

Solution: $\alpha = 5 + 0.2\theta$ rad/s²

when $\theta_0 = 0$, $\omega_0 = 10$ rad/s

Ch.11 : $a = a(x)$

$$\int_{v_0}^{v} v\,dv = \int_{x_0}^{x} a(x)\,dx$$

$$\therefore \int_{\omega_0}^{\omega} \omega\,d\omega = \int_{\theta_0}^{\theta} \alpha(\theta)\,d\theta$$

$$\therefore \left.\frac{\omega^2}{2}\right|_{\omega_0}^{\omega} = \int_{\theta_0}^{\theta} (5 + 0.2\theta)\,d\theta$$

$$\therefore \frac{1}{2}(\omega^2 - 100) = 5\theta + 0.1\theta^2$$

Set $\theta = 50(2\pi) = 100\pi$ (rad)

then $\omega = 151.6$ rad/s

2

At $t_o = 0$, $\omega_o = \theta_o = 0$

Given: $\alpha = \left(\frac{\omega}{16} - 8\right)^2$

Ch.11: $a = a(v)$

$$\int_{v_o}^{v} \frac{dv}{a(v)} = \int_{t_o}^{t} dt$$

RHS: $\int_{t_o}^{t} dt = t$

LHS becomes $\int_{\omega_o}^{\omega} \frac{d\omega}{\left(\frac{\omega}{16} - 8\right)^2}$

$$= -2 - \frac{16}{\frac{\omega}{16} - 8}$$

LHS = RHS and solving for ω in terms of t

then, $\omega = 128 - \frac{256}{2+t}$

$$\therefore \omega = \frac{d\theta}{dt}$$

$$\therefore d\theta = \omega\, dt$$

$$\therefore \int_{\theta_o}^{\theta} d\theta = \int_{t_o}^{t} \omega(t)\, dt$$

LHS = θ

RHS = $\int_{t_o}^{t} \left(128 - \frac{256}{2+t}\right) dt$

$$= 128t - 256 \ln|2+t| + 256 \cdot \ln 2$$

$$\therefore \theta = 128t - 256 \ln|2+t| + 256 \ln 2$$

a) $\theta = 50(2\pi) = 100\pi$ (rad)

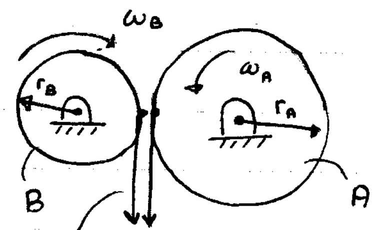
$\therefore t = 4.944$ (s)

2

b) $\omega = 128 - \dfrac{256}{2+t}$

at $t = 4.944\,s$, $\omega = 91.13$ rad/s

SAMPLE PROBLEM 15.4 :



$t = 6$ sec

$r_A\omega_A = r_B\omega_B$ (when no slip)