JAN. 16 /17

§ 11.1 Rectilinear Motion of Particles

11.1A Position, Velocity and Acceleration

1. Rectilinear Motion
2. Position and coordinate setup
3. velocity (average, instantaneous)
4. Average acceleration and istantaneous acceleration

$$t \rightarrow t + \Delta t$$

$$v \rightarrow v + \Delta v$$

average acceleration $= \dfrac{\Delta v}{\Delta t}$

instantaneous acceleration (or simply acceleration)

$$a = \lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t} = \frac{dv}{dt}$$

$$= \frac{d}{dt}\left(\frac{dx}{dt}\right) = \frac{d^2 x}{dt^2}$$

1) units : m/s^2

Ft/s^2 or in/s^2

2) $a > 0 \rightarrow v$ is increasing

$a < 0 \rightarrow v$ is decreasing

3) deceleration : when the speed decreases or particle slows down

4) accelerating / decelerating ?

Particle is accelerating when v and a are of the same sign (+ or −) ; or speed is increasing.

Particle is decelerating when v and a are of opposite signs ; or speed is decreasing.

* See Fig 11.5, and last two paragraphs on page 619.

Example:

Given $x(t) = 6t^2 - t^3$ where t is given in seconds and x is in meters.

1) determine $v(t)$ and $a(t)$
2) is the particle's motion irreversible or reversible?
3) When is the particle accelerating? and when is it decelerating?

Solution

1) $x(t) = 6t^2 - t^3 \quad (m)$

$$\therefore v(t) = \frac{dx}{dt} = 12t - 3t^2 \quad (m/s)$$

$$a(t) = \frac{dv}{dt} = 12 - 6t \quad (m/s^2)$$

2) Set $v = 0$

$12t - 3t^2 = 0$

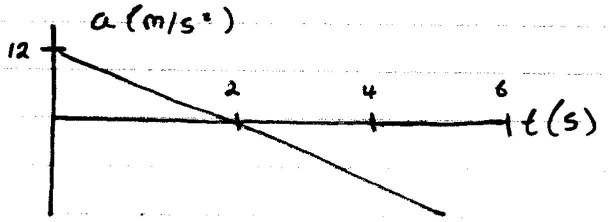
when $t = 0, \quad t = 4$

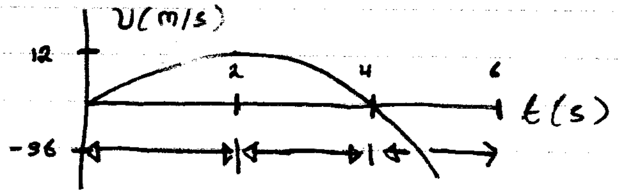
$v > 0$ when $t \in (0, 4)$

$v < 0$ when $t \in (4, \infty)$

$\therefore$ Reversible motion

3)

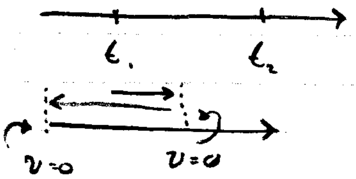




$0 < t < 2$, and $t > 4$: accelerating

$2 < t < 4$: decelerating

Prob. 11.2.



§ 11.1 B Determination of the motion of a particle

a = acceleration, caused by forces

forces : constant, e.g. weight
time - dependent $f(t)$
Position - dependent $f(x)$
velocity - dependent $f(v)$

1) $a = f(t)$ is given
initial conditions $v(t_0) = v_0$, $x(t_0) = x_0$

$a = f(t)$, initial conditions
$v(t_0) = v_0$, $x(t_0) = x_0$

by definition : $a = \frac{dv}{dt} = f(t)$

$$\therefore dv = f(t)dt$$

$$\therefore \int_{v_0}^{v} dv = \int_{t_0}^{t} f(t)\,dt$$

$$v - v_0 = \int_{t_0}^{t} f(t)\,dt$$

$$\therefore v = \int_{t_0}^{t} f(t)\,dt + v_0 = v(t)$$

Again, by definition, $v = \frac{dx}{dt} = v(t)$

$$\therefore \int_{x_0}^{x} dx = \int_{t_0}^{t} v(t)\,dt$$

$$\therefore x - x_0 = \int_{t_0}^{t} v(t)\,dt$$

$$\therefore x = \int_{t_0}^{t} v(t)\,dt + x_0$$

2) $a = f(v)$
aerodynamic forces
hydrodynamic forces } depend on v

by definition, $a = \frac{dv}{dt} = f(v)$

$$d(v) = f(v)\,dt$$

$$\int_{v_0}^{v} \frac{dv}{f(v)} = \int_{t_0}^{t} dt$$

$$\therefore \int_{v_0}^{v} \frac{dv}{f(v)} = t - t_0$$

$v(t)$ (implicit)

