§15.3 Instantaneous Center of Rotation

Sample Prob. 15.9, 10, 11.

From previous question:

$v_C = 19.5$ m/s NOT 24

$x = 0.1345$ m NOT 0.24 m

$\omega_C = 145.0$ rad/s ↻ NOT 100

§15.4 General Plane Motion: Acceleration

15.4A Absolute and Relative Acceleration in Plane Motion

15.4B Analysis of Plane Motion in Terms of a parameter.

Sample Prob. 15.12, ~ 15.16

15.4A Absolute and Relative Acceleration in Plane Motion

General Motion =
translation with A
+ rotation about A (as if A were fixed)

At velocity level:

$$\vec{v}_B = \vec{v}_A + \vec{v}_{B/A}$$
$$= \vec{v}_A + \vec{\omega} \times \vec{r}_{B/A}$$

At acceleration level:

$$\vec{a}_B = \vec{a}_A + \vec{a}_{B/A}$$

$\vec{a}_A$, $\vec{a}_B$: absolute acceleration of A and B, respectively.

$\vec{a}_{B/A}$: relative acceleration of B rotating about A, as if A were fixed.

$$\therefore \vec{a}_{B/A} = (\vec{a}_{B/A})_t + (\vec{a}_{B/A})_n$$
$$(\vec{a}_{B/A})_t = \vec{\alpha} \times \vec{r}_{B/A}$$
$$(\vec{a}_{B/A})_n = \vec{\omega} \times \vec{v}_{B/A} = -\omega^2 \vec{r}_{B/A}$$

works only if $\vec{r}_{B/A}$ is a vector on the x-y plane.

$$\therefore \vec{a}_B = \vec{a}_A + \vec{\alpha} \times \vec{r}_{B/A} - \omega^2 \vec{r}_{B/A} \quad (*)$$

- As with velocity analysis by vectors, A is the base point, and B is the point under consideration.
- Velocity analysis must be performed before acceleration analysis. That is, angular velocity ω must be determined before applying $\vec{a}_B = \vec{a}_A + \vec{a}_{B/A}$
- In general, but not always, the same base point can be used for both velocity analysis and acceleration analysis.

$\vec{a}_A$, $\vec{a}_B$: may have normal and tangential components, depending on the paths traveled by A and B.

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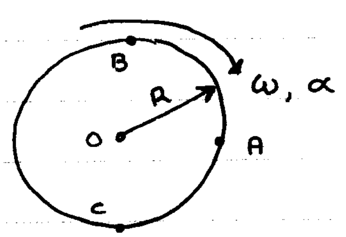


Determine $\vec{a}_o$, $\vec{a}_A$, $\vec{a}_B$, $\vec{a}_c$ in terms of R, ω and α

Solution :

(1) Velocity analysis :

C as the base point.

$$\vec{v}_o = R\omega\vec{i}$$

(2) Acceleration analysis :

O as the base point.

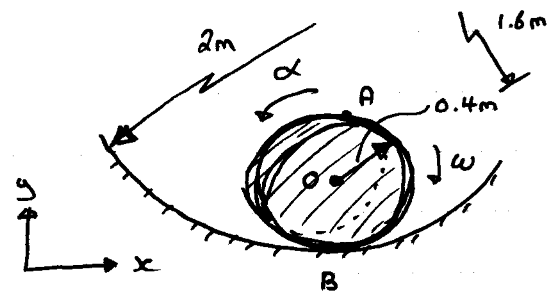
$$\vec{a}_o = \dot{\vec{v}}_o = \frac{d}{dt}(R\omega\vec{i}) = R\frac{d\omega}{dt}\vec{i} = R\alpha\vec{i}$$

$$A: \vec{a}_A = \vec{a}_o + (-\alpha \vec{k}) \times \vec{r}_{A/o} - (-\omega)^2 \vec{r}_{A/o}$$
$$\vec{a}_A = R(\alpha - \omega^2)\vec{i} - R\alpha \vec{j}$$

$$B: \vec{a}_B = \vec{a}_o + (-\alpha \vec{k}) \times \vec{r}_{B/o} - (-\omega)^2 \vec{r}_{B/o}$$
$$\vec{a}_B = 2R\alpha \vec{i} - R\omega^2 \vec{j}$$

$$C: \vec{a}_c = \vec{a}_o + (-\alpha \vec{k}) \times \vec{r}_{c/o} - (-\omega)^2 \vec{r}_{c/o}$$
$$\vec{a}_c = \cancel{2R\alpha i} \; R\omega^2 j \uparrow$$

2



The cylinder rolls on the internal cylindrical surface without slip.

