Bifurcation and Bifurcation Diagram

What is bifurcation?
For a dynamic system defined by $\dot{\boldsymbol{x}} = f(\boldsymbol{x}; c)$, the equilibrium points are those that meet the condition of $\dot{\boldsymbol{x}} = \boldsymbol{0}$, or $f(\boldsymbol{x}_e) = \boldsymbol{0}$, with $\boldsymbol{x}_e$ denoting the equilibrium points (basically you set the LHS equal to zero to solve for $\boldsymbol{x}$).

Equilibrium points are classified as:

- Centers or stable equilibrium points.
- Saddle points/nodes or unstable equilibrium points.

As the system's parameter c changes, the number of equilibrium points and the stability of such points can change as well.

The phenomenon that the number of equilibrium points and the stability of such points can change as the system's parameter changes is known as the bifurcation.

Specifically, it is about the change in the type of *long-term* behaviors of the system when parameters when parameters are varied.

Theory of bifurcation is the study of these changes is nonlinear systems.

What is a bifurcation diagram?
It is a widely used technique for examining the pre- or post-chaotic changes in a dynamic system under parameter variations.

Typically, some measure of the response of the system is plotted against a system parameter.

The measure may be, for example, the maximum amplitude, the local maxima or minima, or data sampled using a Poincaré map.

When the bifurcation diagram loses continuity, it means either quasi-periodic motion or chaotic motion. Bifurcating does not mean being chaotic, it simply means it is bifurcating. Other approaches to identify chaos should be jointly used.

Bifurcation diagram can be drawn through analytical ways or by computation.

Bifurcation diagram by computation

The following outline of computation assumes that the local maxima in position $x(t)$ are the measure to be plotted.

Steps are:

Loop over a parameter range (say, $f_0 = 20$ to 25, at an increment of 0.001).

> Run an ode solver for the response. Make sure that is covers a long period of time, say, ≥ 100 forcing period if excitation is present
>
> Use the second half of the computed time history of position, for plotting the bifurcation diagram. That is, the measure is $y(t)$.
>
> Loop over the length of $y(t)$ to find and store the local maxima
>
>> Find $y(t_{i-1}) < y(t_i)$ and $y(t_i) > y(t_{i+1})$
>>
>> Use the three points (three t −values and three y −values to evaluate the local maximum)
>>
>> Store the local maximum in an array (as a vector)
>
> End of loop
>
> Plot the local maxima vector against the specific parameter value

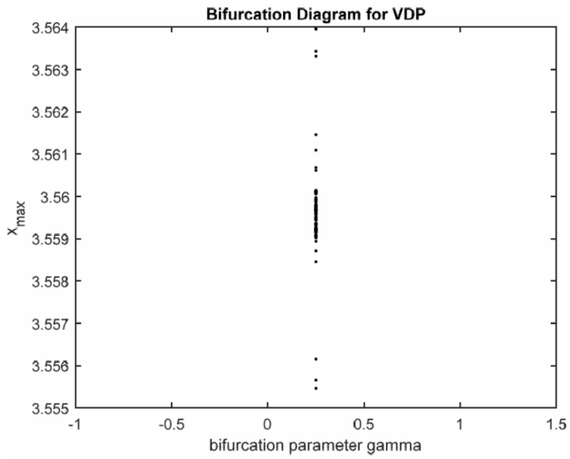
End of loop

Example: Consider the following van der Pol oscillator:

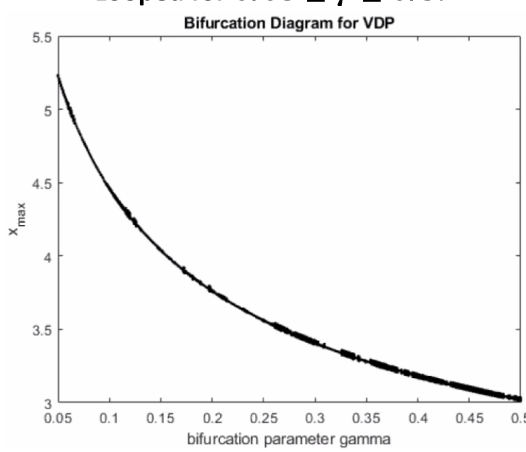
$$\ddot{x} - \gamma\dot{x}(1 - x^2) + \alpha x = f_0 \cos(\omega t)$$

Where $\alpha = 1$, $\gamma = 0.25$, $f_0 = 3$, $\omega = 1.2$, $x(0) = 1$, and $\dot{x}(0) = 0$

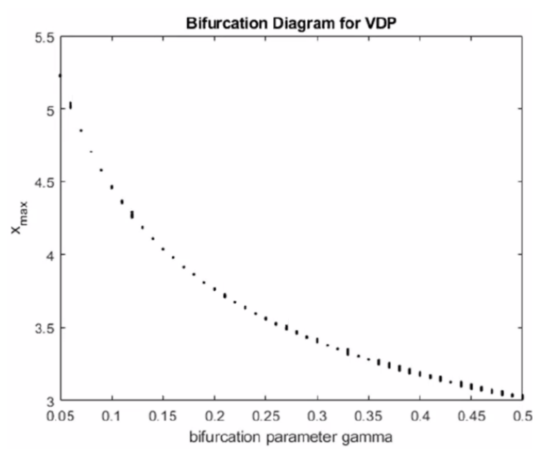
Plot when $\gamma = 0.25$ (plot of 100 points):



