Random Inputs
In experiments and real-life: noise is always present

Numerical simulation: numerical noise exists

Checking for random inputs means to make sure that large non-periodic response does not arise from small input noise.

Time History
Usually the first clue of chaotic vibrations is that the $x(t) - t$ plot (or the $v(t) - t$ plot) shows no visible pattern or periodicity.

Time history test is not foolproof.

The system may exhibit transient chaos or intermittent chaos.

Figure 2-20 Sketch of the time history for intermittent-type chaos.
Courtesy of Chaotic vibrations An Introduction for Applied Scientists and Engineers, Moon, 2004.

Phase Portrait
Trajectory: the curve traded out by points $x(t), v(t)$.

Periodic vibrations: the trajectory is a closed curve.

Chaotic vibrations: the trajectory never closes or repeats, eventually filling up a section of the plane.

Again, the phase portrait alone is not foolproof.

In fact, it is believed that the Poincare map, considered by some the modified phase portrait, should be used instead of the phase portrait, as the Poincare map yields more relevant information.

Pseudo-phase Portrait
In physical experiments or real-life observations, there are times when only one measurement (i.e., one signal) s available. A time-delayed pseudo-phase portrait can be used as an alternative.

For a SDOF system with the signal $x(t)$, one plots the following pairs, $x(t),\ x(t+\sigma)$, where σ is a fixed time constant.

Typically, for the same system, the phase portrait and the pseudo-phase portrait will reveal the same characteristics. Specifically, if the system has a periodic vibration, then both portraits will show closed trajectories. If the system's motion is chaotic, the portraits will show trajectories that do not close or repeat.

The choice of σ is not crucial, expect to avoid the natural or forcing period.

When state variables are more than three, high dimensional pseudo-phase portrait can be constructed using multiple delays. For example, points such as $x(t)$, $x(t + 2\sigma)$ may be plotted in a 3D pseudo-phase portrait.

Fourier Spectrum

One clue of chaotic vibrations is the appearance of broad spectrum of frequencies when the input is of a single frequency.

This broad band spectrum characteristic is more prevalent in low dimension system with degrees of freedom of up to three.

Fourier spectrum is also useful in detecting subharmonics which are a precursor to chaos.

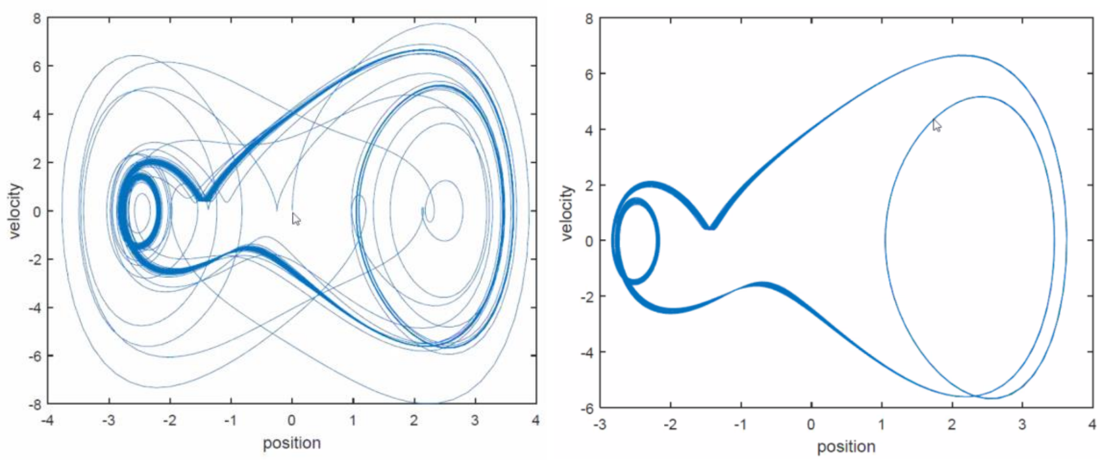
Subharmonic: if ω is the dominant frequency, then ω/n (n is an integer) is a subharmonic.

