Bifurcation Diagram by Analytical Ways

Bifurcation and bifurcation diagram

For a dynamic system defined by $\ddot{\boldsymbol{x}} = \boldsymbol{F}(\boldsymbol{x}, \dot{\boldsymbol{x}}, t; c)$, the number of its equilibrium points and the stability of such points change as the system's parameter c is varied. This phenomenon is known as the bifurcation.

Bifurcation diagram is a widely used technique for examining the pre- or post-chaotic changes in a dynamic system under parameter variations.

Bifurcation diagrams can be drawn through analytical ways or by computation.

Bifurcation diagram by the analytical ways

Focusing on autonomous dynamic systems defined by first order ODEs $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}; c)$;

<u>The existence and uniqueness of theorem</u>

Detail of the existence and uniqueness theorem can be found form Boyce et al, Theorem 2.4.2 and Theorem 2.8.1, for example.

The essence of the theorem governing the existence and uniqueness of the solutions to first order ODEs $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}; c)$ is,

If the functions $\boldsymbol{f}$ and their first order partial derivatives are continuous over a certain domain for x and t, then there exists a unique solution of the system of ODEs that satisfied the initial condition.

<u>The equilibrium points (or critical points)</u>

They are those that meet the condition of $\dot{\boldsymbol{x}} = \boldsymbol{0}$, or $\boldsymbol{f}(\boldsymbol{x_e}; c) = \boldsymbol{0}$, with $\boldsymbol{x_e}$ denoting the equilibrium points.

<u>Definition of stability</u>

There is not a universally agreed upon definition. But the most fundamental definition is attributed to Lyapunov.

Let $\boldsymbol{x_e} \in R^n$ be an equilibrium point,

(1) $\boldsymbol{x_e}$ is stable if, for any $h > 0$, there is a $\delta > 0$ such that if a solution $\boldsymbol{x}(t)$ satisfies $\left\|\boldsymbol{x}(0) - \boldsymbol{x}_e\right\| < \delta$, then:

$$\left\|\boldsymbol{x}(t) - \boldsymbol{x_e}\right\| < h, \qquad for\ all\ t > 0$$

(2) x_e is asymptotically stable is there is a $\delta > 0$ such that if a solution $x(t)$ satisfies $\left\|\boldsymbol{x}(0) - \boldsymbol{x}_e\right\| < \delta$, then:

$$\lim_{t \to \infty} \boldsymbol{x}(t) = \boldsymbol{x_e};$$

(3) $\boldsymbol{x_e}$ is monotonically stable if it is asymptotically stable and $||\boldsymbol{x}(t) - \boldsymbol{x_e}||$ decreases monotonically with time;

(4) $\boldsymbol{x_e}$ is globally asymptotically stable if it is asymptotically stable and $\boldsymbol{x}(t) \to \boldsymbol{0}\ and\ t \to \infty$ for all $\boldsymbol{x}(0)$; and

(5) $\boldsymbol{x_e}$ is unstable if it is not stable as defined above in (1).

Parts (1), (2) and (5) of the definition appear often in the literature.
Detail of (1), (2) and (5) can be found from Boyce et al, Sec. 9.2 for example.

Linearization of nonlinear ODEs

For ODEs $x = f(x; c)$, its Jacobian matrix J evaluated at x_e, is:

$$\boldsymbol{J}(\boldsymbol{x_e}; c) = \boldsymbol{A}(\boldsymbol{x_e}; c) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}_{\boldsymbol{x_e}}$$

Taking the Taylor expansion of $\boldsymbol{f}(\boldsymbol{x}; c)$ at $\boldsymbol{x_e}$ and keeping terms up to the linear,

$$\dot{x} = \boldsymbol{f}(\boldsymbol{x}; c) \approx \boldsymbol{f}(\boldsymbol{x_e}; c) + \boldsymbol{J}(\boldsymbol{x} - \boldsymbol{x_e})$$

Defining $\boldsymbol{u} = \boldsymbol{x} - \boldsymbol{x_e}$, and noting that $\boldsymbol{f}(\boldsymbol{x_e}; c) = 0$ and $\dot{\boldsymbol{x}}_{\boldsymbol{e}} = \boldsymbol{0}$, then linearized ODEs at $\boldsymbol{x_e}$ is:

$$\dot{\boldsymbol{u}} = \boldsymbol{J} \cdot \boldsymbol{u}$$

In other words, $\boldsymbol{x_e}$ is the “reference”, $\boldsymbol{u}$ is the growth or shrinkage from the reference.

