Chapter 15 - Kinematics of Rigid Bodies

7 - Sections in total

Introduction

§ 15.1 } Translation
Rotation about a fixed axis

§ 15.2 → 15.5 } General motion (2D)

§ 15.6 → 15.7 } General motion (not req'd)

What is a rigid body?
A rigid body is a collection of particles. It has mass, and shape and dimension. The distance between any two particles will remain constant regardless of the external forces exerted on the rigid body.

Introduction
5 types of rigid-body motion
1. Translation
2. Rotation about a fixed axis } 2D Motion
3. General plane motion
4. Motion about a fixed point } ~~3D Motion~~
5. General motion

Translation

A rigid body is said to be in translation if any straight line "drawn" on the body keeps the same orientation during the motion.

→ all particles in a translating body move along parallel paths
→ If the path are straight lines, the motion is known as rectilinear translation
→ If the paths are curved lines, the motion is known as curvilinear translation (Fig 15.1, Fig 15.2)

Rotation about a fixed axis (Fig 15.3)

A rigid body is said to be in rotation about a fixed axis if particles in the body travel/move along circles or circular arcs whose centers of curvature form the fixed axis of rotation.

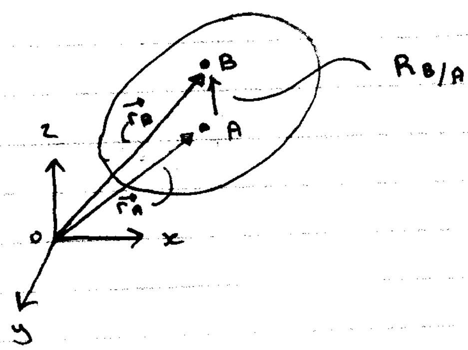
→ the fixed axis can be located within or beyond the physical confines of the rigid body
→ $\dot{\theta}$ and $\ddot{\theta}$ are the same for all radial lines.

General Plane Motion (Fig 15.5)

Any plane motion that is neither a translation nor a rotation (about a fixed axis) is called a general plane motion, or a complex plane motion.

§15.1 Translation and Fixed-Axis Rotation

15.1A Translation



Kinematics of A is known.

$\vec{r}_A$, $\vec{v}_A = \dot{\vec{r}}_A$, and
$\vec{a}_A = \dot{\vec{v}}_A = \ddot{\vec{r}}_A$

$\vec{v}_B$ and $\vec{a}_B$?

$\therefore \vec{r}_B = \vec{r}_A + \vec{r}_{B/A}$

$$\therefore \vec{v}_B = \vec{v}_A + \frac{d}{dt}\left(\vec{r}_{B/A}\right)$$

$\vec{r}_{B/A}$: $\begin{cases} \text{magnitude} \rightarrow \text{constant} \\ \text{direction} \rightarrow \text{unchanged} \end{cases}$

constant → rigid body assumption

unchanged → translation

$\therefore \dot{\vec{r}}_{B/A} = \vec{\emptyset}$

$\therefore \vec{v}_B = \vec{v}_A$

and $\vec{a}_B = \vec{a}_A$

When a rigid body is in translation, all particles of the body will have the same velocity and acceleration at any given time instant;

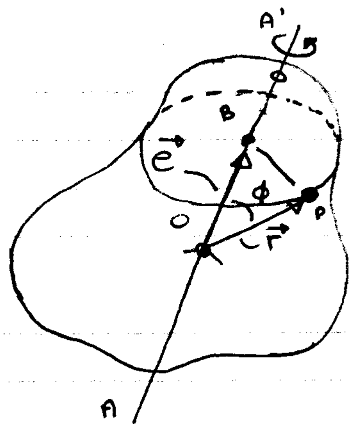
∴ The Kinematics of a translating rigid body can be represented by any particle within the rigid body

∴ Chapter 11 is applicable to translating rigid bodies.

