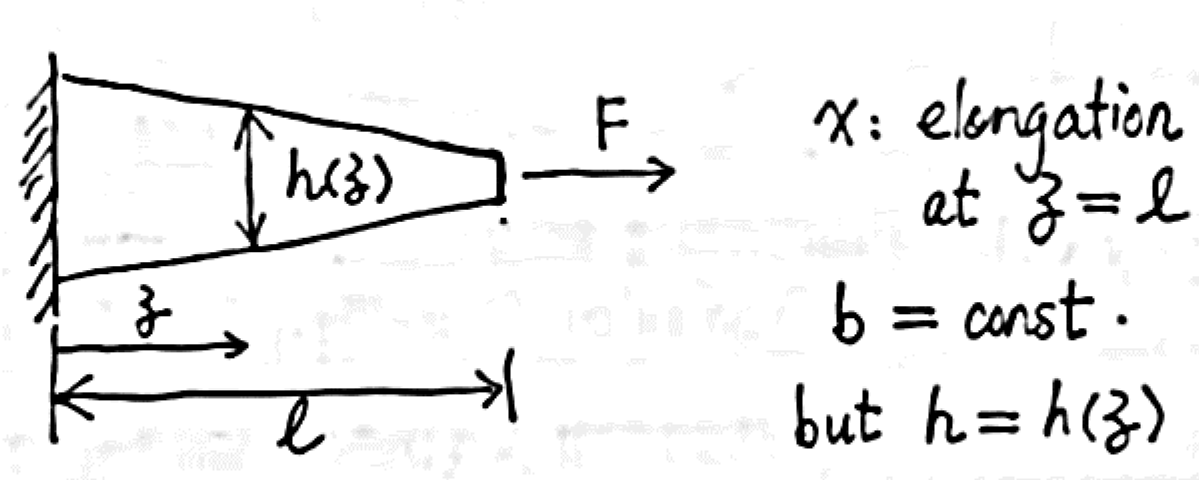
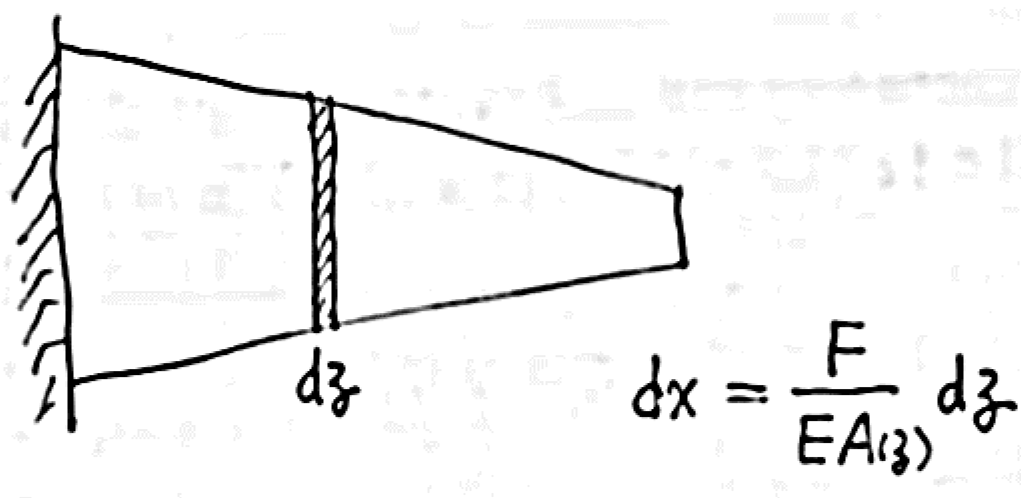
Example 2.4 (d)



$$x = \frac{FL}{EA}$$

This is no longer applicable (EA is no longer constant)



$$\therefore x = \int_0^l \frac{F}{EA(z)}\,dz$$

By integral approach:

$$k = 32.568 \cdot 10^6\ (N/m)$$

Using A_{av}:

$$k = 30.7125 \cdot 10^6\ (N/m)$$

2.3.3 General Combination

The system has various springs, translational and/or torsional. The i-th spring has potential energy $(1/2)k_i x_i^2$, where k_i and x_i should be interpreted in the general sense.

