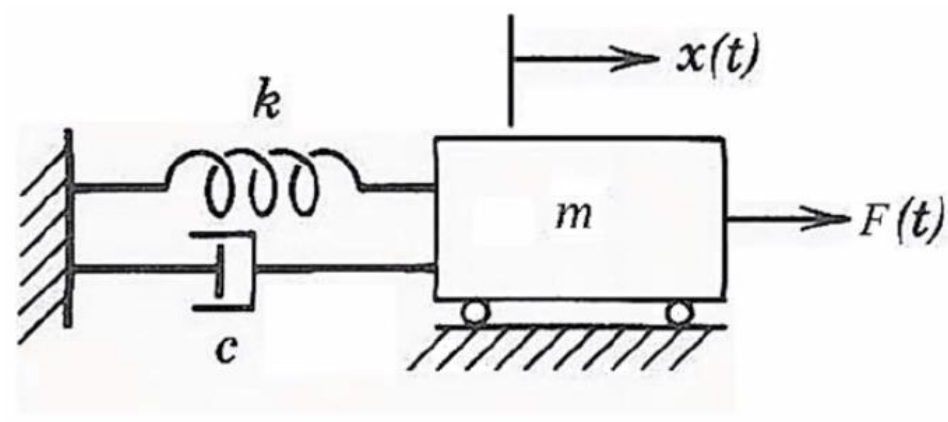
4.2 van der Pol Oscillator

Back to the linear oscillator,



This time the damping is non-linear. This results in the so-called van der Pol oscillator:

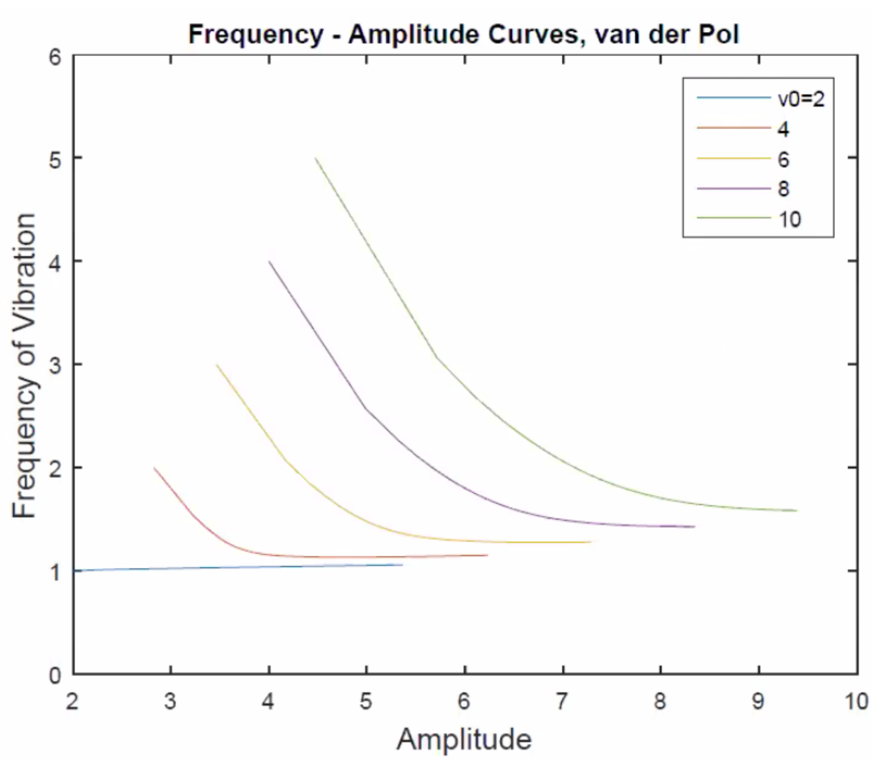
$$\ddot{x} - \delta\dot{x}(1 - \gamma x^2) + \alpha x = f(t)$$

Van der Pol oscillators are used when the so-called stick-slip phenomena, aero-elastic flutter, and biological phenomena are involved.

4.2.1 Unforced van der Pol Oscillator

$$\ddot{x} - \delta\dot{x}(1 - \gamma x^2) + \alpha x = 0$$

Similar to Duffing oscillator, frequency of vibration depends on amplitude of vibration.



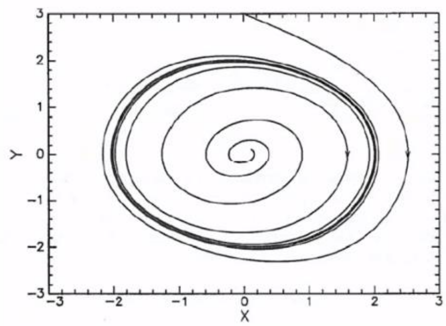
Two situations to focus on: limit cycle and relaxation oscillator.