For example, Duffing oscillator

$$x + \delta x + \alpha x + \beta x^3 = f_0 \cos(\omega t)$$

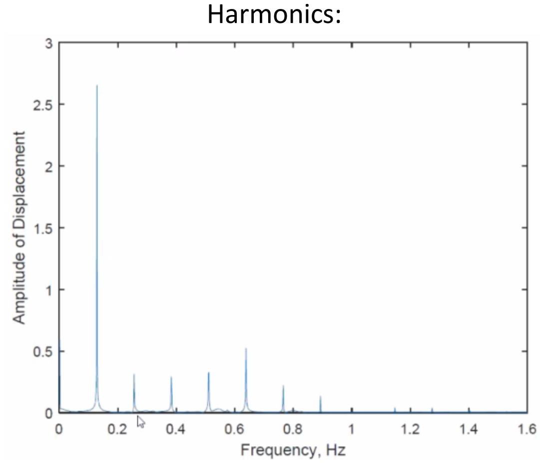
With $\delta = 0.18$, $\alpha = \beta = 1$, $\omega = 0.8$, and $f_0 = 19, 22$ and 22.5

Note that the forcing frequency is $0.127\ Hz$.



Phase portrait (left) and phase portrait with transient removed (right)

Harmonics:



Differential Equations (DEs) and Maps

Flows refer to the differential equations (Des). They are the responses of a dynamic system in continuous time, as represented by the trajectories in phase space.

Maps are algebraic rules for completing the next state of a dynamic system in discrete time.

For example, the ode45 solver in MATLAB is a set of algebraic rules: an explicit Runge-Kutta (4,5) formula with the Dormand-Prince pair.

Therefore, phase portraits that are based on computational responses and plotted as points instead of curves/lines, are in fact maps, or solution maps, to be precise.

Example: The logistic DE is:

$$\dot{x} = x(1 - x)$$

Using the Euler's method to solve the DE, one has:

$$x_{n+1} = x_n + hx_n(1 - x_n) = \lambda x_n(1 - x_n)$$

With h being the time-step size, and $\lambda = 1 + h$. The logistic map is then,

$$x_{n+1} = \lambda x_n(1 - x_n)$$

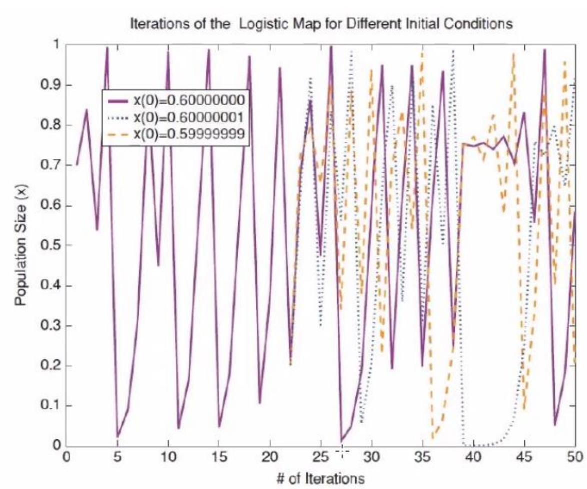


Fig. 2 Sensitivity to initial conditions in the logistic map

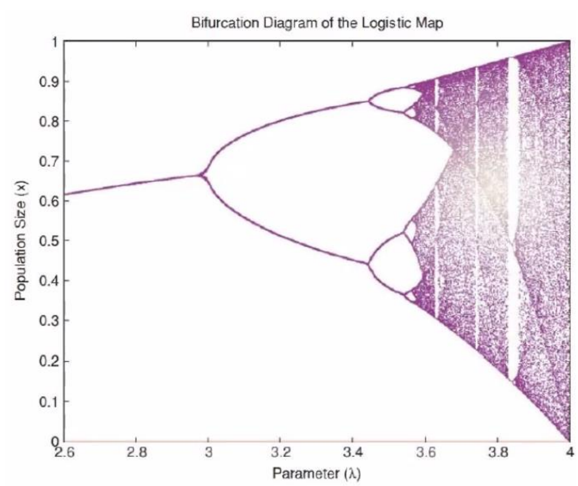


Fig. 3 Bifurcation diagram for the logistic map

Next-Return Maps; Next-Amplitude Maps

The rules for next-return maps are, for example,

- Position is zero
- Velocity is zero
- Position of x is zero and velocity of y is zero, or
- When the excitation has a phase angle of, say, $\omega t = 2n\pi + \pi/2$, or 90°, $450°$, $810°$, …

Poincare map is a next-return map.

Next-amplitude maps employ rules such as a state variable reaches maximum (or minimum for that matter).