Total potential energy in the system is:

$$V = \sum(1/2)k_i x_i^2$$

For the equivalent spring k_{eq}, the generalized coordinate is x. Each x_i is assumed directly proportional to x. Then:

$$V = (1/2)k_{eq}x^2$$

Therefore

$$k_{eq}x^2 = \sum k_i x_i^2$$

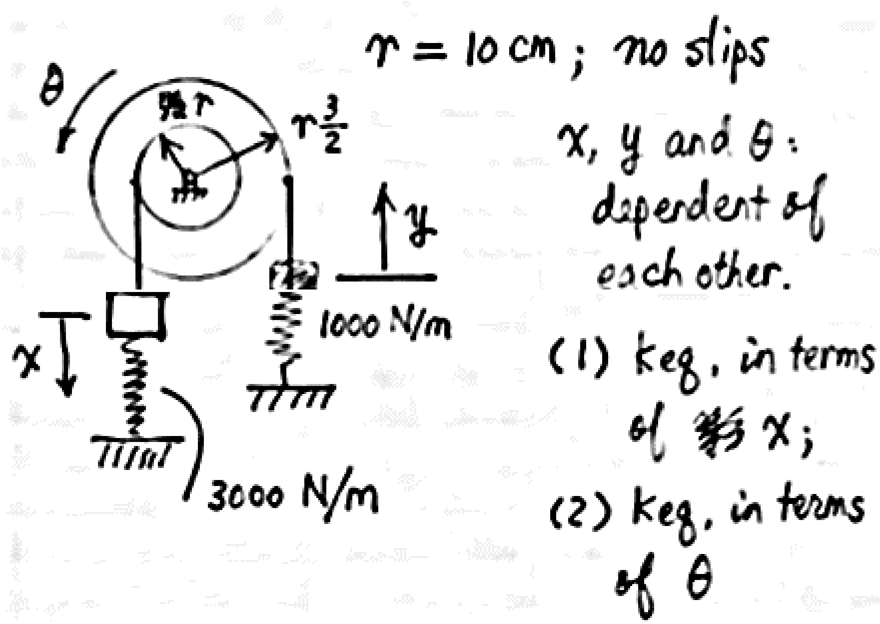
And

$$k_{eq} = \sum k_i \left(\frac{x_i}{x}\right)^2$$

Note: each x_i/x is a constant

Example 2.5

On a horizontal plane



Determine:
(1) k_{eq} in terms of x
(2) k_{eq} in terms of θ

Solution:

$\therefore no\ slips$
$\therefore x = r\theta$
$y = \left(\frac{3}{2}\right) r\theta$
$y = \left(\frac{3}{2}\right) x$

(1) k_{eq} in terms of x

$$V = \left(\frac{1}{2}\right)(3000)x^2 + \left(\frac{1}{2}\right)(1000)y^2$$
$$= \left(\frac{1}{2}\right)(3000)x^2 + \left(\frac{1}{2}\right)(1000)\left(\frac{9}{4}\right)x^2$$
$$= \left(\frac{1}{2}\right)\left(3000 + \frac{9}{4} \cdot 1000\right)x^2$$
$$\therefore k_{eq} = 5{,}250\ (N/m)$$

(2) k_{eq} in terms of θ

$$V = \left(\frac{1}{2}\right)(3000)x^2 + \left(\frac{1}{2}\right)(1000)y^2$$
$$= \left(\frac{1}{2}\right) k_{eq}\theta^2$$
$$k_{eq} = 52.5\ (N \cdot m/rad)$$

2.4 Other sources of potential energy

2.4.1 Gravity

It is a conservative force

V due to gravity is:

$$V = mgh$$

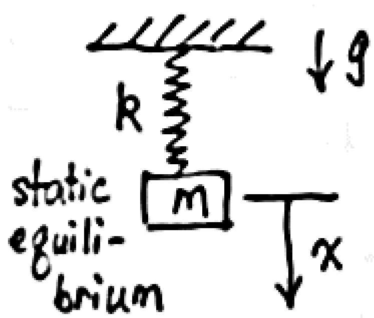
Where h is the positive if the particle relative to the datum. h is positive if the particle is positioned above the datum, and h is negative if positioned below.

Example 2.6 (A pendulum, 3 choices of datum)

V due to spring force as well as gravity:

$$V = V_{spring} + V_{gravity}$$

As an example, a mass (i.e. a particle) is suspended from a spring (Figure 2.15)



- If datum is located at the static equilibrium position:

$$V = \left(\frac{1}{2}\right) k(\Delta_{st} + x)^2 - mgx$$

- The work done (by spring force and gravity, on the particle) from 0 to x is:

$$U_{1\to 2} = V_1 - V_2$$

- The principle of energy conservation can be more conveniently expressed as:

$$T_1 + V_1 = T_2 + V_2$$
$$\to T_1 - T_2 = -U_{1\to 2} = \left(\frac{1}{2}\right) kx^2$$

As long as x is measured from the static equilibrium position.

