Problem 11.29

The acceleration due to gravity at an altitude y above the surface of the earth can be expressed as:

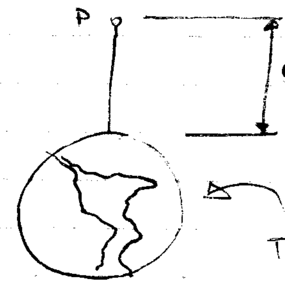
$$a = \frac{-32.2}{[1 + (y/20.9\times10^6)]^2}$$

where $a$ and $y$ are expressed in $ft/s^2$ and Ft respectively. Using this expression, compute the height reached by a projectile fired vertically upward from the surface of the earth if it's initial velocity is:

a) 1800 ft/s

b) 3000 ft/s

c) 36 700 ft/s




Answer:

$$a = \frac{-32.2}{\left[1 + \frac{y}{20.9\times10^6}\right]^2}$$

$\left\{\begin{array}{l}\text{The relation between } y \text{ and } v \\ y_{max} \text{ when } v = 0\end{array}\right.$

Solution:

$$a(y) = \frac{dv}{dt} = \frac{dv}{dy}\cdot\frac{dy}{dt} = \frac{dv}{dy}\cdot v$$

$$\therefore \int v\,dv = \int a(y)\cdot dy$$

$$\therefore \int_{v_0}^{v} v\,dv = \int_0^y a(y)\cdot dy$$

$$\therefore (1/2)(v^2 - v_0^2) = \int_0^y \frac{-32.2}{\left[1 + \frac{y}{20.9\times10^6}\right]^2}\,dy$$

$$I_2 = \int \frac{-32.2}{\left[1 + \frac{y}{20.9\times10^6}\right]^2}\,dy \Rightarrow (672.98\times10^6)\frac{1}{1 + y/20.9\times10^6} + C$$

$$\therefore \tfrac{1}{2}(v^2 - v_0^2) = (672.98 \times 10^6) \left. \frac{1}{1 + y/20.9 \times 10^6} \right|_0^y$$

$$= (672.98 \times 10^6) \left[ \frac{1}{1 + y/20.9 \times 10^6} - 1 \right]$$

$y_{max}$ occurs when $v = 0$

$$\therefore y_{max} = 20.9 \times 10^6 \frac{v_0^2}{13.46 \times 10^8 - v_0^2}$$

$v_0 = 1800$ ft/s $\longrightarrow$ $y_{max} = 50,430$ ft

$30000$ ft/s $\longrightarrow$ $y_{max} = 14.07 \cdot 10^6$ ft

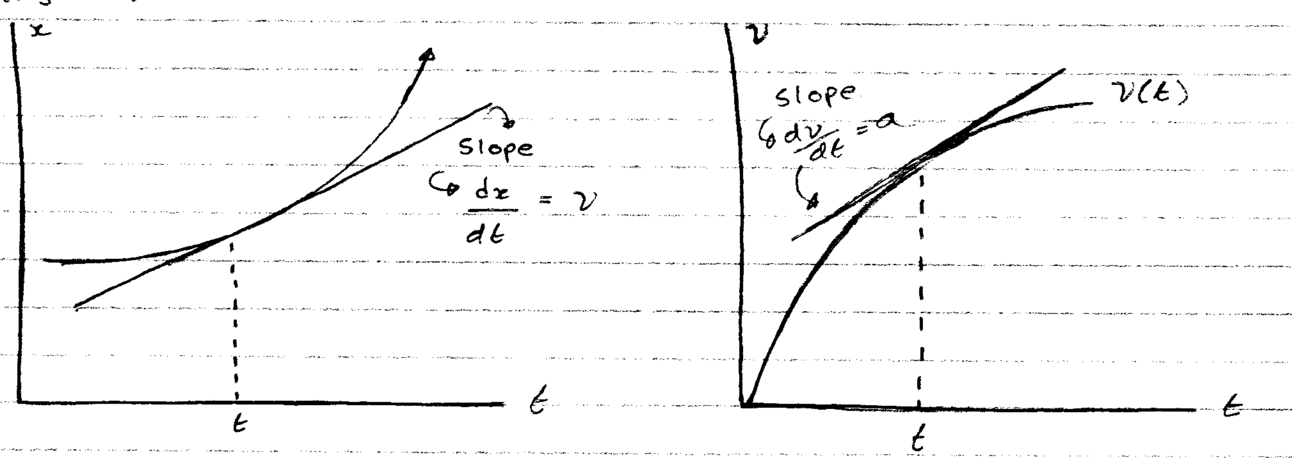
$36,700$ ft/s $\longrightarrow$ $y_{max} = -3.163 \cdot 10^{10}$ ft