15.1 B Rotation about a Fixed axis

A. General (3D) cases

Fig. 15.8 and revision



AA' - Fixed axis of rotation

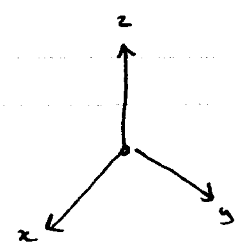
O - chosen Fixed point on AA'

$\vec{e}$ - directed A to A'

$\vec{v}_P$, $\vec{a}_P$ - on the plane passing P normal to AA'

$\vec{r}$ - drawn From O to P

BP - radial line $(\theta, \dot{\theta}, \ddot{\theta})$



$\dot{\theta}$ = angular velocity, $\dot{\theta} = \omega$

$\ddot{\theta}$ = angular acceleration, $\ddot{\theta} = \alpha$

then $\vec{\omega} = \omega \vec{e}$, $\vec{\alpha} = \alpha \vec{e}$

and $\vec{v}_P = \vec{\omega} \times \vec{r}$

$$\vec{a}_P = \vec{\alpha} \times \vec{r} + \vec{\omega} \times (\vec{\omega} \times \vec{r})$$

$$\vec{a}_P = \underbrace{\vec{\alpha} \times \vec{r} + \vec{\omega} \times \vec{v}_P}_{(\overrightarrow{a_t})_P + (\overrightarrow{a_n})_P}$$

15.1B Rotation about a fixed axis

A. General (3D) Cases

The axis of rotation is not coincidental with x, or y, or z-axis.

$\vec{e}$: unit vector directed along axis of rotation
$\vec{\omega}, \vec{\alpha}$: by right-hand rule
$\vec{r}$: directed from any point on axis of rotation to particle of interest, P.

then :
$$\vec{v_p} = \vec{\omega} \times \vec{r}$$
$$\vec{a_p} = \vec{\alpha} \times \vec{r} + \vec{\omega} \times \vec{v_p}$$
$$= (\vec{a_t})_p + (\vec{a_n})_p$$

Example 15.14 : (ω = 26 rad/s, $\therefore \alpha = 0$)
B → A axis of rotation

Point A (0, 80, 120)
Point B (0, 180, -120)
Point E (120, 0, 0)

AB = (0, -100, 240)

$$\therefore \vec{e} = \frac{\vec{r}_{A/B}}{|\vec{r}_{A/B}|} = \frac{-100\vec{j} + 240\vec{k}}{260} = \frac{-10\vec{j} + 24\vec{k}}{26}$$

$$\therefore \vec{\omega} = 26 \cdot \vec{e} = -10\vec{j} + 24\vec{k} \quad (\text{rad/s})$$

$$\vec{r}_{E/A} = 120\vec{i} - 80\vec{j} - 120\vec{k} \quad (\text{mm})$$

$$\therefore \vec{v}_E = \vec{\omega} \times \vec{r} = \begin{array}{c|ccc} & i & j & k \\ \hline \vec{\omega} & 0 & -10 & 24 \\ \vec{r} & 120 & -80 & -120 \end{array} = 3.120\vec{i} + 2.880\vec{j} + 1.2\vec{k} \quad (\text{m/s})$$

$$\therefore \vec{a}_E = \cancel{\vec{\alpha}}^{\,0} \times \vec{r} + \vec{\omega} \times \vec{v}_E$$

$$= -81.12\vec{i} + 74.88\vec{j} + 31.20\vec{k} \quad (\text{m/s}^2)$$

Problem 15.15

At the given time instant, $\omega = 26$ rad/s, and increases at a rate of 65 rad/s^2

$\therefore \quad \vec{\alpha} = 65\,\vec{e}$

$\therefore \quad \vec{v}_E = 3.120\,\vec{i} + 2.880\,\vec{j} + 1.200\,\vec{k}$ m/s

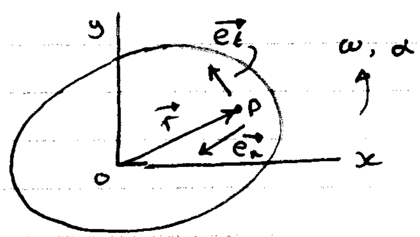
and $\vec{a}_E = \vec{\alpha} \times \vec{r} + \vec{\omega} \times \vec{v}_E$