Again, by definition :

$$v = \frac{dx}{dt} = \frac{dx}{dv} \cdot \frac{dv}{dt} = \frac{dx}{dv} \cdot f(v)$$

$$\therefore \int_{v_0}^{v} \frac{v\,dv}{f(v)} = \int_{x_0}^{x} dx$$

$$\therefore \int_{v_0}^{v} \frac{v\,dv}{f(v)} = x - x_0$$

$x(v)$ $\qquad$ $x(v) \Rightarrow x(v(t)) = x(t)$

3) $a = f(x)$

Spring Forces depend on position x.

by definition $\quad a = \dfrac{dv}{dt} = \dfrac{dv}{dx} \cdot \dfrac{dx}{dt} = \dfrac{dv}{dx} \cdot v$

$$\therefore f(x) = \frac{d(v)}{d(x)} \cdot v$$

$$\therefore \int_{x_0}^{x} f(x)\,dx = \int_{v_0}^{v} v \cdot dv$$

$$\therefore \int_{x_0}^{x} f(x)\,dx = \frac{1}{2}(v^2 - v_0^2)$$

$v(x)$ $v \rightarrow$ (implicit)

again, by definition $\quad v = \dfrac{d(x)}{d(t)} = v(x)$

$$\therefore \int_{x_0}^{x} \frac{dx}{v(x)} = \int_{t_0}^{t} dt \Rightarrow \int_{x_0}^{x} \frac{dx}{v(x)} = t - t_0$$

$x(t)$ and $v(x) \Rightarrow v(x(t)) \Rightarrow v(t)$

JAN 18/17

11.1 B Determination of motion of a particle

$a(t)$
$a(x)$
$a(v)$

For tutorial: Prob 11.2,
Prob 11.20,
Prob 11.29

Problem 11.10 (From text)
The acceleration of a particle is defined by the relation $a = 3e^{-0.2t}$, where a and t are expressed in ft/s² and seconds, respectively. Knowing that $x = 0$, and $v = 0$ at $t = 0$, determine the velocity and position of the particle when $t = 0.5$ s.

Find: $v(t)$, $x(t)$

Solution:

$$a(t) = 3e^{-0.2t}$$

$$\int_0^v dv = \int_0^t a(t)\,dt$$

let $u = e^{-t}$
$du = e^{-t}dt$
$dt = \dfrac{du}{e^{-t}} \Rightarrow du\,e^{t}$

$$v\Big|_0^v = \int_0^t 3e^{-0.2t}\,dt = \frac{3}{(-0.2)}e^{-0.2t}\Big|_0^t$$

$$v - 0 = \frac{3}{(-0.2)}e^{-0.2t} - \frac{3}{(-0.2)}(1)$$

$$= 15 - 15e^{-0.2t}$$

$$\therefore v(t) = 15 - 15e^{-0.2t} \quad (\text{ft/s})$$

$$\int_0^x dx = \int_0^t v(t)\,dt$$

$$x\Big|_0^x = \int_0^t v(t)\,dt$$

$$x = 15t + 75e^{-0.2t} - 75 \quad (\text{ft})$$

$$\therefore v(0.5) = v\Big|_{t=0.55} = 1.427 \quad (\text{ft/s})$$ ▲ Ans.

$$x\Big|_{t=0.55} = 0.3628 \text{ ft}$$ ▲ Ans.

Example: The acceleration of a particle is directly proportional to the time t. At $t = 0$, the velocity of the particle is $v = 16$ in/s. Knowing that $v = 15$ in/s and that $x = 20$ in when $t = 1$s, determine the velocity and position when $t = 7$s.