Classification of equilibrium point

Earlier and in a not-too-specific way, equilibrium point, or stability, is classified as:

- Centers or stable equilibrium points
- Saddle points/nodes or unstable equilibrium points.

A few terminologies are in order.

Linear stability:

- The stability of a system of linear ODEs
- The stability of a system of nonlinear ODEs which is linearized as $\dot{\boldsymbol{u}} = \boldsymbol{J} \cdot \boldsymbol{u}$

The latter is also known as the local stability.

Non-linear stability:

The stability of a system of nonlinear ODEs, typically by making use of the Lyapunov function.

Proper nodes and improper nodes:

When $\boldsymbol{J}$ has identical eigenvalues, if the corresponding eigenvectors are independent of each other, the node is proper; otherwise it is improper.

The more specific classification:

If the eigenvalues associated with $\boldsymbol{J}$ evaluated at $\boldsymbol{x_e}$ are, $\lambda_1, \ldots, \lambda_n$,

$n = 2$:

$\boldsymbol{\lambda_1}$ **and** $\boldsymbol{\lambda_2}$	**Classification**	**Linear Stability**
Both real and positive	Source	Unstable
Both real and negative	Sink	Asymptotically stable
Both real, one positive, one negative	Saddle	Unstable
Identical, real and positive	Improper Node	Unstable
Identical, real and negative	Improper Node	Asymptotically stable
Complex, with positive real part	Outward spiral	Unstable
Both imaginary*	Center	Stable

* Stability is indeterminate if local stability is concerned.

Detail can be found from Boyce et al, Theorem 9.3.3 and Table 9.3.1, for example.

$n > 2$:
All eigenvalues have negative real parts, then $\boldsymbol{x_e}$ is a stable equilibrium point;

If at least one of the eigenvalues has a positive real part, then $\boldsymbol{x_e}$ is an unstable equilibrium point.

For other cases, nonlinear stability analysis is required.

<u>Classification of bifurcation</u>
Saddle point bifurcation or fold bifurcation (two equilibrium points move towards each other, collide, and become one; or the opposite)

Tranncritical bifurcation (a pair of equilibrium points exchange stability; i.e., one point goes from stable to unstable while the other does the opposite; but the change takes place at the same parameter value)

Pitchfork bifurcation (equilibrium points go from one to three, or the opposite; the former is known as supercritical pitchfork bifurcation and the latter subcritical pitchfork bifurcation)

Andronov-Hopf bifurcation or simply *Hopf bifurcation* (bifurcation from periodic solutions; e.g., the creation or destruction of a limit cycle)

<u>Saddle point Bifurcation</u>
Consider:

$$\dot{x} = a - x^2$$

where x and a are real

The equilibrium points:

$$x = \begin{cases} 0 & a \le 0 \\ \pm\sqrt{a} & a > 0 \end{cases}$$

The Jacobian at $\boldsymbol{x_e}$:

$$J = \begin{cases} [0] & x_e = 0 \\ [-2\sqrt{a}] & x_e = \sqrt{a} \\ [2\sqrt{a}] & x_e = -\sqrt{a} \end{cases}$$

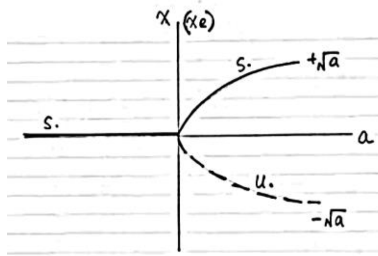
Eigenvalues:

$$\lambda = \begin{cases} [0] & x_e = 0 \\ [-2\sqrt{a}] & x_e = \sqrt{a} \\ [2\sqrt{a}] & x_e = -\sqrt{a} \end{cases}$$

The solution to $\dot{\boldsymbol{u}} = \boldsymbol{J} \cdot \boldsymbol{u}$:

$$\boldsymbol{u}(t) = \begin{cases} \alpha e^{0 \cdot t} & x_e = 0 \\ \alpha e^{-2\sqrt{a} \cdot t} & x_e = \sqrt{a} \\ \alpha e^{2\sqrt{a} \cdot t} & x_e = -\sqrt{a} \end{cases}$$

The bifurcation diagram is:



Transcritical Bifurcation

Consider:

$$\dot{x} = ax - bx^2$$

where x, a and b are real, $a \neq 0$, $b > 0$. The parameter is a/b.

