Chapter 2 Modeling of SDOF Systems

2.1 Introduction

Key components in a SDOF system:

Inertia element that has mass and stores kinetic energy;
Stiffness element that stores and releases potential energy;
Damping element that dissipates energy.
And source of work or energy (i.e., the excitation)

The chapter cover the principles behind the determination of the key components in a SDOF system, with the objective of modeling a SDOF system with equivalent mass, equivalent stiffness, equivalent damping, and equivalent excitation.

Principles reviewed in Ch. 1: Newton's laws; Work & energy; and Impulse & Momentum.

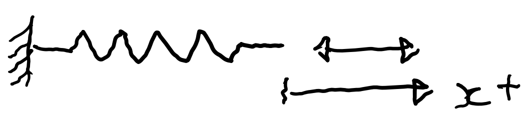
Topics covered:
2.2-2.4: stiffness
2.5-2.6: damping
2.7: inertia/mass
2.8: external sources
2.9: FBD method (or Newton's 2nd law)
2.12: equivalent system method
2.14: further examples

Additional topics:
2.10: static deflection & gravity
2.13: benchmark example
2.15: chapter summary

2.2 Springs

2.2.1 Introduction

<u>1. translational springs</u>
$F = f(x)$: spring force



Where x is the stretch or compression from the natural length.

For spring materials having the same properties in tension and under compression, $f(x)$ is an odd function.
Or: $f(-x) = -f(x)$

Taylor expansion of $f(x)$ about $x = 0$, then

$$F = f(x) \approx k_1 x + k_3 x^3 + k_5 x^5 + \cdots \quad (2.3)$$

re. Eq. (2.3):
i) All springs are inherently nonlinear; and
ii) Linear springs result from the assumption of small x

For linear springs,

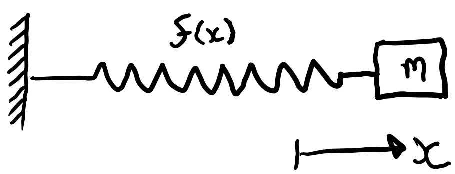
$$F = f(x) \approx k_1 x \tag{2.4}$$

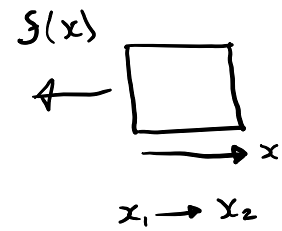
2. work and energy for translational springs

A particle is attached to a spring. The work done by the spring force, on the particle, when the particle moves from x_1 to x_2 (for the particle) is:

$$U_{1\to 2} = \left(\frac{1}{2}\right) k x_2^2 - \left(\frac{1}{2}\right) k x_1^2 \tag{2.5}$$

Consider:





$$U_{1\to 2} = \int_{x_1}^{x_2} -f(x)\, dx$$

If $f(x) = k_1 x + k_3 x^3$

If $f(x) = kx$, then $U_{1\to 2}$ is by Eq. (2.5)

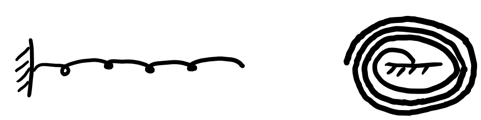
The potential function V (for the spring) is:

$$V(x) = \left(\frac{1}{2}\right) k x^2 \tag{2.6}$$

Eqs. (2.5) and (2.6): for linear translational springs only.

3. torsional springs

Eqs. (2.7) and (2.8): for linear torsional springs only.



$$M = k_t \theta \tag{2.7}$$

$$V = \frac{1}{2} k_t \theta^2 \tag{2.8}$$

2.2.2 Helical coil springs

Eq. (2.11) relates spring constant k to coil spring material and dimensions.

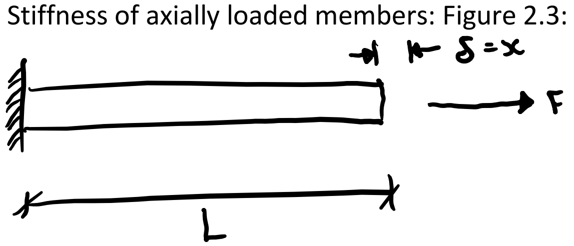
$$k = \frac{GD^4}{64Nr^3} \tag{2.11}$$

It is a formula commonly seen in machine design texts.

2.2.3 Elastic elements as springs

This sub-section deals with spring constant (or stiffness) when the spring is, for example, an axially loaded rod, a straight member in torsion, a simply supported beam, and so on.

Stiffness of axially loaded members: Figure 2.3:



$$\delta = \frac{FL}{EA} = x$$

$$k = \frac{F}{x} = \frac{EA}{L} \tag{2.16}$$

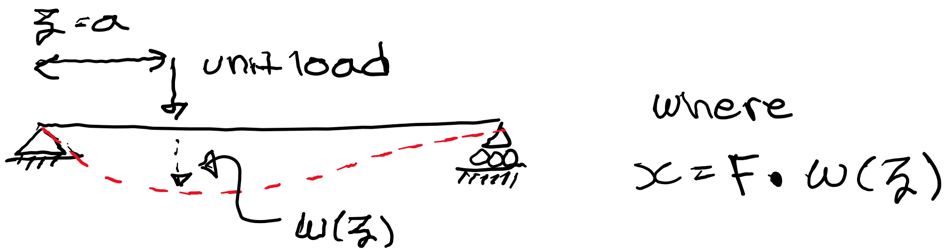
Stiffness of beams and frames:

i) $\mathrm{k} = \mathrm{F}/\Delta$: Δ must be the deflection in the same direction of F, and at the point of application of F;

ii) Generalized interpretation of $k = F/\Delta$: F can be replaced by moment M; Δ represents the angular deflection at the point of application of M and in the same sense of M;

iii) Eqs. (2.19) and (2.20): both involve $\omega(z)$ which is the deflection due to a unit load at z; and

iv) Table $D.2$.: list of $\omega(z)$, given as $y(z)$ on Table D.2



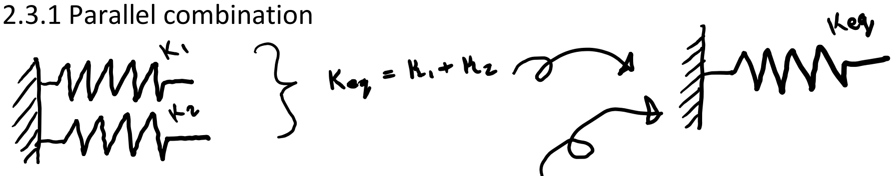
Stiffness of circular shafts:

i) See top portion of p. 62

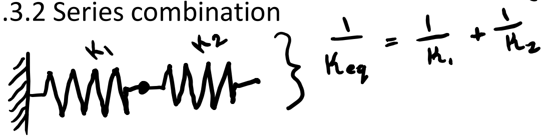
2.2.4 Static deflection

p. 61 Figure 2.5, Eq. (2.23)

2.3.1 Parallel combination



2.3.2 Series combination



Notes on Table D.2

Table D.2 (uniform beam, unit concentrated load at $z = a$)

$$y(z) = \frac{1}{EI}\left[\frac{1}{6}(z-a)^3 u(z-a) + \frac{1}{6}\sum_{i=1}^{n} R_i (z-z_i)^3\, u(z-z_i) + C_1\frac{z^3}{6} + C_2\frac{z^2}{2} + C_3 z + C_4\right]$$

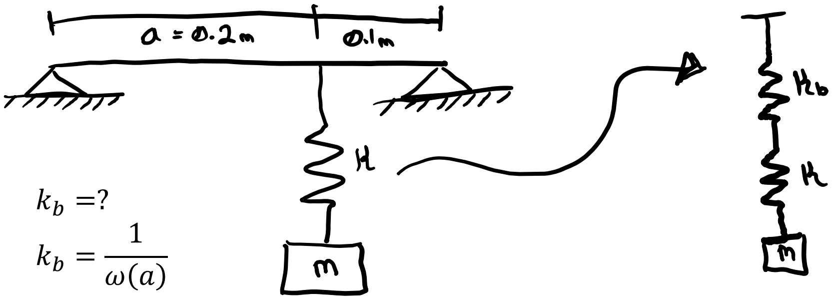
Referring to Figure D.1

z_i is the z –coordinate of the i –th support (excluding the first of the left support)

$u(z-a)$ is the unit step; i.e.

$$u(z-a) = \begin{cases} 1 & z > a \\ 0 & z \le a \end{cases}$$

Example 2.4



$k_b = ?$

$k_b = \dfrac{1}{\omega(a)}$

$E = 210\ GPa$

$I = 5 \cdot 10^{-4}\ m^4$

Using Table D.2

Case 6:

$a = 2$

$z_1 = 3$

And:

$R_1 = -\dfrac{2}{3}$

$C_1 = -\dfrac{1}{3}$

$C_2 = C_4 = 0$

$C_3 = -\left(1 - \dfrac{a}{z_1}\right)\dfrac{z_1^2}{6}\left[\left(1 - \dfrac{a}{z_1}\right)^2 \cdot u(z_1 - a) - 1\right]$

$\therefore u(3-2) = u(1) = 1$

$\therefore C_3 = \left(\dfrac{4}{9}\right)$

$$\therefore y(z) = \frac{1}{EI}\left[\frac{1}{6}(z-2)^3 u(z-2) + \frac{1}{6}\left(-\frac{2}{3}\right)(z-3)^2 u(z-3) + \left(-\frac{1}{3}\right)\frac{z^3}{6} + 0 + \left(\frac{4}{9}\right)z + 0\right]$$

Now $z = a = 2$ (terms highlighted turn to zero)

$$\therefore y(z) = \frac{1}{EI}\left[\frac{1}{6}(0)^3 u(0) + \frac{1}{6}\left(-\frac{2}{3}\right)(-1)^2 u(-1) + \left(-\frac{1}{3}\right)\frac{(2)^3}{6} + 0 + \left(\frac{4}{9}\right)(2) + 0\right]$$

$$\therefore \omega(z) = y(z) = \frac{1}{EI}\left[0 + 0 - \frac{4}{9} + 0 + \frac{8}{9} + 0\right]$$

$$\therefore \frac{4}{9EI}$$

$$k_b = \frac{1}{\omega(2)} = \frac{9EI}{4} = 2.3625 \cdot 10^8\ \left(\frac{N}{m}\right)$$

Choose

$$k = 1 \cdot 10^8\ \left(\frac{N}{m}\right)$$

Then

$$k_{eq} = \left(\frac{1}{k_b} + \frac{1}{k}\right)^{-1} = 0.7026 \cdot 10^8\ (N/m)$$