Solution:

$a(t) = kt$ (directly proportional to)

$a(t) = kt + d$ (linear relation)

$$\int_{16}^{v} dv = \int_{0}^{t} a(t)\,dt$$

$$v - 16 = \int_{0}^{t} kt\,dt = \tfrac{1}{2}kt^2 \Big|_{0}^{t} = \tfrac{1}{2}kt^2$$

$$\therefore\ v = 16 + \tfrac{1}{2}kt^2$$

$$\therefore\ v\big|_{t=1s} = 15, \quad \therefore\ k = -2$$

$$\therefore\ v = 16 - t^2 \quad (\text{in/s})$$

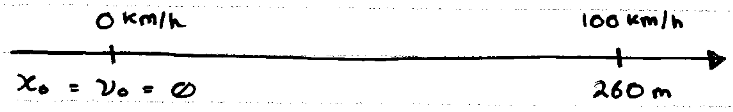
$$\int_{20}^{x} dx = \int_{1}^{t} v(t)\,dt$$

$$\therefore\ x = -\frac{t^3}{3} + 16t + \frac{13}{3} \quad (\text{in})$$

Finally, $v\big|_{t=7s} = -33$ in/s ▲ Ans.

and $x\big|_{t=7s} = 2$ in ▲ Ans.

Example: A human-powered vehicle (HPV) team wants to model the acceleration during the 260m sprint race (the first 60m is called a flying start) using $a = A - Cv^2$, where a is acceleration in m/s² and v is the velocity in m/s. From wind tunnel testing, they found that $C = 0.0012\,m^{-1}$. Knowing that the cyclist is goint 100 km/h at the 260 meter mark, what is the value of A?



Solution:

$a(v) = A - Cv^2$

$C = 0.0012 \ (m^{-1})$

$x_0 = v_0 = 0$

$$a = \frac{dv}{dt} = \frac{dv}{dx} \cdot \overset{\text{velocity by definition.}}{\frac{dx}{dt}} = v \cdot \frac{dv}{dx}$$

$$\Rightarrow a = v \cdot \frac{dv}{dx}$$

$$\therefore = dx = \frac{v\,dv}{a(v)}$$

$$\therefore \int_0^x dx = \int_0^v \frac{v\,dv}{A - Cv^2}$$

$u = A - Cv^2$

$du = 1 - 2v\,dv$

$dv = \frac{du}{1-2v}$

$$x = \frac{-1}{2C}\left[\ln|A - Cv^2|\right]\Big|_0^v$$

$$= \frac{-1}{2C}\left[\ln|A - Cv^2| - \ln|A|\right]$$

$$e^{-2Cx} = e^{\ln\left|\frac{A - Cv^2}{2}\right|} = \left|\frac{A - Cv^2}{A}\right|$$

$$\therefore \left|\frac{A - Cv^2}{A}\right| = e^{-2Cx}$$

Assume $A > 0$ (Positive)

$A - Cv^2 > 0$

then $\frac{A - Cv^2}{A} = e^{-2Cx}$

where:
$C = 0.0012 \ (m^{-1})$
$x = 260m$
$v = 100$ km/h => 27.78 m/s

then $A = 1.995$ m/s² ▲ Ans.

$a(v) = A - Cv^2$

§11.2 Special cases and Relative Motion

11.2A - Uniform rectilinear motion
11.2B - Uniformly accelerated rectilinear motion
11.2C (won't be covered)
↳ relative motion

Uniform Rectilinear Motion

v = constant
$a = 0$
$$x \rightsquigarrow \int dx = \int \underset{\hookrightarrow v}{v(t)}\, dt$$

Uniformly accelerated rectilinear motion

a = constant
$$v - v_0 = a(t - t_0)$$
$$x - x_0 = \frac{1}{2} a (t - t_0)^2 + v_0 (t - t_0)$$
$$v^2 - v_0^2 = 2a(x - x_0)$$

JAN. 19/17

§ 11.1 Rectilinear Motion of Particles

11.1 A Concept application 11.1

11.1 B Sample Problem 11.1 ~ 4

Table 11.1

§ 11.2 Special cases and Relative Motion

Prob 11.33, 11.34, 11.36

Prob 11.2

Given $x = 2t^3 - 9t^2 + 12t + 10$ (ft)

Find 1) the time, position and acceleration of the particle when $v = 0$

2) the total distance traveled from $t = 0s$ and $t = 3s$

3) show that the particle is accelerating when $t \in (1, 1.5)$, and $t \in (2, \infty)$

Solution:

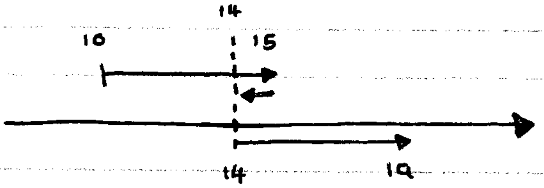
1)
$$x = 2t^3 - 9t^2 + 12t + 10$$
$$v = 6t^2 - 18t + 12$$
$$a = 12t - 18$$

$$v = 0 \quad \therefore \quad 6t^2 - 18t + 12$$
$$\Rightarrow 6(t^2 - 3t + 2)$$
$$\Rightarrow 6(t-1)(t-2) = 0$$
$$\therefore \quad t_1 = 1s, \quad t_2 = 2s$$

$\therefore$ $t = 1s$, $x(1) = 15$ ft, $a(1) = -6$ ft/s^2 ▲ Ans.