The equilibrium points:

$$x_e = 0 \text{ and } \frac{a}{b}$$

The Jabocian at x_e:

$$\boldsymbol{J} = \begin{cases} [a] & x_e = 0 \\ [-a] & x_e = \dfrac{a}{b} \end{cases}$$

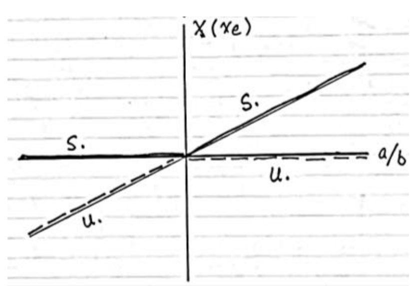
Eigenvalues:

$$\lambda = \begin{cases} a & x_e = 0 \\ -a & x_e = \dfrac{a}{b} \end{cases}$$

The solution to $\dot{\boldsymbol{u}} = \boldsymbol{J}\boldsymbol{u}$:

$$u(t) = \begin{cases} \alpha e^{at} & x_e = 0 \\ \alpha e^{-at} & x_e = \dfrac{a}{b} \end{cases}$$

The bifurcation diagram is:



Pitchfork bifurcation

Consider:

$$\dot{x} = ax - bx^3$$

Where x, a and b are real, $a \neq 0$, $b > 0$. The parameter is a/b.

The equilibrium points:

$$x_e = \begin{cases} 0 & any\ a \\ \pm\sqrt{\frac{a}{b}} & a > 0 \end{cases}$$

The Jabocian at x_e:

$$\boldsymbol{J} = \begin{cases} [a] & x_e = 0 \\ [-2a] & x_e = \pm\sqrt{\frac{a}{b}} \end{cases}$$

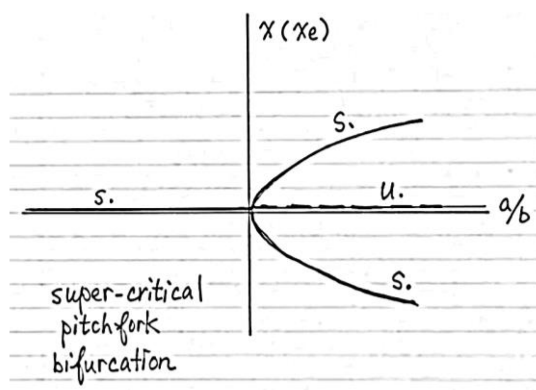
Eigenvalues:

$$\lambda = \begin{cases} a & x_e = 0 \\ -2a & x_e = \pm\sqrt{\frac{a}{b}} \end{cases}$$

The solution to $\dot{\boldsymbol{u}} = \boldsymbol{J}\boldsymbol{u}$:

$$u(t) = \begin{cases} \alpha e^{at} & x_e = 0 \\ \alpha e^{-2at} & x_e = \pm\sqrt{\frac{a}{b}} \end{cases}$$

The bifurcation is:

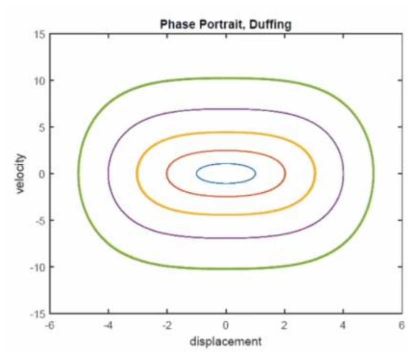


Unforced and Undamped Duffing Oscillator:

$$\ddot{x} + ax + \beta x^3 = 0$$

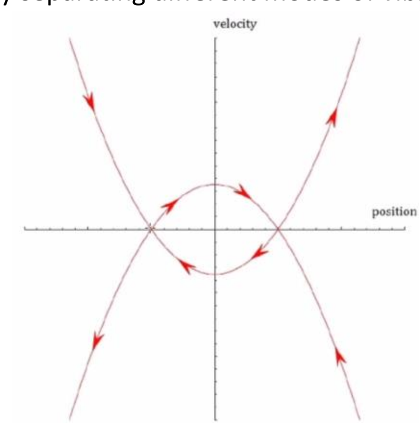
$\beta > 0$ or hardening:

- O is a center, or a stable equilibrium point.



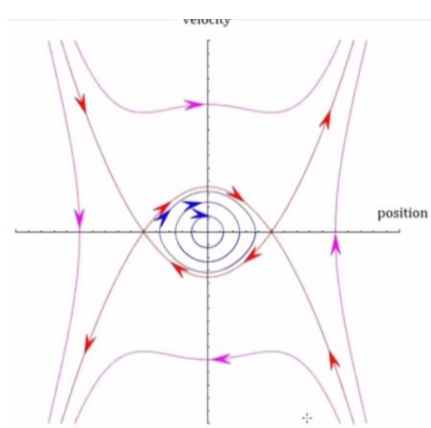
$\beta < 0$ or softening:

- Saddle points (or nodes) and separatrices:
  Saddle points (or nodes) are unstable equilibrium points.
  Separatrix refers to the boundary separating different modes of vibrations



- The two situations:

1) Continuous, or closed curves inside the separatrices; or
2) Curves "running off" to infinity outside the separatrices



Softened Duffing oscillator:

$$\dot{x} + ax + \beta x^3 = 0, \quad \text{where } \alpha \geq 0, \beta < 0$$

Define:

$$x = \begin{Bmatrix} x \\ \dot{x} \end{Bmatrix} = \begin{Bmatrix} x \\ y \end{Bmatrix}$$

The first order ODEs

$$x = \begin{Bmatrix} \dot{x} \\ \dot{y} \end{Bmatrix} = \begin{Bmatrix} y \\ -\alpha x - \beta x^3 \end{Bmatrix}$$

The equilibrium points:

$$\boldsymbol{x_e} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}, \qquad \boldsymbol{x_e} = \begin{Bmatrix} \sqrt{\dfrac{\alpha}{\beta}} \\ 0 \end{Bmatrix}, \qquad \boldsymbol{x_e} = \begin{Bmatrix} -\sqrt{\dfrac{\alpha}{\beta}} \\ 0 \end{Bmatrix}$$

Jacobians:

$$\boldsymbol{J} = \begin{bmatrix} 0 & 1 \\ -\alpha - 3\beta x^2 & 0 \end{bmatrix}$$

$$\therefore \boldsymbol{J} = \begin{bmatrix} 0 & 1 \\ -\alpha & 0 \end{bmatrix}, \qquad \boldsymbol{J} = \begin{bmatrix} 0 & 1 \\ 2\alpha & 0 \end{bmatrix}, \qquad \boldsymbol{J} = \begin{bmatrix} 0 & 1 \\ 2\alpha & 0 \end{bmatrix}$$

Eigenvalues:

$$\lambda = \begin{Bmatrix} -\sqrt{\alpha} j \\ \sqrt{\alpha} j \end{Bmatrix}, \qquad \lambda = \begin{Bmatrix} -\sqrt{2\alpha} j \\ \sqrt{2\alpha} j \end{Bmatrix}, \qquad \lambda = \begin{Bmatrix} -\sqrt{\alpha} 2j \\ \sqrt{2\alpha} j \end{Bmatrix}$$

Classification:

$\boldsymbol{x_e} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$ Center

$\boldsymbol{x_e} = \begin{Bmatrix} \sqrt{\dfrac{\alpha}{\beta}} \\ 0 \end{Bmatrix}, \boldsymbol{x_e} = \begin{Bmatrix} -\sqrt{\dfrac{\alpha}{\beta}} \\ 0 \end{Bmatrix}$ Saddle Points