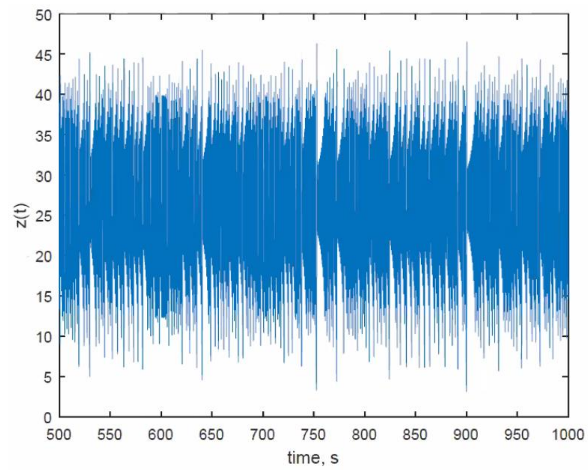
Next-amplitude maps were owing to Lorenz (of the Loren attractor fame). He intuitively constructed the first next-amplitude map, it is believed.

Example: The Lorenz attractor is:

$$\begin{aligned}\dot{x} &= \sigma(y - x)\\ \dot{y} &= \rho x - y - xz\\ \dot{z} &= xy - \beta z\end{aligned}$$

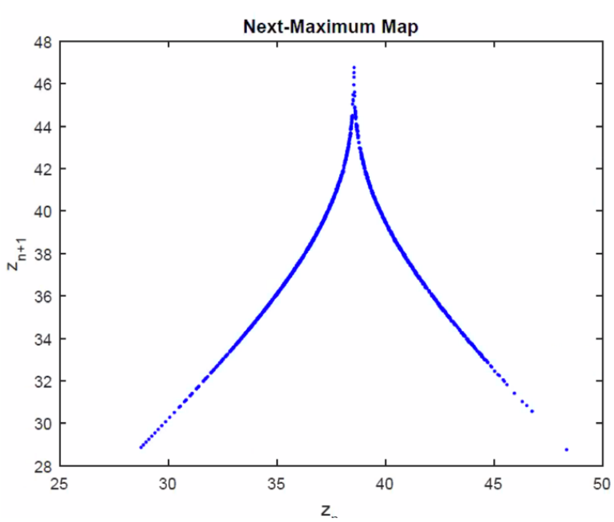
Chaos is observed when $\sigma = 10$, $\rho = 28$, $\beta = 8/3$.

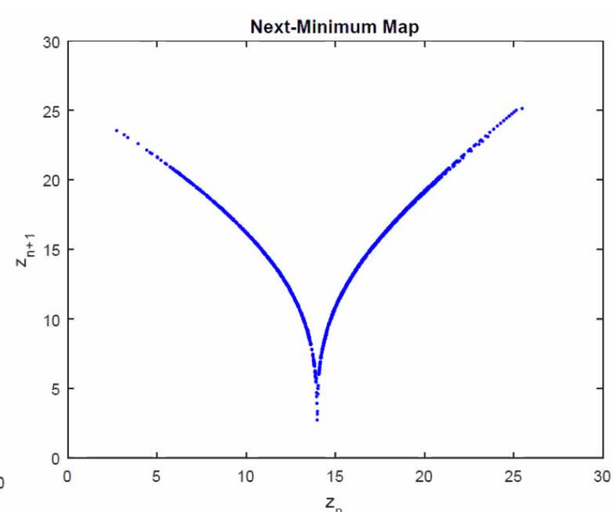
<u>Time History</u> (run it for a while, remove transient – which could just be the beginning of the signal):



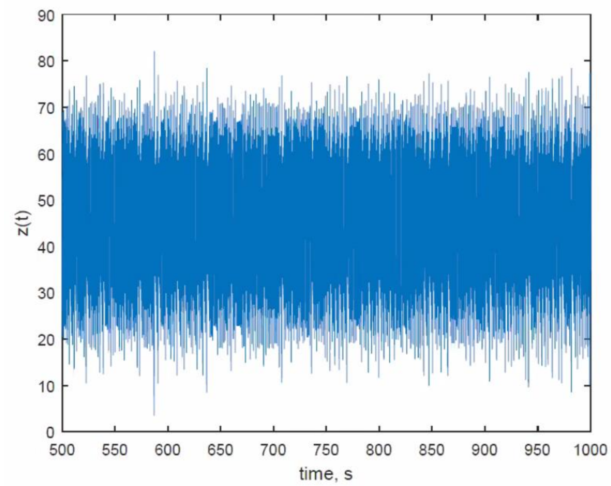
Strong indication of chaos because there is no discernable pattern by inspection.

