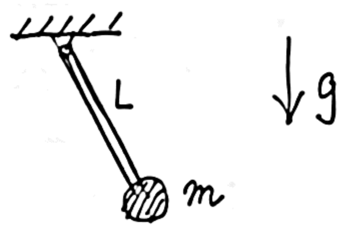
2.11 Small Angle or Displacement Assumption

Structural element (shaft, rod, beam, etc.) as a spring:

If the material is metal, the linear spring assumption is valid.

What if the material is, say, plastic or composites, or others? The linear assumption needs to be thoroughly explained. For example, plastic is nonlinear even when deformation is small.

Example 2.23
(a pendulum)



Without any assumption, the E.O.M. is:

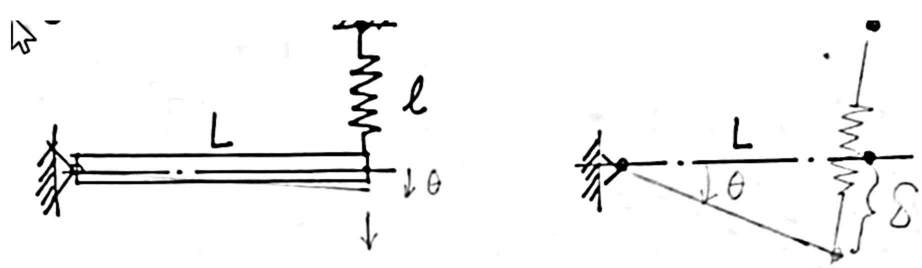
$$\ddot{\theta} = \frac{g}{L}\sin\theta = 0$$

Which is nonlinear. The Taylor expansion on $\sin\theta$ is:

$$\sin\theta = \theta - \frac{1}{6}\theta^3 + \frac{1}{120}\theta^5 \ldots$$

If the first two terms are kept, the result is the softened Duffing oscillator.

Figure 2.42



L: length of the bar
l: natural length of the spring

Under small angle assumption, the deformation of the spring is $\delta = L\theta$.

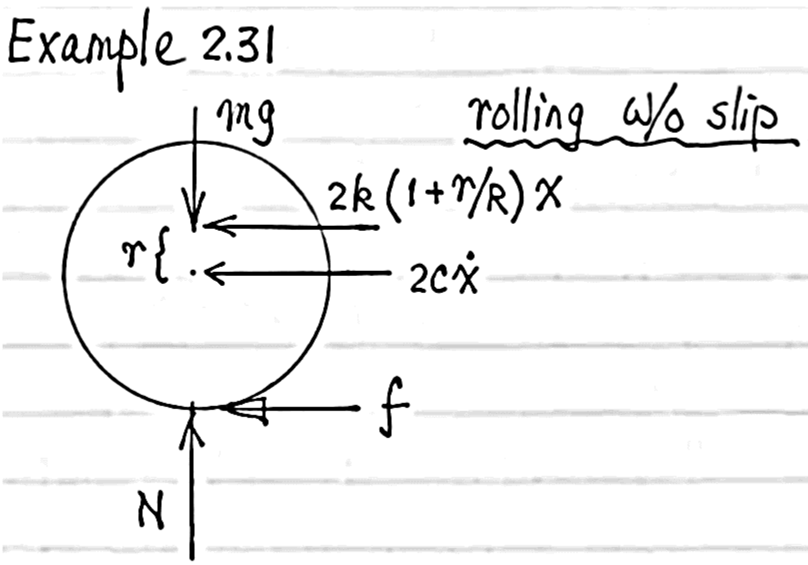
If θ is large, then the deformation will be:

$$\delta = \sqrt{(L - \mathrm{L}\cos\theta)^2 + (l + Lsin\theta)^2} - l \qquad (2.76)$$

Textbook Corrections (for this Chapter)

A2
Table D.2, case 5

$$C_2 = \frac{z_1}{2}\left(1 - \frac{a}{z_1}\right)\left[1 - \left(1 - \frac{a}{z_1}\right)^2 u(z_1 - a)\right]$$



$E.O.M$ is:

$$\left(\frac{I}{R^2} + m\right)\ddot{x} + 3c\dot{x} + 2k\left(1 + \frac{r}{R}\right)^2 x = 0$$

(c) is correct

Rolling w/ slip

Spring force is:

$$2k(x + r\theta)$$

And:

$$\theta \neq x/R$$

Friction is $f = \pm\mu mg$

$\therefore$ it's a 2 DOF system; DOFs are x and θ

$$\begin{cases} m\ddot{x} + 2c\dot{x} + 2kx + 2kr\theta = \pm\mu mg \\ \frac{1}{2}mR^2\ddot{\theta} + 2k(x + r\theta)r = \pm\mu mgR \end{cases}$$

Rolling w/ slip but keeping SDOF.