$\therefore y_{max} = \infty$ (y cannot be negative)

$\hookrightarrow$ exceeds escape velocity.

## § 11.3 Graphical Solution of Rectilinear Motion

(Fig 11.10)




$x = x(t)$ is given

$$v = \frac{dx}{dt}$$

$\therefore$ slope of $x(t) - t$ curve at $t$ gives the velocity

$v = v(t)$ is known

$$a = \frac{dv}{dt}$$

$\therefore$ slope of $v(t) - t$ curve at $t$ gives the acceleration

Solution :

(1) $v - t$

$$v - v_0 = \text{area under } a-t \text{ from } t_0 \text{ to } t$$

$\emptyset < t < 2$ $\quad v(2) = v\big|_{t=2s} = v_2$

$$v_2 - v_0 = 3 \times 2$$

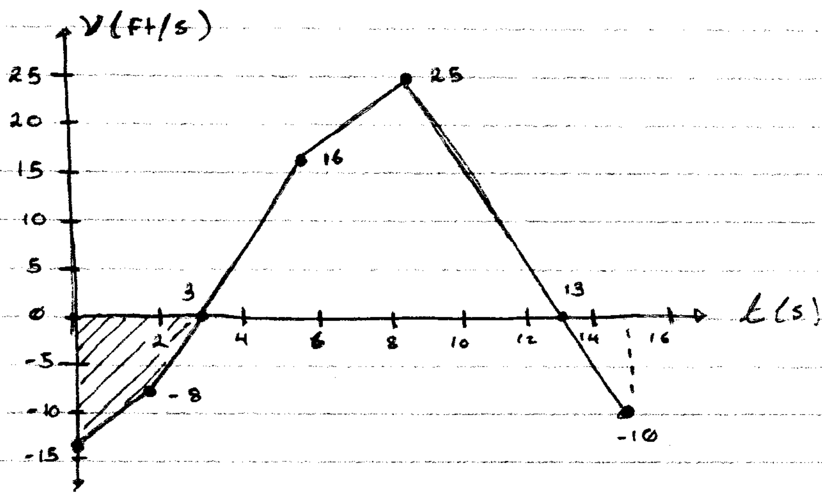
$$\therefore v_2 = 6 + (-14) = -8 \ (ft/s)$$

$2 < t < 5 \quad \Rightarrow v_5 = 16 \ (ft/s)$

$5 < t < 8 \quad \Rightarrow v_8 = 25 \ (ft/s)$

$8 < t < 15 \quad \Rightarrow v_{15} = (-1\emptyset) \ (ft/s)$

Construct $v - t$ plot :




$$x - x_0 = \text{area under } v(t) \text{ curve from } t_0 \text{ to } t.$$

$x_0 = \emptyset$

$\emptyset < t < 2 \quad \Rightarrow x_2 = -22 \ ft$

$2 < t < 3 \quad \Rightarrow x_3 = -26 \ ft$

$3 < t < 5 \quad \Rightarrow x_5 = -1\emptyset \ ft$

$5 < t < 8 \quad \Rightarrow x_8 = 51.5 \ ft$

$8 < t < 13 \quad \Rightarrow x_{13} = 114 \ ft$

$13 < t < 15 \quad \Rightarrow x_{15} = 1\emptyset 4 \ ft$

JAN. 25/17

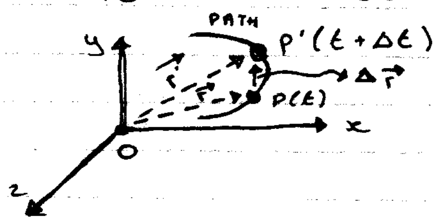
Classlist Index Amended

Tutorial Problems for tomorrow will be posted tonight.