$\omega = 1$ rad/s

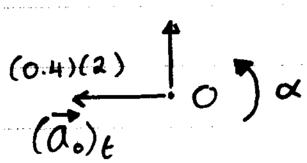
$\alpha = 2$ rad/s²

Determine : $\vec{a}_A$, $\vec{a}_B$

Solution :

(1) O as base point.

$$\vec{a}_o = (\vec{a}_o)_t + (\vec{a}_o)_n$$



$$\therefore (\vec{a}_o)_t = -0.8\vec{i} \; (m/s^2)$$

$$(\vec{a}_o)_n = \frac{V_o^2}{\rho} = \frac{[(0.4)(1)^2]^2}{1.6}$$

$$(\vec{a}_o)_n = 0.1 \; m/s^2$$

$$\therefore (\vec{a}_o)_n = 0.1\vec{j} \; (m/s^2)$$
$$\therefore (\vec{a}_o) = -0.8\vec{i} - 0.1\vec{j} \; (m/s^2)$$

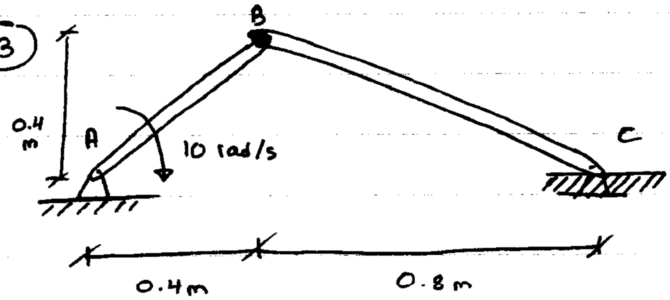
(2) $\vec{a}_A$, $\vec{a}_B$

$$\vec{a}_A = \vec{a}_o + (2\vec{k}) \times (0.4\vec{j}) - (-1)^2(0.4\vec{j})$$
$$= -1.6\vec{i} - 0.3\vec{j} \; (m/s^2)$$

$$\vec{a}_B = \vec{a}_o + (2\vec{k}) \times (-0.4\vec{j}) - (-1)^2(-0.4\vec{j})$$
$$= 0.5\vec{j} \; (m/s^2)$$

$$\vec{a}_B = \vec{a}_o + (2\vec{k}) \times (-0.4\vec{j}) - (-1)^2(-0.4\vec{j})$$
$$= 0.5\vec{j} \quad (m/s^2)$$

(3)



C in contact with the surface ;

$\omega_{AB} = 10$ rad/s and constant ;

Find α_{BC} and a_c

Solution

(1) Velocity analysis to find ω_{BC}

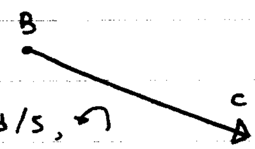
B as base point

C as point under consideration

$\vec{v}_B = \vec{\omega}_{AB} \times \vec{r}_{B/A} = 4\vec{i} - 4\vec{j} \quad (m/s)$

$\vec{v}_c = v_c \vec{i} \quad (\longrightarrow)$

$\therefore \vec{v}_c = \vec{v}_B + \vec{\omega}_{BC} \times \vec{r}_{C/B}$



Solving leads to $\omega_{BC} = 5$ rad/s, ↺

$v_c = 6$ m/s, ⟶

(2) acceleration analysis to find α_{BC} and a_c

$\vec{a}_B = (\vec{\alpha}_{AB} \times \vec{r}_{B/A} - \omega_{AB}^2 \vec{r}_{B/A}) \quad (\because \omega_{AB} = \text{const.} \ \therefore \alpha_{AB} = 0)$