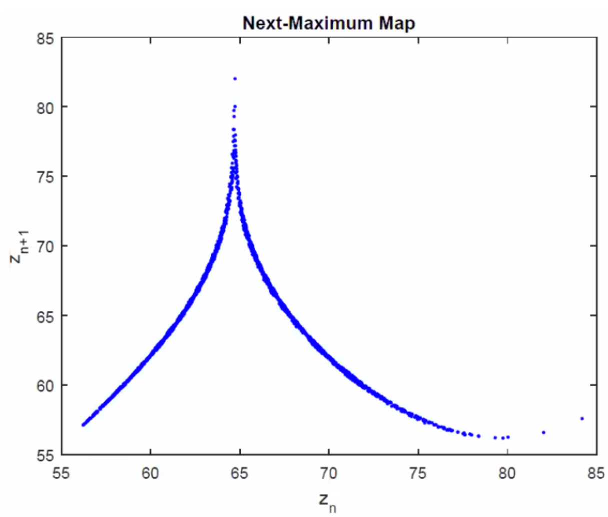
<u>Next-Maximum and Next-Minimum Map</u>

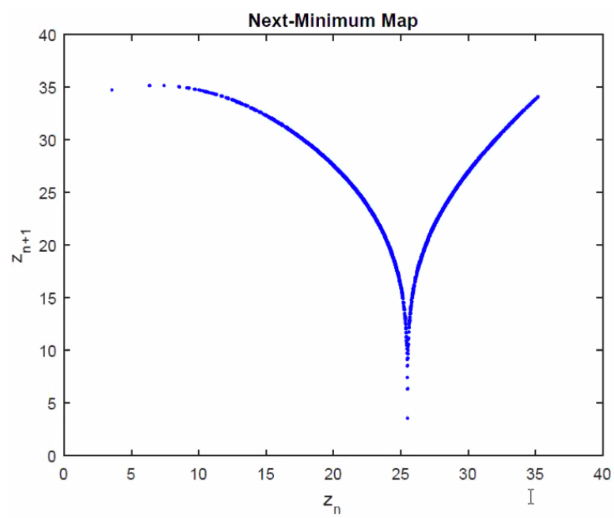




Another set of parameters – Chaos is observed when $\sigma = 28$, $\rho = 46.92$, and $\beta = 4$.







Autonomous DEs and Non-autonomous DEs

- An autonomous differential equation (DE) is one such that $\ddot{x} = f(x, \dot{x})$.
- A non-autonomous DE is one such that $\ddot{x} = f(x, \dot{x}, t)$

Basically, whether the function depends on time is the difference between autonomous and non-autonomous DEs.

NOTE:

(1) Although a non-autonomous DE indicated forced oscillation, an autonomous DE does not suggest free oscillation, because the excitation may be a constant force, for example.

(2) For non-autonomous DEs, t is a state variable, in addition to position and velocity.

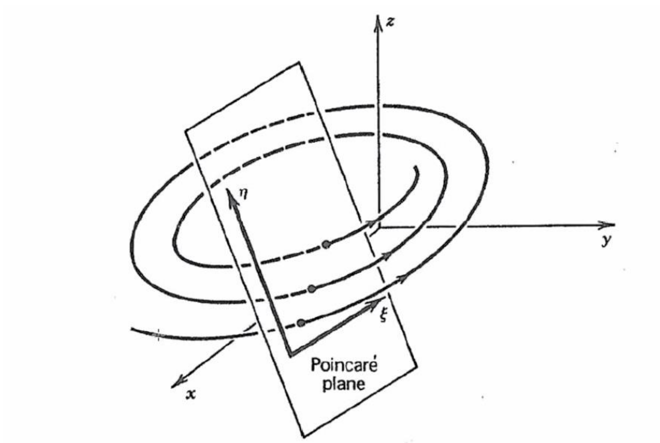
(3) Therefore, the dimension of the state space, n, of $\ddot{x} = f(x, \dot{x})$ is 2; but it is 3 for $\ddot{x} = f(x, \dot{x}, t)$, where x is a scalar.