Looped for $0.05 \le \gamma \le 0.5$:



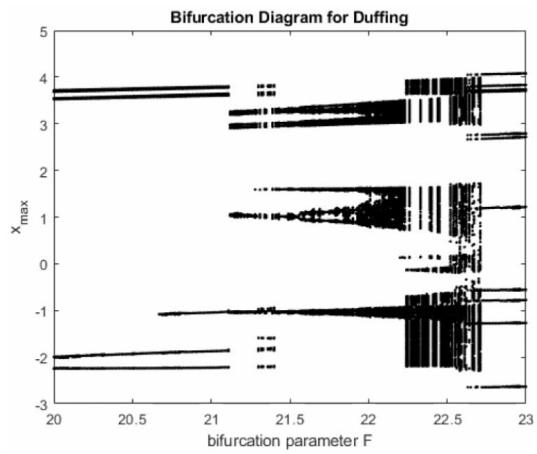
With 10x the increment as above:



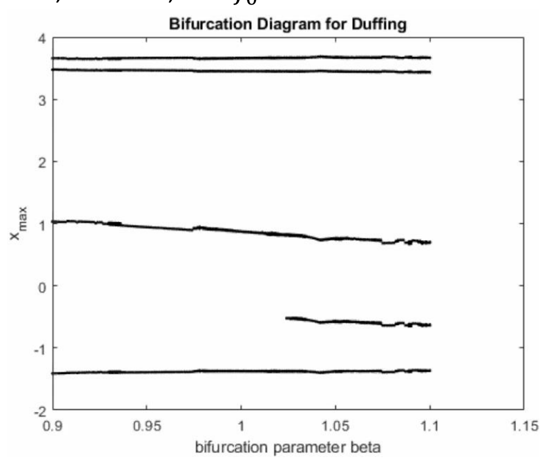
Example: Consider the following Duffing oscillator:

$$\ddot{x} - \gamma\dot{x} + \alpha x + \beta x^3 = f_0 \cos(\omega t)$$

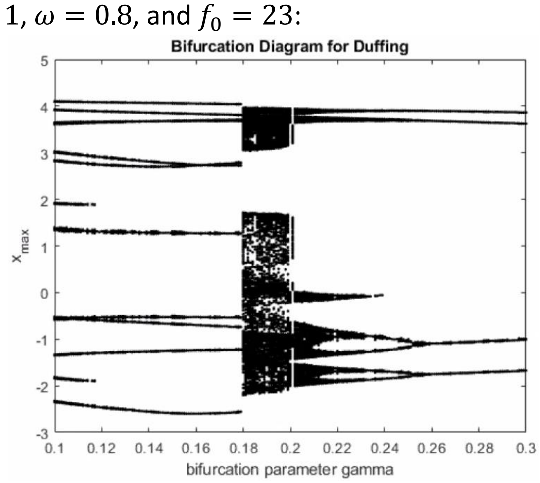
(1) $\gamma = 0.18$, $\alpha = \beta = 1$, $\omega = 0.8$, and $\boldsymbol{f_0} = \mathbf{20}$ **to** $\mathbf{23}$:



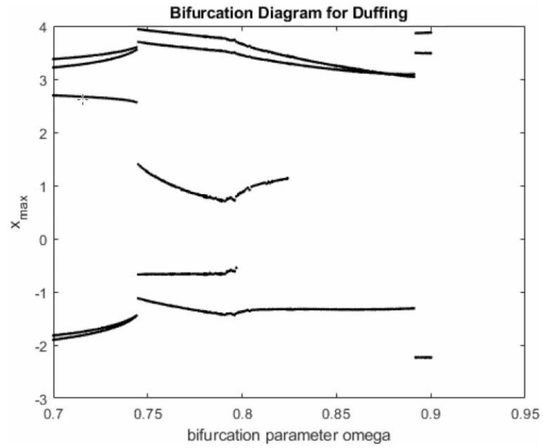
(2)TODO FIX $\gamma = 0.18$, $\alpha = \beta = 1$, $\omega = 0.8$, and $f_0 = 20$ to 23:



(3) $\boldsymbol{\gamma} = \mathbf{0.1}$ **to** $\mathbf{0.3}$, $\alpha = \beta = 1$, $\omega = 0.8$, and $f_0 = 23$:



(4) $\gamma = 0.18$, $\alpha = \beta = 1$, $\boldsymbol{\omega} = \mathbf{0.7}$ **to** $\mathbf{0.9}$, and $f_0 = 23$:



NOTE: Some potential topics for final exam: Perturbation, harmonic balance, modeling, bifurcation by computation or by theory