$= -\omega_{AB}^2 \vec{r}_{B/A}$

$= -40\vec{i} - 40\vec{j} \quad (m/s^2)$

$\vec{a}_c = a_c \vec{i}$

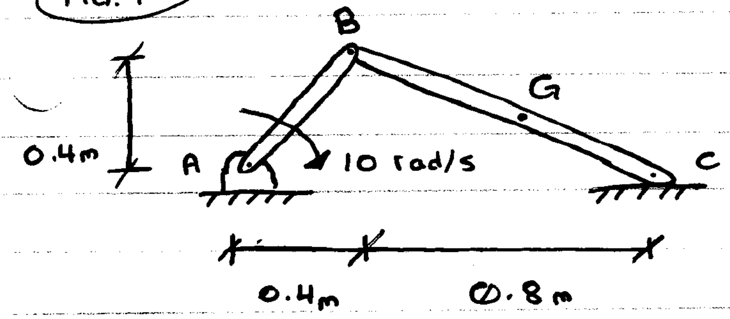
assume CCW α_{BC} $\therefore \vec{\alpha}_{BC} = \alpha_{BC} \vec{k}$

$\therefore \vec{a}_c = \vec{a}_B + \vec{\alpha}_{BC} \times r_{B/C} - \omega_{BC}^2 r_{C/B}$

$\alpha_{BC} = 37.5 \ rad/s^2$ ↺

$a_c = 45 \ m/s^2 \ \longleftarrow$

FIG: 1



C in contact with the surface.

$\omega_{AB} = 10$ rad/s (and constant)

$$\vec{a}_c = a_c \vec{i}$$

assume ccw α_{BC} $\therefore \vec{\alpha}_{BC} = \alpha_{BC}\vec{k}$

✱

$$\therefore \vec{a}_c = \vec{a}_B + \vec{\alpha}_{BC} \times \vec{r}_{c/B} - \omega_{BC}^2 \vec{r}_{c/B}$$

$$\begin{cases} \alpha_{BC} = 37.5 \text{ rad/s}^2 \circlearrowright \\ a_c = 45 \text{ m/s}^2 \leftarrow \end{cases}$$

where $\vec{r}_{c/B} = 0.8\vec{i} - 0.4\vec{j}$ (m)

$$\therefore \vec{a}_c = a_c\vec{i} = -40\vec{i} - 40\vec{j} + (\alpha_{BC}\vec{k}) \times (0.8\vec{i} - 0.4\vec{j}) - \omega_{BC}^2(0.8\vec{i} - 0.4\vec{j})$$

$$= -40\vec{i} - 40\vec{j} + 0.4\alpha_{BC}\vec{i} + 0.8\alpha_{BC}\vec{j} - 0.8\omega_{BC}^2\vec{i} + 0.4\omega_{BC}^2\vec{j}$$

$$\left.\begin{aligned} \therefore a_c &= -40 + 0.4\alpha_{BC} - 0.8\omega_{BC}^2 \\ 0 &= -40 + 0.8\alpha_{BC} + 0.4\omega_{BC}^2 \end{aligned}\right\}$$

$\therefore \omega_{BC} = 5$ rad/s $\circlearrowright$

$\therefore \alpha_{BC} = 37.5$ rad/s^2

$a_c = -45$ m/s^2

$\therefore \alpha_{BC} = 37.5$ rad/s^2, $\circlearrowright$

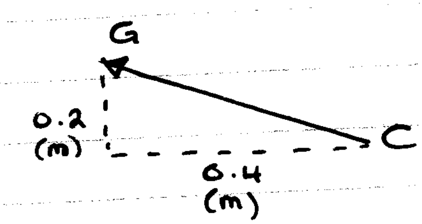
$a_c = 45$ m/s^2 $\leftarrow$

Find: α_{BC} and a_c; and $\vec{v}_G$, $\vec{a}_G$ (Fig: 1)

(3) $\vec{v}_G$ and $\vec{a}_G$

$$\vec{v}_G = \vec{v}_c + \vec{\omega}_{BC} \times \vec{r}_{G/c}$$

$$= 5\vec{i} - 2\vec{j} \text{ (m/s)}$$



↓

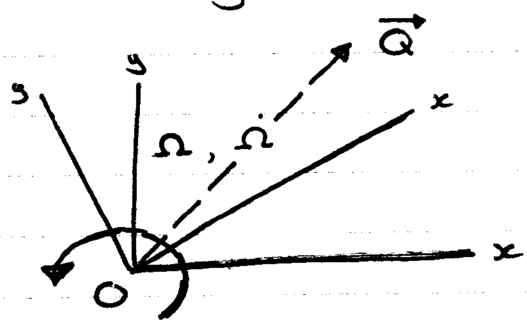
$$\vec{a}_G = \vec{a}_c + \vec{\alpha}_{BC} \times \vec{r}_{G/C} - \omega_{BC}^2 \, r_{G/C}$$
$$= -42.5\vec{i} - 20\vec{j} \quad (m/s^2)$$