They occur as long as one initial condition (the initial displacement or initial velocity) is not zero.

They are periodic response of the oscillator, signifying sustained vibration.

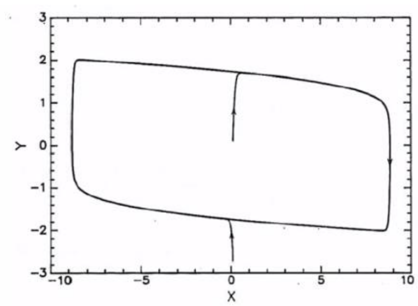
Differences?

- Limit cycle takes place when $\delta/\sqrt{\alpha}$ is small
- Relaxation oscillation occurs when $\delta/\sqrt{\alpha}$ is large



Limit Cycle Solution for the van der Pol Oscillator

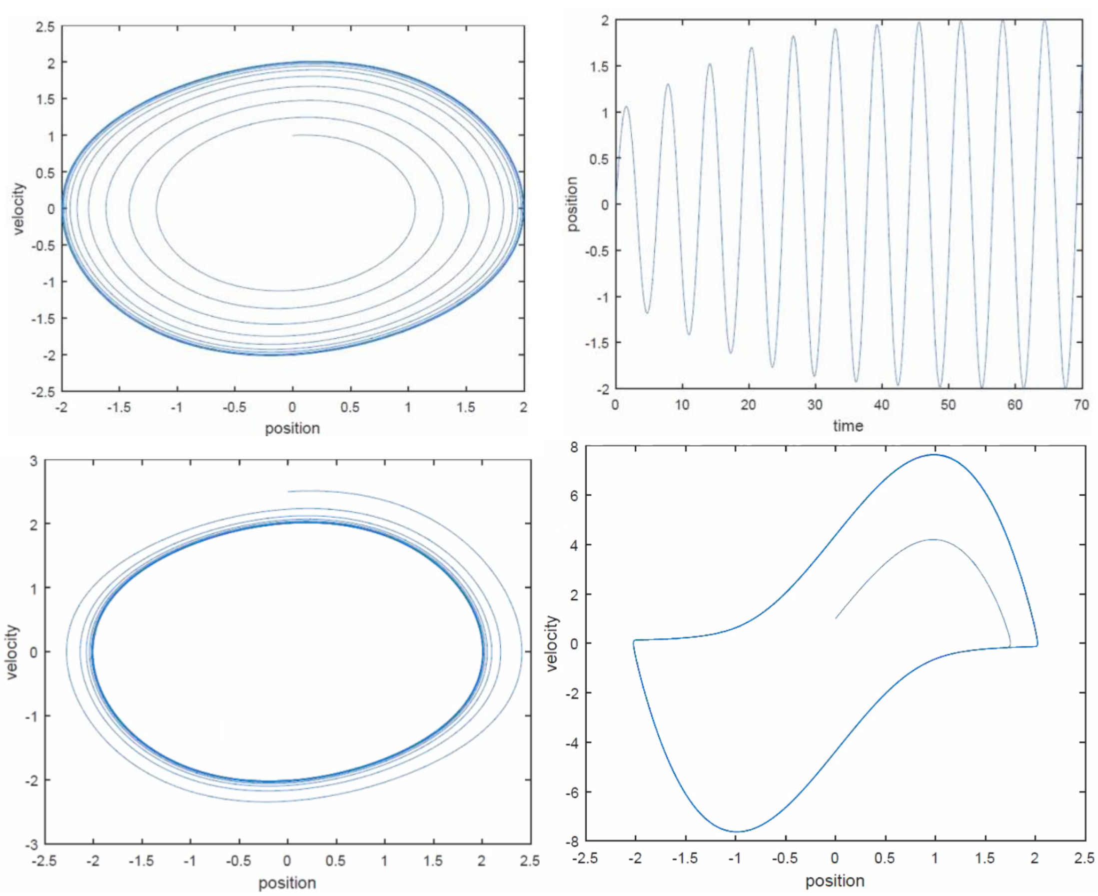
Courtesy of *Chaotic vibrations An Introduction for Applied Scientists and Engineers*, Moon