$t = 2s$, $x(2) = 14$ ft, $a(2) = 6$ ft/s^2 ▲ Ans.

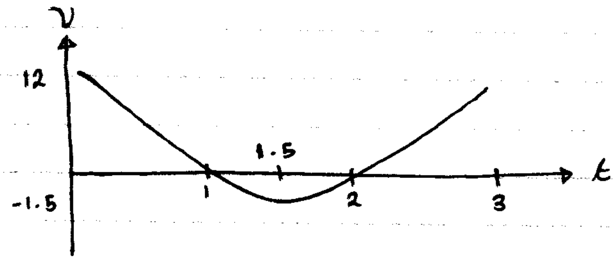
2)
$$x|_{t=0s} = 10 \text{ ft}$$
$$x|_{t=1s} = 15 \text{ ft}$$
$$x|_{t=2s} = 14 \text{ ft}$$
$$x|_{t=3s} = 19 \text{ ft}$$

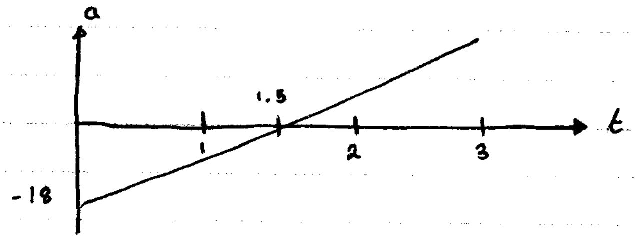


$\therefore$ distance traveled

$$= \left| x|_{t=1s} - x|_{t=0s} \right| + \left| x|_{t=2s} - x|_{t=1s} \right| + \left| x|_{t=3s} - x|_{t=2s} \right| = 11 \text{ ft}$$

3)





∴ accelerating when $t \in (1, 1.5)$ and $t \in (2, \infty)$

Prob 11.20 → see textbook.

$$a(x) = -100\left(x - \frac{l x}{\sqrt{l^2+x^2}}\right)$$

at $t = 0$, $v_0 = 0$, x_0 as given

Find : v when $x = 0$

Solution :

$$a = \frac{dv}{dt} = \frac{dv}{dx}\cdot\frac{dx}{dt} = v\cdot\frac{dv}{dx}$$

$$\therefore a\,dx = v\,dv$$

$$\int_0^v v\,dv = \int_{x_0}^{x} (-100)\left(x - \frac{l x}{\sqrt{l^2+x^2}}\right) dx$$

$$\tfrac{1}{2}v^2 = (-100)\int_{x_0}^{x}\left[x - \frac{l x}{\sqrt{l^2+x^2}}\right) dx$$

$$I_1 = \int\left(x - \frac{l x}{\sqrt{l^2+x^2}}\right)dx$$

$$\therefore \int_{x_0}^{x}\left(x - \frac{l x}{\sqrt{l^2+x^2}}\right)dx \Rightarrow \left[\frac{x^2}{2} - l\sqrt{l^2+x^2}\right]\Bigg|_{x_0}^{x}$$

$$\Rightarrow \left[\frac{x^2}{2} - l\sqrt{l^2+x^2}\right] - \left[\frac{x_0^2}{2} - l\sqrt{l^2+x_0^2}\right]$$

$$\therefore \tfrac{1}{2}v^2 = (-100)\left[\frac{x^2}{2} - l\sqrt{l^2+x^2} - \frac{x_0^2}{2} + l\sqrt{l^2+\ldots \ldots x_0^2}\right]$$

v when $x = 0$

$$\therefore v^2 = (-200)\left[-l^2 - \frac{x_0^2}{2} + l\sqrt{l^2+x_0^2}\right]$$

$$= (200)\left[l^2 + \frac{x_0^2}{2} - l\sqrt{l^2 + x_0^2}\right]$$

$$\therefore\ v = -\sqrt{}$$