Rolling w/ slip but keeping SDOF

Spring force is:

$$2k\left(1+\frac{r}{R}\right)x$$

$$\therefore m\ddot{x}+2c\dot{x}+2kx+2k\left(1+\frac{r}{rR}\right)x=\pm\mu mgR$$

Then:

$$\frac{1}{2}mR^2\ddot{\theta}=-2kr\left(1+\frac{r}{R}\right)x\pm\mu mgR$$

(g) should read:

$$35\ddot{x}+2000\dot{x}+3.422(10^5)x$$
$$=\begin{cases}-85.84 & ; \quad \dot{x}>0\\ +85.84 & ; \quad \dot{x}<0\end{cases}$$

The following presentation is based on *Chaotic vibrations An Introduction for Applied Scientists and Engineers*, F.C. Moon, Wiley, 2004.

Definition of chaos

Chaos is one of the scientific concepts that enter the popular culture.

In the non-scientific world, chaos means without pattern, out of control.

In the scientific world, there is no universally agreed definition of chaos.

However, a widely accepted working definition is:

> *Chaos is the aperiodic time-asymptotic behavior in a deterministic system which exhibits sensitive dependence on initial conditions.*

Misconceptions

- *Is chaotic vibration a random vibration?*

Random vibration means that the true values of input forces and/or systems parameters are unknown. In other words, probability and its distribution are needed for solving random motion.

Example: earthquakes, environmental sounds

Chaotic vibration is a deterministic phenomenon. The key characteristics is the sensitivities.

- *What is the necessary condition for chaotic motions?*

Nonlinearity in the system.

However, not all nonlinear systems will be chaotic.

- *Is chaotic motion associated with high-dimension, and/or high-order differential equations (DEs)?*

Not necessarily.

For example, the three well-studied chaotic systems are,

Duffing oscillator:

$$\ddot{x} + 2\gamma\dot{x} + \alpha x + \beta x^3 = f(x)$$

van der Pol (VDP) oscillator:

$$\ddot{x} - \gamma\dot{x}(1 - \beta x^2) + \alpha x = f(t)$$

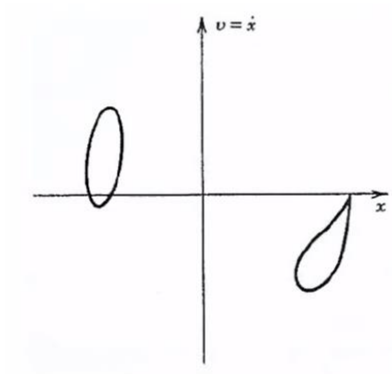
Lorenz attractor:

$$\begin{aligned} \dot{x} &= \sigma(y - x) \\ \dot{y} &= \rho x - y - xz \\ \dot{z} &= xy - \beta z \end{aligned}$$

These well-known systems will be examined later during the course of this course (EMEC 5671 FC).

Three main elements in the working definition of chaos

1. *Aperiodic time-asymptotic behavior.* This implies the existence of phase-space trajectories that do not settle down to fixed points or periodic orbits.



Courtesy of Chaotic vibrations An Introduction for Applied Scientists and Engineers, F.C. Moon (Figure 2.10)

2. *Deterministic*. This implies that the equations of motion of the system possess no random inputs or parameters. As a result, the irregular behavior of the system arises from non-linear dynamics.

3. *Sensitive dependence on initial conditions.* This implies that nearby trajectories in phase-space separate exponentially fast in time: *i.e.*, that the system has a positive Lyapunov exponent.