## § 11.4 Curvilinear Motion of Particles

### 11.4A Position, Velocity and Acceleration Vectors

When a particle moves along a curve, it is said to be in curvilinear motion.




Position: Vector $\vec{r} = \vec{r}(t)$

Velocity: $\vec{r}, t$

$\vec{r}', t + \Delta t$

Change in time: $\Delta t$

Change in position: $\overrightarrow{\Delta r}$

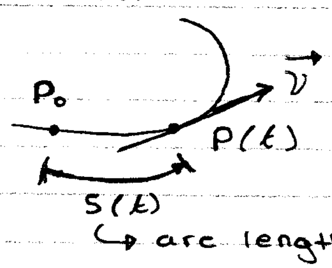
average velocity: $\dfrac{\overrightarrow{\Delta r}}{\Delta t}$

Velocity: $\vec{v} = \lim_{\Delta t \to 0} \dfrac{\Delta \vec{r}}{\Delta t} = \dfrac{d\vec{r}}{dt}$

magnitude of $\vec{v}$: $v$, $|\vec{v}|$, speed

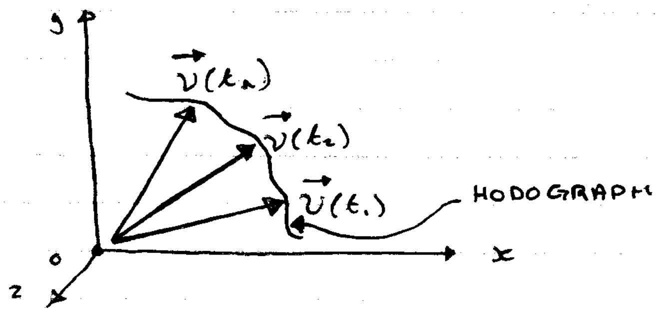
direction of $\vec{v}$: tangent to the path at P directed along increasing arc-length $s(t)$

PATH OF PARTICLE:



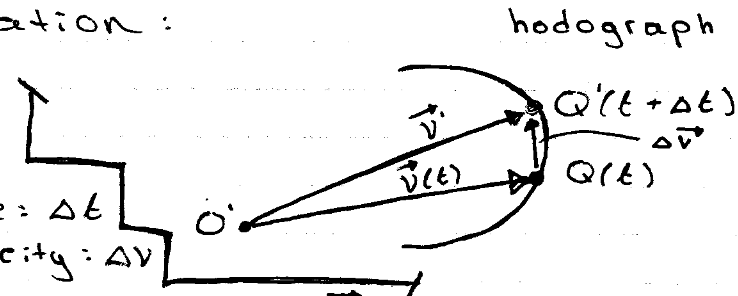

Hodograph: the curve traced by the tip of $\vec{v}$ is called the hodograph of the particle's motion.

$\vec{v}(t_1)$, $\vec{v}(t_2)$, ..., $\vec{v}(t_n)$




acceleration :

at $t$ : $\vec{v}$
at $t+\Delta t$ : $\vec{v}'$
change in time : $\Delta t$
Change in velocity : $\Delta v$