Note: Static deflection is not always by $\Delta_{st} = mg/k$, see for example, Example 2.8 and Problem 2.18.

2.4.2 Buoyancy

If a floating or submerged object has constant cross-section, buoyancy functions very much like the linear translational spring.

ρ: mass density of fluid per unit volume, in kg/m^3
A: cross-sectional area of the object

Then spring constant is:

$$k = \rho g A$$

The work done by buoyancy and gravity on the object is:

$$U_{1\to 2} = -\left(\frac{1}{2}\right) kx^2$$

Where x is measured from the static equilibrium position.

Note: static deflection is not by $\Delta_{st} = mg/k$ when buoyancy is involved.

2.5 Viscous Damping

Viscous damping force has a magnitude that is directly proportional to the velocity.

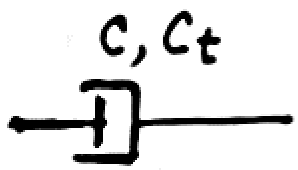
$F = cv$: c is the (translational) damping coefficient;
Figure 2.20; eq. (2.37)

Or,

$M = c_t\dot{\theta}$: c_t is the (torsional) damping coefficient;
Figure 2.21; eq. (2.42)

Direction of viscous damping force: opposite to v or $\dot{\theta}$

Schematic representation:



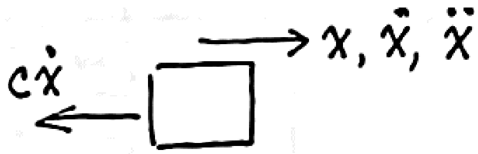
Devices to achieve viscous damping: the dashpot (Figure 2.19); the piston-cylinder damper (Figure 2.20); the torsional viscous damper (Figure 2.21).

Example 2.9

2.6 Energy Dissipated by Viscous Damping

Viscous damping force is non-conservative.

The dissipated energy is measured by work done, see eq. (2.44)



$$U_{1\to 2} = \int_0^x -c\dot{x}\,dx$$

Energy dissipated by a system of dampers: eq. (2.45)

$$U_{1\to 2} = \sum \int_0^{x_i} -c_i\dot{x}_i\,dx_i$$

Equivalent damping coefficient c_{eq} in terms of generalized coordinate x:

- $\dot{x}$ and $\dot{x}_\iota$ are directly proportional to each other (i.e., $\frac{\dot{x}_i}{\dot{x}} = constant = \gamma_y$)

$$\int_0^{x_i} -c_i\dot{x}_i\,dx_i$$

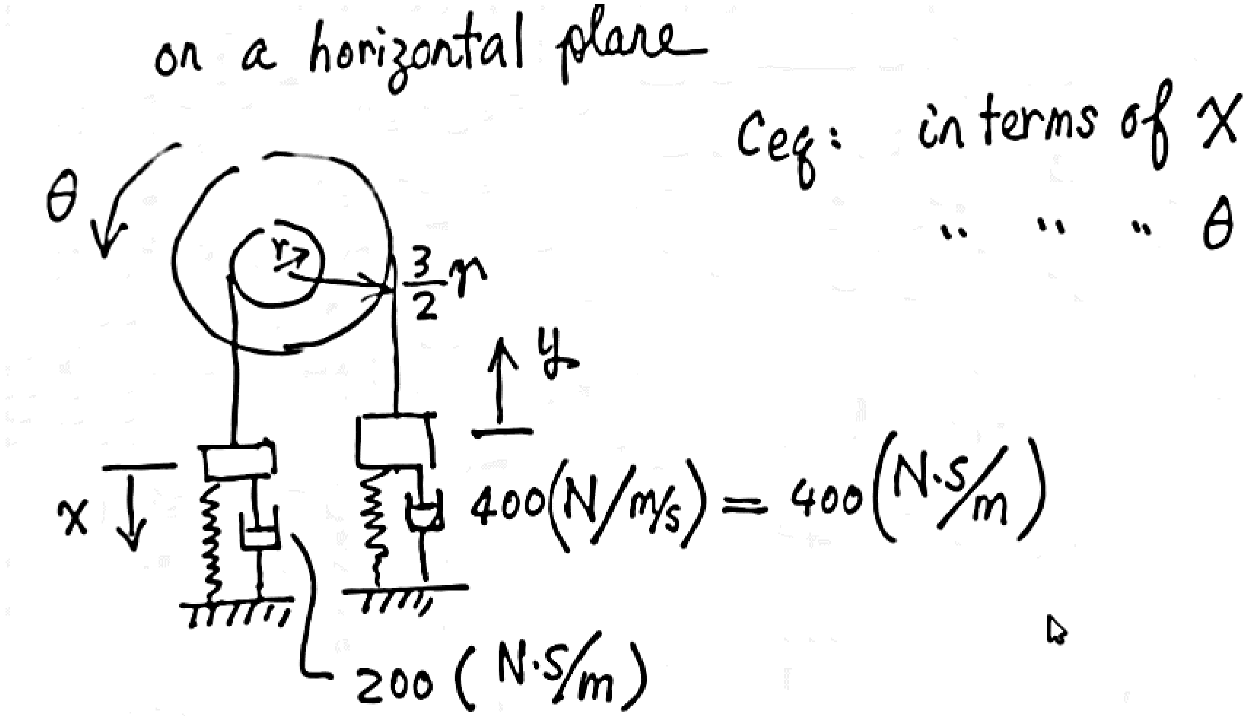
Consider:
$\dot{x}_i = \gamma_i \cdot \dot{x}$
$x_i = \gamma_i \cdot x$