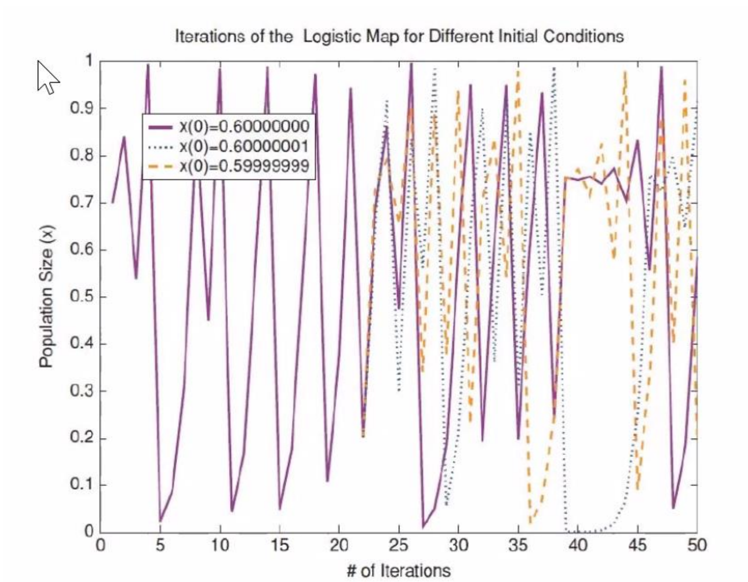


Fig. 2 Sensitivity to initial conditions in the logistic map

Courtesy of Chaos and Its Computing Paradigm, D. Kuo, IEEE Potentials, April/May 2005, pp 13-15

There are systems whose dynamic responses are sensitive to *parametric changes in system parameters.*

Bifurcation is a means to investigate the effects of parametric changes on a system's dynamics, and if parametric changes lead to chaos.

What is bifurcation? The definition will be given later.

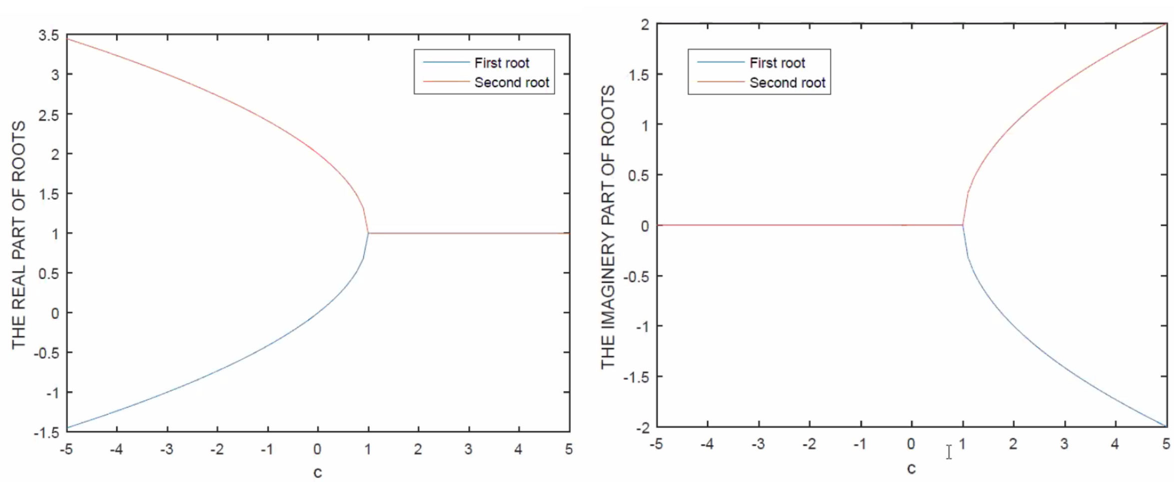
But as a simple example, lets consider the roots of a quadratic equation $ax^2 + bx + c = 0$

$$(where\ a \neq 0)$$

(1) Two identical roots, if $\sqrt{b^2 - 4ac} = 0$
(2) Two distinct real roots, if $\sqrt{b^2 - 4ac} > 0$
and
(3) Two complex roots, if $\sqrt{b^2 - 4ac} < 0$

Let $a = 1, b = 2, c = [-5, 5]$
$c < 1$, two distinct real roots
$c = 1$, two identical real roots
$c > 1$, two complex roots

Bifurcation diagram:



Cascade bifurcation:

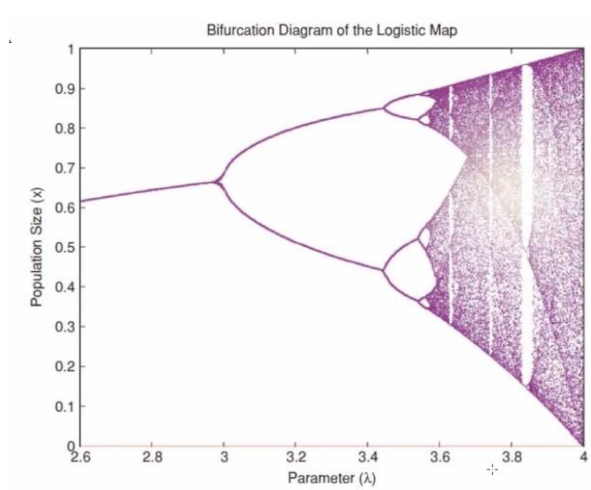


Fig. 3 Bifurcation diagram for the logistic map

Courtesy of Chaos and Its computing paradigm, D. Kuo, IEEE Potentials, April/May 2005, pp. 13-15

Why should engineers study chaos?
To know the source of chaos. Chaos can arise in low-order deterministic nonlinear systems.

To learn the tools within the chaos theory to (1) detect chaotic vibrations in physical systems, and (2) to quantify the chaos.

The know the flipped side of chaos suppression. Chaos can be suppressed, but some feedback control forces are known to cause chaos.

To incorporate chaos in design. Engineers have been using factor of safety to account for unknowns in engineering design. The unknowns can be caused by noises which in turn can lead to long term unpredictability.

Why the title *nonlinear vibrations and chaos?*
The necessary condition for chaotic vibration is nonlinearity in the system.

However, not all nonlinear vibration is chaotic.

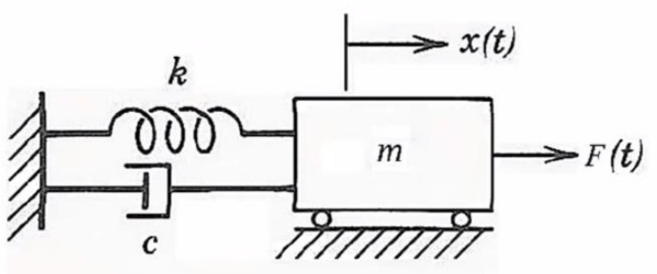
Why linear vibrations first?
Linear vibrations are the foundation for nonlinear vibrations.

The emphasis has been on the modeling of SDOF linear vibration systems; i.e., the mass, spring, damper, and excitation. Another emphasis was the use of energy (kinetic, potential, …) and work done (by non-conservative forces in particular) in problem-solving.

It is also important to understand the simplifications or assumptions made to obtain linear systems.

Examples of Nonlinear Vibrations

1. Duffing Oscillator



Revised from Chaotic vibrations An introduction for Applied Scientists and Engineers, Moon

Starting with the classical mass-spring damper oscillator subject to a periodic force, the equation of motion is, *after normalization*,

$$\ddot{x} + \delta\ddot{x} + \alpha x = f(t)$$

Now considering a cubic (hence nonlinear) spring,

$$\ddot{x} + \delta\dot{x} + \alpha x + \beta x^3 = f(t)$$

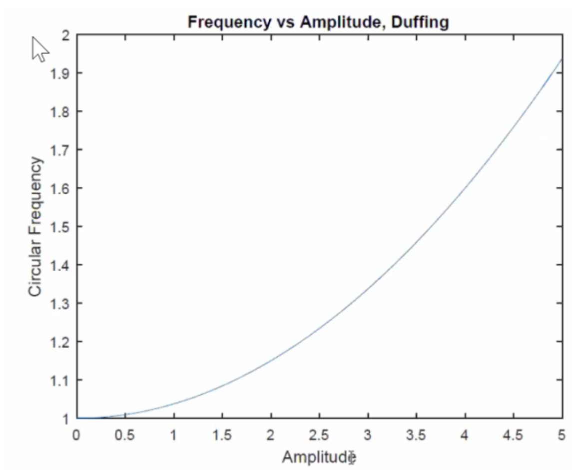
It then becomes the Duffing oscillator, and the E.O.M. is known as the Duffing equation.

Duffing equation is often used in structural analysis involving nonlinear restoring forces.

1.1 Unforced and Undamped Duffing Oscillator

$$\ddot{x} + \alpha x + \beta x^3 = 0$$

- Frequency of vibration depends on amplitude of vibration. (This is true for other nonlinear oscillators.)
- Approximate solutions of period (or frequency) are available.
- Exact solutions of period (or frequency) is only available for a few special cases, typically in the form of elliptic integrals.