$= -73.32\,\vec{i} + 82.08\,\vec{j} + 34.20\,\vec{k}$ m/s^2

15.1B

B. Rotation of a Representative Slab

(2D cases)



Figs. 15.10, 15.11

Fixed-axis of rotation = z-axis

$\vec{e} = \vec{k}$

$\vec{r}$: O → P

x and y-components only

Vector Expressions: $\vec{v}_P = \vec{\omega} \times \vec{r}$

$\vec{a}_P = \vec{\alpha} \times \vec{r} + \vec{\omega} \times \vec{v}_P$

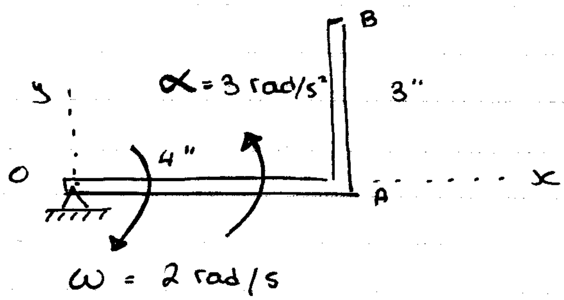
Scalar Expressions: normal-tangential

$$\vec{v}_P = r\omega$$

$$(a_n)_P = r\omega^2 \longrightarrow \text{NOTE: } \frac{(r\omega)^2}{r}$$

$$(a_t)_P = r\alpha = \frac{d}{dt}(r\omega) = r \cdot \frac{d\omega}{dt}$$

$\vec{v}_B$
$\vec{a}_B$



Solution :

(1) Vector expressions

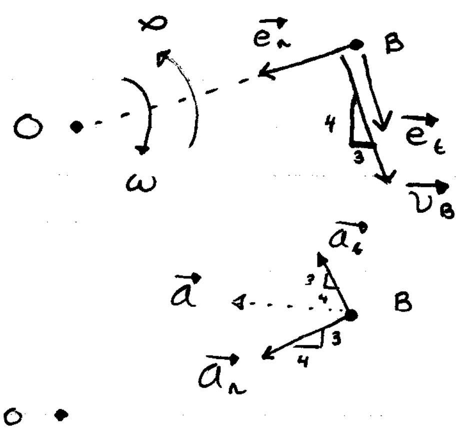
$$\vec{\omega} = -2\vec{k}$$
$$\vec{\alpha} = 3k$$
$$\vec{r} = \vec{r}_{B/O} = 4\vec{i} + 3\vec{j} \quad (in)$$

$$\vec{v}_B = \vec{\omega} \times \vec{r} \Rightarrow 6\vec{i} - 8\vec{j} \; (in/s)$$
$$\vec{a}_B = \vec{\alpha} \times \vec{r} + \vec{\omega} \times \vec{v}_B \Rightarrow -25i \quad (in/s^2)$$

(2) Scalar Expressions

$$\vec{v}_B = 5(2) = 10 \; in/s$$
$$(a_t)_B = 5(3) = 15 \; in/s^2$$
$$(a_n)_B = 5(2)^2 = 20 \; in/s^2$$

(3) Interpretation of scalar results



15.1 B - Rotation about a Fixed axis

A. General (3D Cases)

B. Rotation of a Representative slab (2D cases)

Both deal with Velocity and acceleration of a particle in a rigid body rotating about a fixed axis.

2D Cases: axis of rotation coincides with the z-axis

vector expressions $\vec{v}_P$, $\vec{a}_P$

scalar expressions $\vec{v}_P$, $(a_t)_P$, $(a_n)_P$

3D Cases: axis of rotation: $\vec{e}$

vector expressions only.