§15.5 Analysing Motion w.r.t. a Rotating Frame

15.5A Rate of Change of a vector w.r.t. a Rotating Frame

15.5B Plane motion of a Particle Relative to a Rotating Frame

15.5A



OXY : Fixed (translating)

Oxy : rotating with Ω, $\dot{\Omega}$

"O" as base pt.

$\vec{v}_o$, $\vec{a}_o$: non-zero

$(\dot{\vec{Q}})_{Oxy}$ (known)

$$(\dot{\vec{Q}})_{OXY} = (\dot{\vec{Q}})_{Oxy} + \vec{\Omega} \times \vec{Q}$$

15.5B:

At velocity level, P is the particle under consideration

$$\vec{v}_P = \vec{v}_o + \vec{\Omega} \times \vec{r} + v_{rel}$$

$\vec{v}_P$: Velocity of P

$\vec{v}_o$: Velocity of O

$\vec{r} = \vec{r}_{P/O}$: vector drawn from O to P' where P' occupies the same location as P, at the given instant / position / configuration.

$\vec{v}_{rel}$: Velocity of P relative to Oxy.

at acceleration level :

$$\vec{a}_P = \vec{a}_O + \dot{\vec{\Omega}} \times \vec{r} + \vec{\Omega} \times (\vec{\Omega} \times \vec{r}) + 2\vec{\Omega} \times \vec{v}_{rel} + \vec{a}_{rel}$$

$\vec{a}_P$, $\vec{a}_O$: accelerations of P and O

$\dot{\vec{\Omega}} \times \vec{r}$: tangential component of P' rotating about O.

$\vec{\Omega} \times (\vec{\Omega} \times \vec{r})$: normal component of P' rotating about O.

if $\vec{r}$ is on XY, or xy plane

then $\vec{\Omega} \times (\vec{\Omega} \times \vec{r}) = -\Omega^2 \vec{r}$

$\vec{a}_{rel}$: acceleration of P relative to Oxy.

and $2\vec{\Omega} \times \vec{v}_{rel} = \vec{a}_c$ is the Coriolis acceleration; it measures the difference in accelerations of P as measured from Oxy and from OXY.

Problem - Solving :

$\vec{v}_{rel}$, $\vec{a}_{rel}$: known 15.172 (*)

$\vec{v}_{rel}$, $\vec{a}_{rel}$: unknown 15.176 (*)

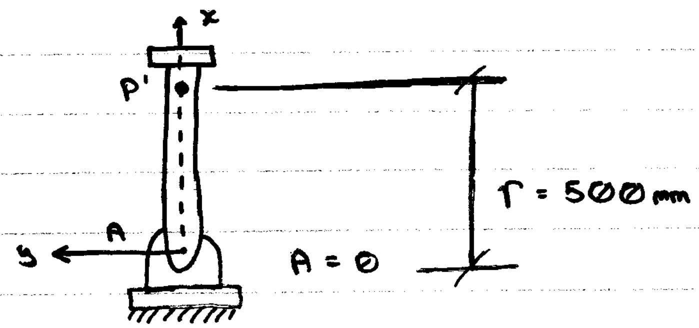
Problem 15.172 :

$\omega = 20$ rpm , and constant , $\omega \Rightarrow \Omega$,

$\theta = 0°$, $r = 250$ mm

$\theta = 90°$, $r = 500$ mm

u = Const , outward . $u = v_{rel}$



$\vec{r} = \vec{r}_{P/O}$

$= 500 i$ (mm)

$\Omega = 20$ rpm

$= 2.094$ rad/s

$\therefore \dot{\Omega} = 0$

$\therefore a_{rel} = 0$

$v_{rel} = u =$ const.