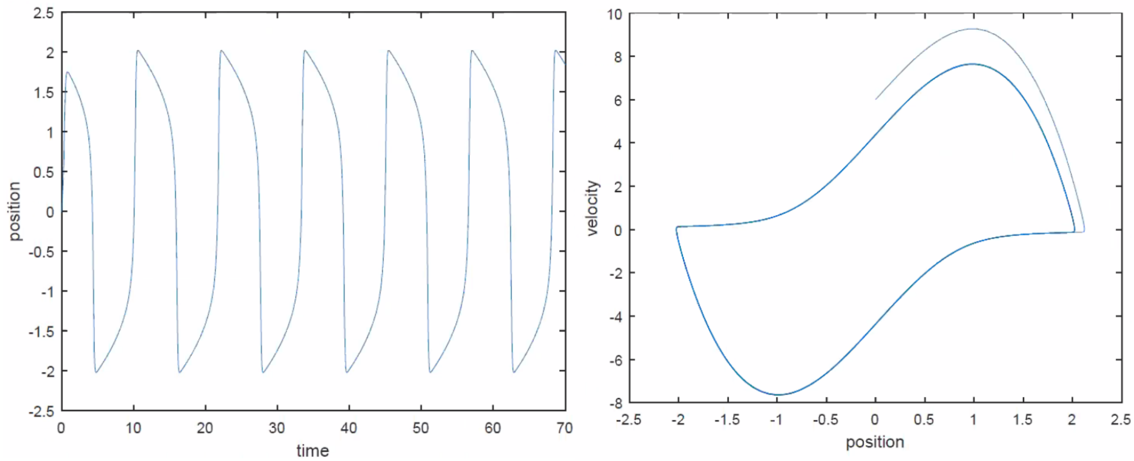


Relaxation Oscillation for the van der Pol Oscillator

Courtesy of *Chaotic vibrations An Introduction for Applied Scientists and Engineers*, Moon

Simulations from Dr. M. Liu (for reference):