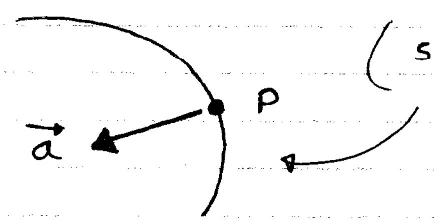



average acceleration : $\dfrac{\Delta\vec{v}}{\Delta t}$

acceleration : $\vec{a} = \lim\limits_{\Delta t \to 0} \dfrac{\Delta\vec{v}}{\Delta t} = \dfrac{d\vec{v}}{dt} = \dfrac{d}{dt}\left(\dfrac{d\vec{r}}{dt}\right)$

$$= \frac{d^2\vec{r}}{dt^2}$$

Magnitude : $|\vec{a}|$
direction : tangent to hodograph at Q
↳ not tangent to the path at P




11.4B Derivatives of Vector Functions
over-dot notation (with respect to)
when differentiation w.r.t. time, over-dot notation is typically used.

given $x(t)$ $\quad v = \dfrac{dx}{dt} = \dot{x}$

Given $v(t)$ $\quad a = \dfrac{dv}{dt} = \dot{v}$

$$= \frac{d^2x}{dt^2} = \ddot{x}$$

but... $y = f(x)$

$$\frac{dy}{dx} = y' = f'$$

$$\vec{v} = \frac{d\vec{r}}{dt} = \dot{\vec{r}}$$

$$\vec{a} = \dot{\vec{v}} = \ddot{\vec{r}}$$

## 11.4C Rectangular Components of Velocity And Acceleration

$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ (in the text)

We Focus on $\vec{r} = x\vec{i} + y\vec{j}$

$$\therefore \vec{v} = \dot{x}\vec{i} + \dot{y}\vec{j}$$

$$\therefore \vec{a} = \ddot{x}\vec{i} + \ddot{y}\vec{j}$$

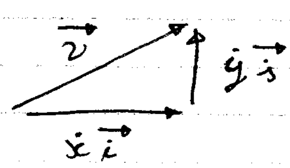
$$\vec{r} = x\vec{i} + y\vec{j}$$

$$\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j}$$

$$\vec{v} = \dot{\vec{r}} = \frac{d}{dt}\left[x(t)\vec{i} + y(t)\vec{j}\right]$$

$$= \dot{x}\vec{i} + \dot{y}\vec{j}$$

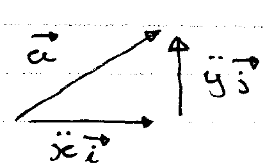
for $\vec{v}$ :




$\left\{\begin{array}{l} \dot{x} \text{ is the scalar } x\text{-component} \\ \dot{x}\vec{i} \text{ is the vector } x\text{-component} \end{array}\right.$

$\left\{\begin{array}{l} \dot{y} \text{ is the scalar } y\text{-component} \\ \dot{y}\vec{j} \text{ is the scalar } y\text{-component} \end{array}\right.$

Similarly, For $\vec{a}$ :




$\left\{\begin{array}{l} \ddot{x} \text{ " } \quad \text{"} \\ \ddot{x}\vec{i} \text{ " } \quad \text{"} \end{array}\right.$

$\left\{\begin{array}{l} \ddot{y} \text{ " } \quad \text{"} \\ \ddot{y}\vec{j} \text{ " } \quad \text{"} \end{array}\right.$

$$\vec{v} = v_x \vec{i} + v_y \vec{j}$$
$$= \dot{x}\vec{i} + \dot{y}\vec{j}$$
$$\therefore v_x = \dot{x}, \quad v_y = \dot{y}$$

$$\vec{a} = a_x \vec{i} + a_y \vec{j}$$
$$= \ddot{x}\vec{i} + \ddot{y}\vec{j}$$
$$\therefore a_x = \ddot{x} = \dot{v}_x$$
$$a_y = \ddot{y} = \dot{v}_y$$