C. D. } entire rigid body or bodies.

PROB 12.7

$a(v) = b - kv$

(1) $\int_{t_0}^{t} dt = \int_{v_0}^{v} \frac{dv}{a(v)} \Rightarrow \int_{v_0}^{v} \frac{dv}{b - kv}$

$= \frac{1}{-k} \int \frac{-k\,dv}{b - kv} = -\frac{1}{k} \int_{v_0}^{v} \frac{d(b - kv)}{b - kv}$

by substitution

$u = b - kv$

$du = 0 - k\,dv$

$dv = \frac{du}{-k} = \left(\frac{-1}{k}\right) du$

(or something like that.)

I got

$\int_{t_0}^{t} dt = \frac{-m}{k}\left[\ln|u| \Big|_{v_0}^{v} \right]$

where $u = b - kv$

(2) $\int_{x_0}^{x} dx = \int_{v_0}^{v} \frac{v \cdot dv}{a(v)} = \int_{v_0}^{v} \frac{v\,dv}{b - kv}$

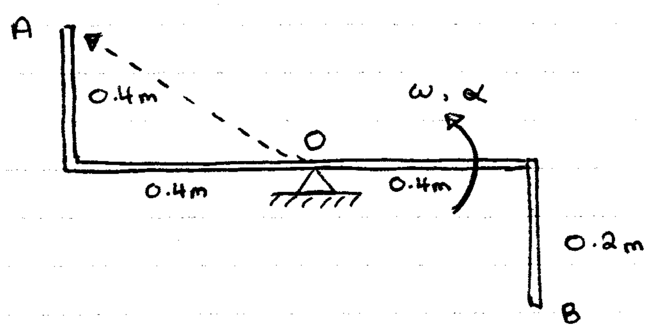
$\frac{v}{b - kv} = \frac{1}{-k} \cdot \frac{-kv + b - b}{b - kv}$

$= \frac{-1}{k} \left(\frac{b - kv}{b - kv}\right) - b$

$= \frac{-1}{k} \left[1 - \frac{b}{b - kv} \right]$

$\therefore \text{RHS} = -\int_{v_0}^{v} \left(1 - b \frac{1}{b - kv}\right) dv$

$= \frac{-1}{k} \left[\int_{v_0}^{v} dv - b \int_{v_0}^{v} \frac{dv}{b - kv} \right]$



Given that $v_A = 3$ m/s and $|\vec{a}_A| = 28$ m/s²

Determine: v_B and $|\vec{a}_B|$.

Solution:

Assume CCW ω, α

$\therefore \quad \vec{\omega} = \omega \vec{k}, \quad \vec{\alpha} = \alpha \vec{k}$

Pt. A: $\vec{r} = -0.4\vec{i} + 0.4\vec{j}$

$$\vec{v}_A = \vec{\omega} \times \vec{r}$$

$$= \begin{vmatrix} i & j & k \\ 0 & 0 & \omega \\ -0.4 & 0.4 & 0 \end{vmatrix} = -0.4\omega\vec{i} - 0.4\omega\vec{j}$$

but, $v_A = \sqrt{(0.4\omega)^2 + (-0.4\omega)^2}$

$$= 0.4\sqrt{2} \cdot \omega = 3$$

$$\therefore \quad \omega = 5.303 \text{ rad/s}$$

Further, $\vec{a}_A = \vec{\alpha} \times \vec{r} + \vec{\omega} \times \vec{v}_A$

$$= -0.4\alpha\vec{i} - 0.4\alpha\vec{j} + 0.4\omega^2\vec{i} - 0.4\omega^2\vec{j}$$

$$= -0.4\alpha\vec{i} - 0.4\alpha\vec{j} + 11.25\vec{i} - 11.25\vec{j}$$

$$= (11.25 - 0.4\alpha)\vec{i} - (11.25 + 0.4\alpha)\vec{j}$$

$$|\vec{a}_A|^2 = (11.25 - 0.4\alpha)^2 + (11.25 + 0.4\alpha)^2$$

$$(28)^2 = 2(11.25)^2 + 2(0.4\alpha)^2$$

$$\alpha = \pm 40.73$$

Pt. B: $v_B = 2.732$ m/s

$|\vec{a}_B| = 22.14$ m/s^2