time taken for Oxy , or AB, to rotate 90°

$\therefore t = \frac{\pi/2}{\Omega} = 0.75$ (s)

$\therefore v_{rel} = \frac{500 - 250}{0.75} = 333.3$ mm/s ↑

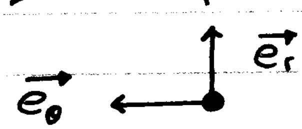
$\therefore \vec{v}_{rel} = 333.3\, i$

$\vec{a}_{rel} = \vec{0}$

$$\vec{a}_P = \underbrace{\cancel{\vec{a}_O}}_{=0} + \underbrace{\cancel{\vec{\dot{\Omega}}}}_{=0} \times \vec{r} - \Omega^2 r + 2\vec{\Omega} \times \vec{v}_{rel} + \underbrace{\cancel{\vec{a}_{rel}}}_{=0}$$

$$= -2192\,\vec{i} + 1396\,\vec{j} \quad (\text{mm/s}^2)$$

as a particle by $\vec{e}_r - \vec{e}_\theta$ components.



$\vec{i} \Leftrightarrow \vec{e}_r$

$\vec{j} \Leftrightarrow \vec{e}_\theta$

$r = 500$ mm

$\dot{r} = v_{rel}$, $\ddot{r} = 0 = a_{rel}$

$\dot{\theta} = \Omega$, $\ddot{\theta} = \dot{\Omega}$

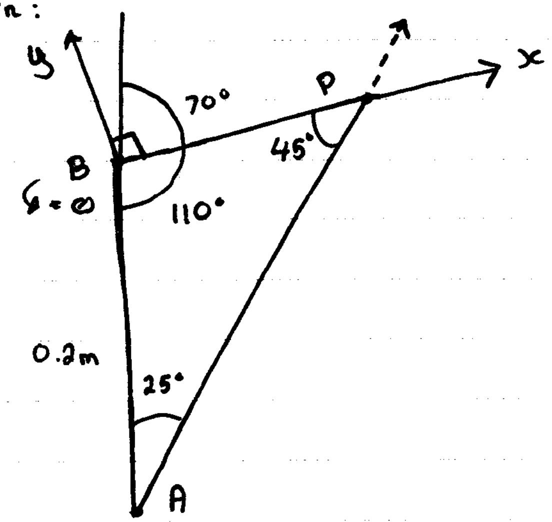
Problem 15.176:

$\omega_{AP} = 5\,rad/s$, CCW

$\alpha_{AP} = 2\,rad/s^2$, CW

Find: $\omega_{BD} = \Omega$, $\alpha_{BD} = \dot{\Omega}$

Sol'n:



by law of Sines

$AP = 0.2658\,m$

$BP = 0.1195\,m$

$\vec{r}_{P/A} = (0.2658\,m)\cdot(\cos 45°\,\vec{i} + \sin 45°\,\vec{j})$

$\vec{r}_{P'/O} = (0.1195)\,\vec{i}$

(1) Velocity analysis:

P on AP: $\vec{v}_P = \vec{\omega}_{AP} \times \vec{r}_{P/A}$

$= -0.9397\,\vec{i} + 0.9397\,\vec{j}$ (m/s)

P slider w.r.t. BD

$\vec{v}_P = \cancel{\vec{v}_O}^{=0} + \vec{\Omega} \times \vec{r} + \vec{v}_{rel}$

$= (\Omega\,\vec{k}) \times (0.1195\,\vec{i}) + v_{rel}\,\vec{i}$

$\therefore -0.9397\,\vec{i} + 0.9397\,\vec{j}$

$= 0.1195\,\Omega\,\vec{j} + v_{rel}\,\vec{i}$

$\therefore \Omega = 7.864\,rad/s$, ↺

$V_{rel} = 0.9397\,m/s$ ←

or $\vec{\Omega} = 7.864\,\vec{k}$, $\vec{v}_{rel} = -0.9397\,\vec{i}$

(acceleration analysis)

(2) P on AP

$\vec{a}_p =$

P as slider w.r.t. BD

$$\vec{a}_P = \vec{a}_o + \dot{\vec{\Omega}} \times \vec{r} - \Omega^2 \vec{r} + 2\vec{\Omega} \times \vec{v}_{rel} + \vec{a}_{rel}$$