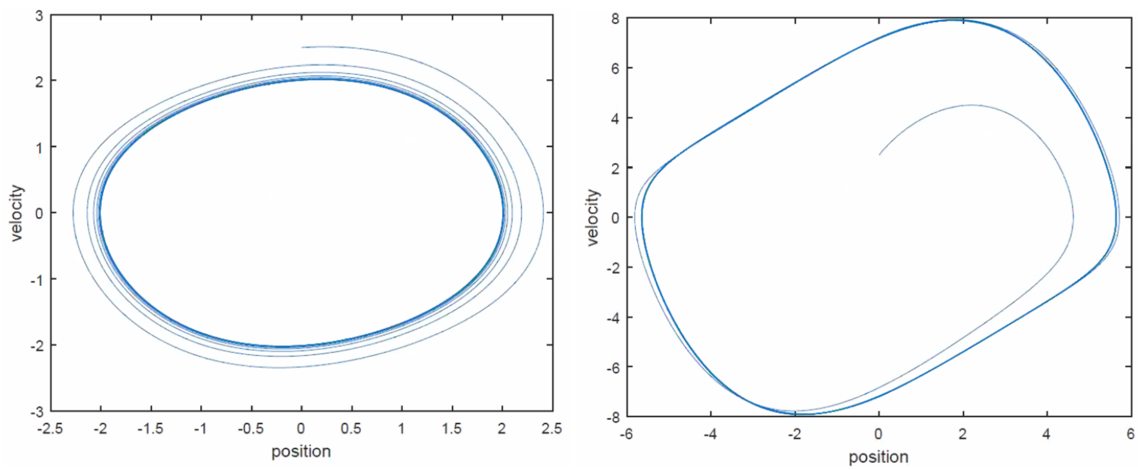




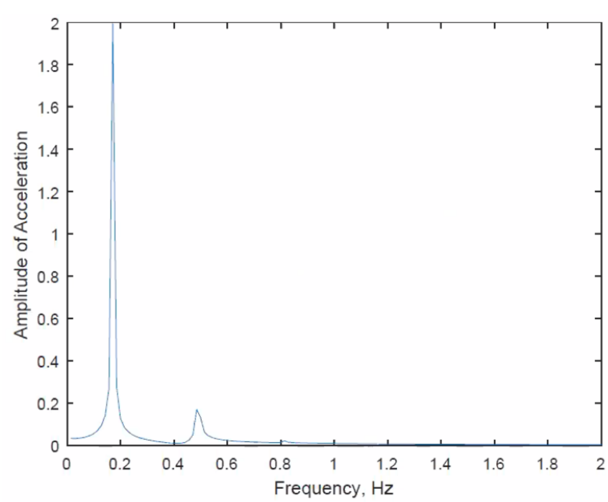
4.2.2 Forced van der Pol Oscillator

$$\ddot{x} - \delta\dot{x}(1 - \gamma x^2) + \alpha x = f_0 \cos(\omega t)$$

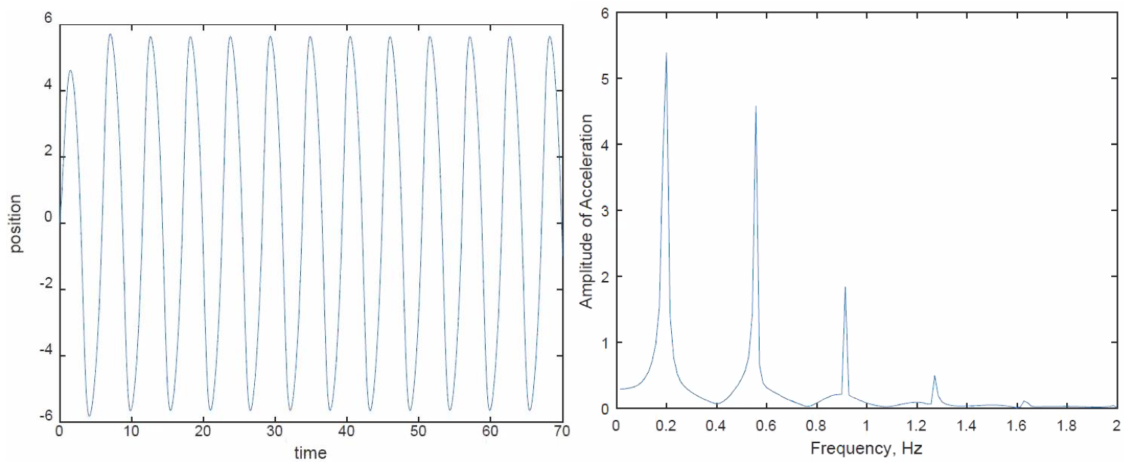
If the forcing is strong, and the forcing frequency and limit cycle are close to each other, the excitation entrains or enslaves the limit cycle. This phenomenon is known as entrainment.



Free vibration showing the limit cycle (left) and Forced vibration showing entrainment (right)



Simulations from Dr. M. Liu (for reference):



4.2.3 Typical analytic Approaches for van der Pol Oscillator:

The method of averaging

Perturbation methods (Lindstedt-Poincare methods, multiple time scales, …)

Harmonic balance method

…

Checklist for Identifying Chaotic Vibrations

- Identify nonlinear elements in the system
- Check for (so as to rule out) random inputs
- Observe the time history of a system variable
- Examine the phase portraits
- Examine the Fourier spectrum of system variables
- Examine the Poincare map of a state variable

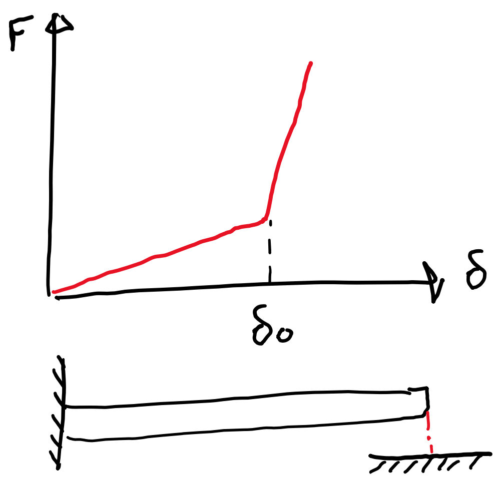
Just a single item from the checklist is not sufficient for conclusively identifying chaos. Typically, a combination is required.

There are more advanced techniques or measures, such as bifurcation diagram, Lyapunov exponents and fractal dimension.

Nonlinear elements

- Nonlinear elasticity, such a nonlinear springs, contact between elastic objects, and so on
- Nonlinear damping, such as stick-slip (dry friction)
- Most systems with fluids
- Nonlinear boundary conditions
- Backlash, play, or bilinear springs

What's a bilinear spring?



non-linear
≠ k, in essence it's piecewise

$\begin{cases} k_1 < \delta_0 \\ k_2 > \delta_0 \end{cases}$

Where have chaotic vibrations been observed?

- Vibrations of buckles elastic structures
- Mechanical systems with play or backlash
- Aeroelastic problems
- Wheel-rail dynamics
- Large-amplitude vibrations of structures such as beams, plates, and shells
- Systems with sliding friction
- Rotation and gyroscopic force
- Feedback control devices

The common thread is strong non-linearity. Other factors include electric magnetic and fluid related forces, and nonlinear boundary conditions.

Random inputs
In experiments and real-life: noise is always present

Numerical simulation: numerical noise exists

Checking for random inputs means to make sure that large non-periodic response does not arise from small input noise.