Poincare Maps

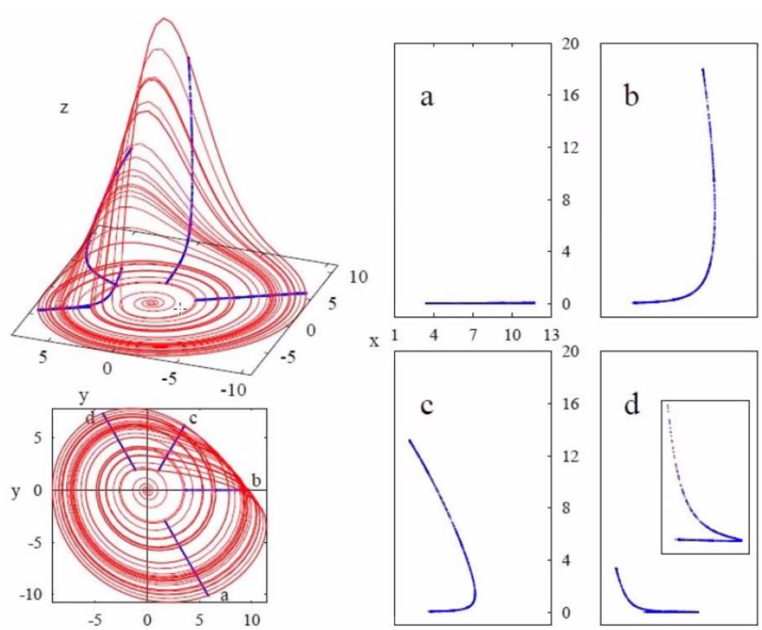
Poincare map is a classical technique for analyzing dynamic systems, conceived by Poincare. When an n −dimensional continuous-time system is replaced with an $(n-1)$ − dimensional discrete-time map, the result is the so-called Poincare map.

In other words, the continuous-time response is sampled according to certain rules. The rules are such that the dimension of the map is one less the dimension of the DE.

The rules can be reflected upon by the Poincare section, the choice of which is different for autonomous and non-autonomous DEs.



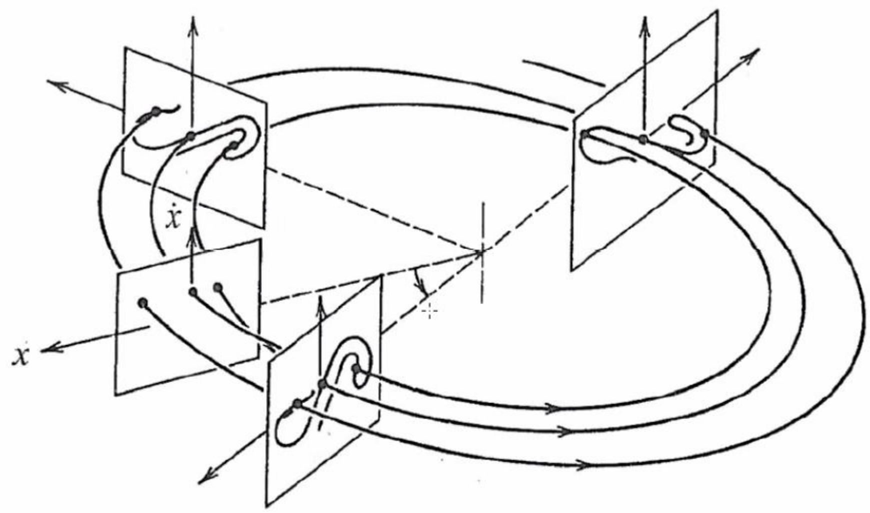
Courtesy of *Chaotic vibrations An Introduction for Applied Scientists and Engineers*, Moon, 2004.



Poincare Maps for Non-Autonomous DEs

Since t is a dimension, a convenient chouce of Poincare section would be $t = kT + t_0$, where T is the forcing/driving period, and $t_0 < T$ is an arbitrary time.

In other words, of every driving period T, one discrete point is sampled and plotted. The collection of such discrete points forms the Poincare map of the oscillator represented by non-autonomous DEs.



Courtesy of *Chaotic vibrations An Introduction for Applied Scientists and Engineers*, Moon, 2004.

Consider the forced vibration of an oscillator given by the following ODEs, where $y = \dot{x}$

$$\dot{x} = y$$
$$\dot{y} = F(x, y) + f_0 \cos(\omega t + \Phi_0)$$

By introducing a variable $z = \omega t + \Phi_0$, the first-order ODEs for the non-autonomous system is,

$$\dot{x} = y$$
$$\dot{y} = F(x, y) + f_0 \cos(z)$$
$$\dot{z} = \omega$$

The samples for the Poincare map may then be collected at $z = 0$, or Φ_0, or any angle within $360°$.