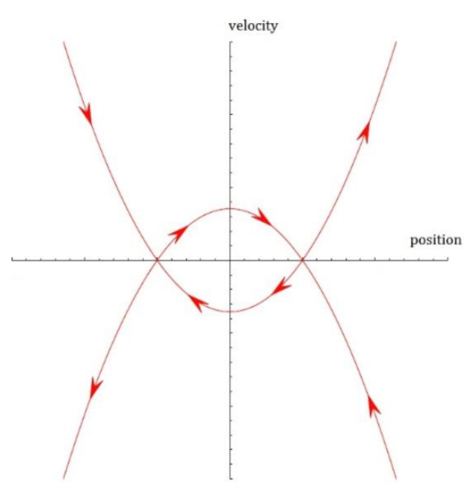
1.1.1 $\beta > 0$ or hardening:

- Phase portraits show continuous, closed curves surrounding the origin O.
- O is a center, or a stable equilibrium point.

1.1.2 $\beta < 0$ or softening:

Two situations may arise depending on the amplitude of vibration:

- Saddle points (or nodes) and separatrices:

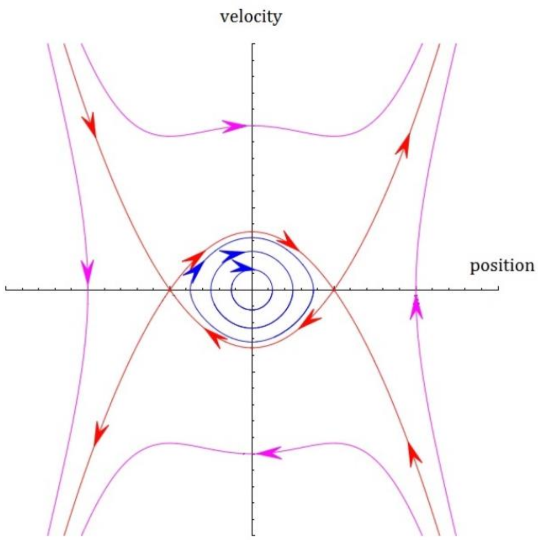


Courtesy of Alfred Clark, Jr., Professor Emeritus of Mechanical Engineering, Mathematics, and Biomedical Engineering, University of Rochester

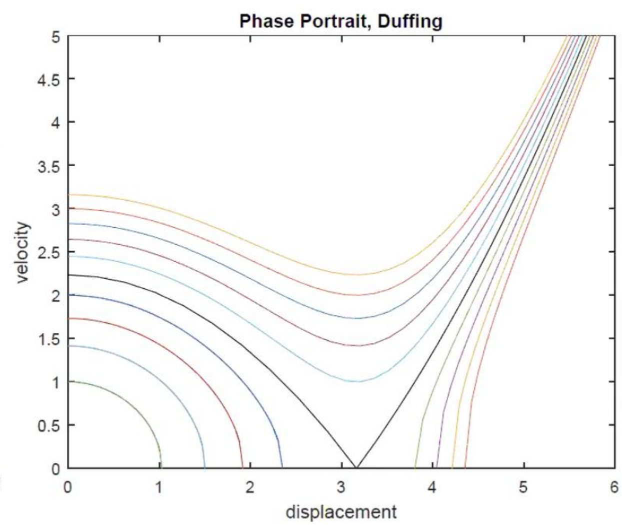
Saddle points (or nodes) are unstable equilibrium points.

Separatrix refers to the boundary separating different modes of vibration.

- The two situations:
1) Continuous, closed curves inside the separatrices; or
2) Curves "running off" to infinity outside the separatrices.



Courtesy of Alfred Clark, Jr., Professor Emeritus of Mechanical Engineering, Mathematics, and Biomedical Engineering, University of Rochester



1.2 Forced Duffing Oscillator

$$\ddot{x} + \delta\dot{x} + \alpha x + \beta x^3 = f_0 \cos(\omega t)$$

In addition to having nonlinear restoring forces, the forces Duffing oscillators are often used when the system demonstrates hysteresis (or the state variables' dependence of history)

The amplitude-frequency relation if $f(t) = f_0 \cos(\omega t)$:
- The jump phenomenon
- The upsweep and downsweep paths

1.3 Typical Analytical Approaches for Duffing Oscillator

Perturbation methods (straightforward expansion, Lindstedt-Poincare method, …)
Harmonic balance method
Averaging method
…