Problem 11.89 $\quad x = 5t \quad (m)$

$y = 2 + 6t - 4.9t^2 \quad (m)$

$t$ in seconds

Find : (a) $\vec{v}\,|_{t=1s}$

(b) horizontal distance

Solution : (a) $x = 5t$

$\therefore v_x = \dot{x} = 5$

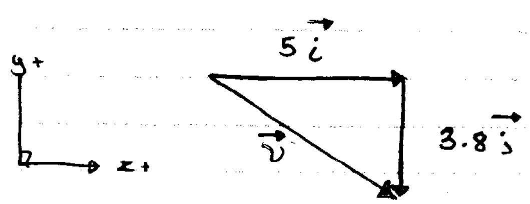
$y = 2 + 6t - 4.9t^2$

$\therefore v_y = \dot{y} = 6 - 9.8t$

@ $t = 1s$, $\quad v_x = 5 \text{ m/s}$

$v_y = 6 - 9.8(1)$

$v_y = -3.8 \text{ m/s}$




$$\vec{v}\,|_{t=1s} = 5\vec{i} - 3.8\vec{j} \quad (m/s)$$

(b) Set $y = 0$

Solving For $t$ : $\quad t_1 = -0.2726 \text{ s}$

$t_2 = 1.497 \text{ s}$

$\therefore$ horizontal distance $= x\,|_{t=t_2} = 7.486 \text{ m}$

JAN. 26/17

Tutorial →

Q1: (1) $t_1 = 10s$

(2) $t_2 = 40s$

(3) Area under $10s \to 40s = (50 \cdot 10)/2 + \ldots$ etc.

(4) NO – it is reversible.

(5) $10 \to 20s$ (because acceleration + velocity (–))

$40 \to 50s$ (because acceleration + velocity (+))

Q2: $$x = \frac{b}{2}\left(\sin\frac{\pi t}{4t_o} + \sin\frac{3\pi t}{4t_o}\right)$$

$$y = \frac{b}{2}\left(\cos\frac{\pi t}{4t_o} - \cos\frac{3\pi t}{4t_o}\right)$$

$$x = \frac{b}{2}\left(\sin\frac{\pi t}{4t_o} + \sin\frac{3\pi t}{4t_o}\right)$$

$$\dot{x} = \frac{b}{2}\left(\cos\frac{\pi t}{4t_o}\cdot\frac{\pi}{4t_o} + \cos\frac{3\pi t}{4t_o}\cdot\frac{3\pi}{4t_o}\right.$$

$$\ddot{x} = \frac{b}{2}\left[\left(\frac{\pi}{4T_o}\right)^2\left(-\sin\left(\frac{\pi t}{4t_o}\right) + \left(\frac{3\pi}{4t_o}\right)^2\left(-\sin\frac{3\pi t}{4t_o}\right)\right]$$

$$y = \frac{b}{2}\left(\cos\frac{\pi t}{4t_o} - \cos\frac{3\pi t}{4t_o}\right)$$

$$\dot{y} = \frac{b}{2}\left(-\frac{\pi}{4t_o}\sin\frac{\pi t}{4t_o} + \frac{3\pi}{4t_o}\sin\frac{3\pi t}{4t_o}\right)$$

$$\ddot{y} = \frac{b}{2}\left[\left(-\left(\frac{\pi}{4t_o}\right)^2\left(\cos\frac{4t}{4t_o}\right) + \left(\frac{3\pi}{4t_o}\right)^2\left(\cos\frac{3\pi}{4t_o}\right)\right]$$

at $t = t_o$: $\sin\frac{\pi t}{4t_o} = \frac{\sqrt{2}}{2}$, $\sin\frac{3\pi t}{4t_o} = \frac{\sqrt{2}}{2}$

$\cos\frac{\pi t}{4t_o} = \frac{\sqrt{2}}{2}$, $\cos\frac{3\pi t}{4t_o} = \frac{\sqrt{2}}{2}$

then $\dot{x} = \frac{-\sqrt{2}\,\pi b}{8t_o}$, $\dot{y} = \frac{\sqrt{2}\,\pi b}{8t_o}$

$\ddot{x} = -\frac{5\sqrt{2}\,\pi^2 b}{32t_o^2}$, $\ddot{y} = -\frac{5\sqrt{2}\,\pi^2 b}{32t_o^2}$