$$= \int_0^x -c_i(\gamma_i \cdot \dot{x})d(\gamma_i \cdot x) = \int_0^x -c_i\gamma_i^2\dot{x}\,dx$$

- Energy dissipation

$$U_{1\to 2} = \sum \int_0^{x_i} -c_i\,\dot{x}_i\,dx_i = \sum \int_0^x -c_i\gamma_i^2\,\dot{x}dx$$
$$U_{1\to 2} = \int_0^x -c_{eq}\dot{x}\,dx$$

Example 2.10



Solution:

$$x = r\theta, \quad y = \left(\frac{3}{2}\right) r\theta$$

$$\dot{x} = r\dot{\theta}, \quad \dot{y} = \left(\frac{3}{2}\right) r\dot{\theta}$$

C_{eq} in terms of x:

$$y = \left(\frac{3}{2}\right) x, \quad \dot{y} = \left(\frac{3}{2}\right) \dot{x}$$

$$\therefore U_{1\to 2} = \int_0^x -(400)\left(\frac{3}{2}\right)\dot{x}\, d\left(\frac{3}{2}x\right) + \int_0^x -(200)\dot{x}\, dx$$

$$\therefore U_{1\to 2} = \int_0^x \underbrace{\left(-\frac{3600}{4} - 200\right)}_{-C_{eq}} \dot{x}\, dx$$

c_{eq} in terms of θ:

$$U_{1\to 2} = \int_0^\theta (-400)\left(\frac{3}{2}\right) r\dot{\theta}\, d\left(\frac{3}{2}r\theta\right) + \int_0^\theta (-200) r\dot{\theta}\, d(r\theta)$$

$$U_{1\to 2} = \int_0^\theta -\underbrace{\left(400 \cdot \frac{9}{4} r^2 + 200 r^2\right)}_{C_{eq}} \theta\ \dot{d}\theta$$

2.7 Inertia Elements

2.7.1 Equivalent mass

The kinetic energy of a system of rigid bodies is

$$T = \sum\left(\frac{1}{2}m_i v_i^2 + \frac{1}{2}I_i\omega_i^2\right)$$

Note:

(i) Table 2.1 for centroidal moments of inertia

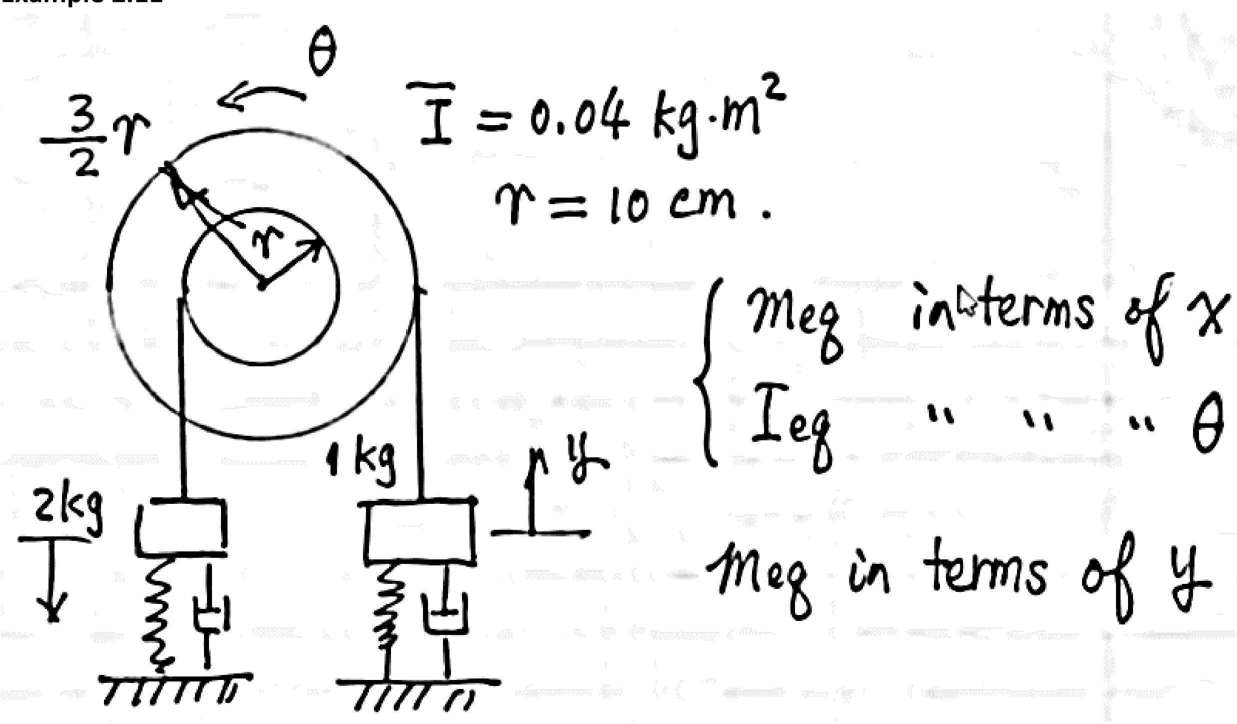
(ii) If v_i and ω_i are directly proportional to a generalized coordinate x, the kinetic energy is then, eq. (2.50)

$$T = \left(\frac{1}{2}\right)m_{eq}\dot{x}^2$$

(iii) Or eq. (2.51) if the generalized coordinate is θ.

$$T = \left(\frac{1}{2}\right)I_{eq}\dot{\theta}^2$$

Example 2.11



Solution:
x in terms of y
θ in terms of y

$$\therefore \dot{x} = \frac{2}{3}\dot{y} \quad ; \quad \dot{\theta} = \frac{2}{3r}\dot{y}$$

$$T = \left(\frac{1}{2}\right)(2)\dot{x}^2 + \left(\frac{1}{2}\right)(1)\dot{y}^2 + \left(\frac{1}{2}\right)(0.04)\dot{\theta}^2$$

$$T = \left(\frac{1}{2}\right)(2)\left(\frac{4}{9}\right)\dot{y}^2 + \left(\frac{1}{2}\right)(1)\dot{y}^2 + \left(\frac{1}{2}\right)(0.04)\left(\frac{4}{9r^2}\right)\dot{y}^2$$

$$T = \left(\frac{1}{2}\right)\left[\frac{8}{9} + 1 + \frac{0.16}{9(0.1)^2}\right]\dot{y}^2$$

And $m_{eq} = \frac{11}{3}(kg)$

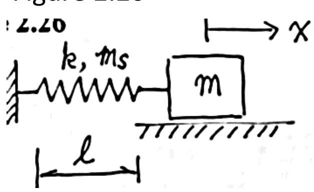
2.7.2 Inertia effects of springs

In reality, springs are structural components. They have mass.

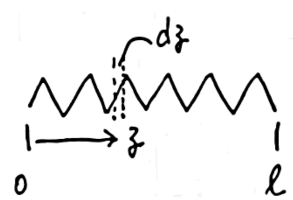
When the mass of a spring is small but not negligible, the mass of the spring is typically added to that of the particle or rigid body.

$$T = T_s + \frac{1}{2}mv^2 = \frac{1}{2}m_{eq}\dot{x}^2$$

Figure 2.26



ρ =mass density per unit length
$\rho \cdot l = m_s$



at z, displacement is:
$\frac{z}{l}x = u(z)$

$$\therefore dT_s = \frac{1}{2}(\rho\, dz)[\dot{u}(z)]^2$$

$$\therefore T_s = \int_0^l \frac{\rho}{2}\left(\frac{\dot{x}}{l}z\right)^2 dz$$

$$T_s = \frac{1}{2}\left(\frac{m_s}{3}\right)\cdot \dot{x}^2$$

$$\therefore T = \frac{1}{2}\left(\frac{m_s}{2}\right)\dot{x}^2 + \left(\frac{1}{2}\right)m\dot{x}^2$$

$$T = \frac{1}{2}\left(m + \frac{m_s}{3}\right)\dot{x}^2$$