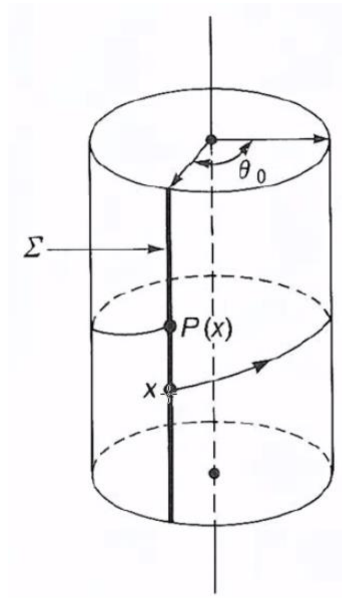
What can be revealed by Poincaré Maps?
Some math/definitions/explanations first.

Mathematically speaking, a Poincaré Map is a mapping $x_{k+1} = P(x_k), x_k, x_{k+1} \in P, k = 1, 2, 3, \dots$

1) x is a fixed point is $x = P(x)$.



Courtesy of Practical Numerical Algorithms for Chaotic Systems, Parker & Chua, 1989

2) The set of sampled points $\{x_1, x_2, \dots, x_k\}$ is a period$-K$ closed orbit, if $x_{k+1} = P(x_k)$ for $k = 1, 2, \dots, K-1$ and $x_1 - P(x_K)$

3) Quasi-periodic solution is the sum of finite numbers of periodic solutions, each having a frequency that is an integer combination of frequencies taken from a finite base set.

   For example, $x(t) = Acos(2t) + Bcos(\sqrt{2}t)$ is quasiperiodic. So is $x(t) = Acos(t) + Bcos(3\sqrt{2}t)$, as ω_1/ω_2 is an irrational number. Quasi-periodic is not periodic.

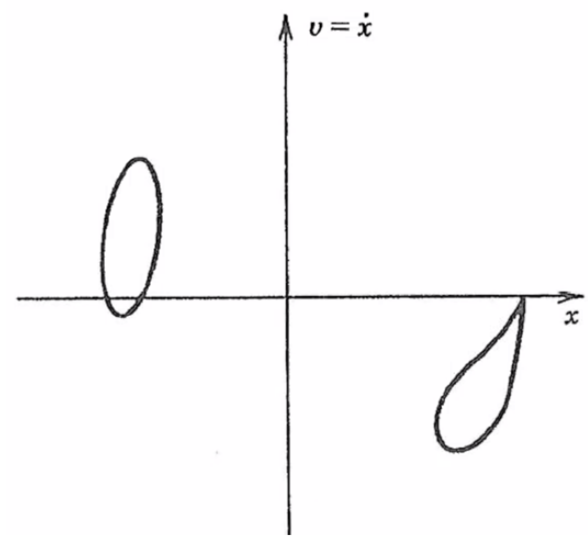
4) Attractor is a set of values that a dynamic system tends to evolve toward. Examples of attractor include, but not limited to, a fixed point, a finite number of points, and a limit cycle.

5) Strange attractor is a set of values showing fractal structure.

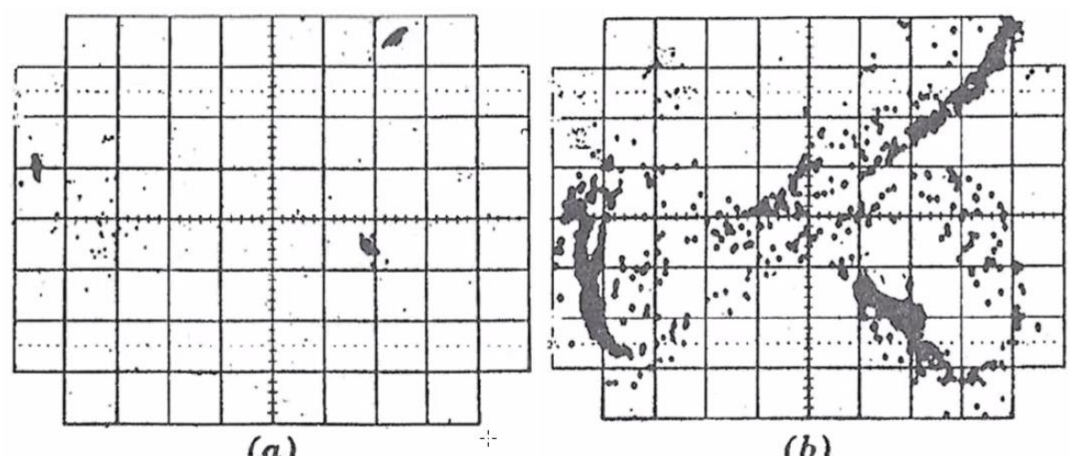
6) Cantor set refers to the embedding structure within structure, usually appearing at finer and finer scales.

Classification of Poincaré Maps (non-autonomous DEs):
- A fixed point on the Poincaré map indicates a period-1 solution (periodic solution, only 1 point!)
- K distinct points on the map, indicates a $K - th$ subharmonic
- One or more closed curves on the map indicate a quasi-periodic solution.
- Fractal collection of points, strange attractor, or Cantor set indicate chaos.

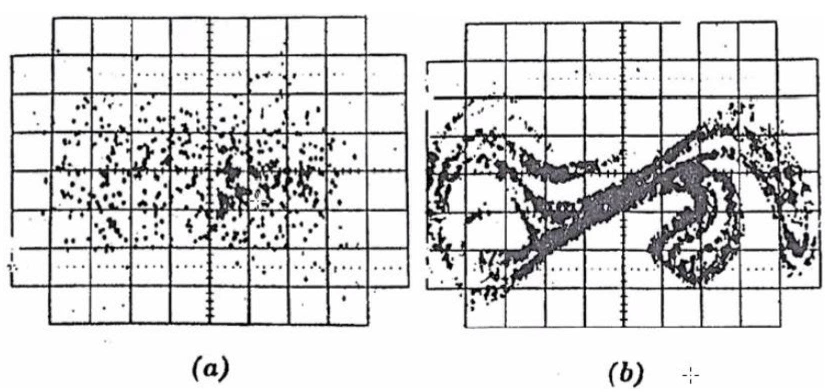
Poincaré Map Examples



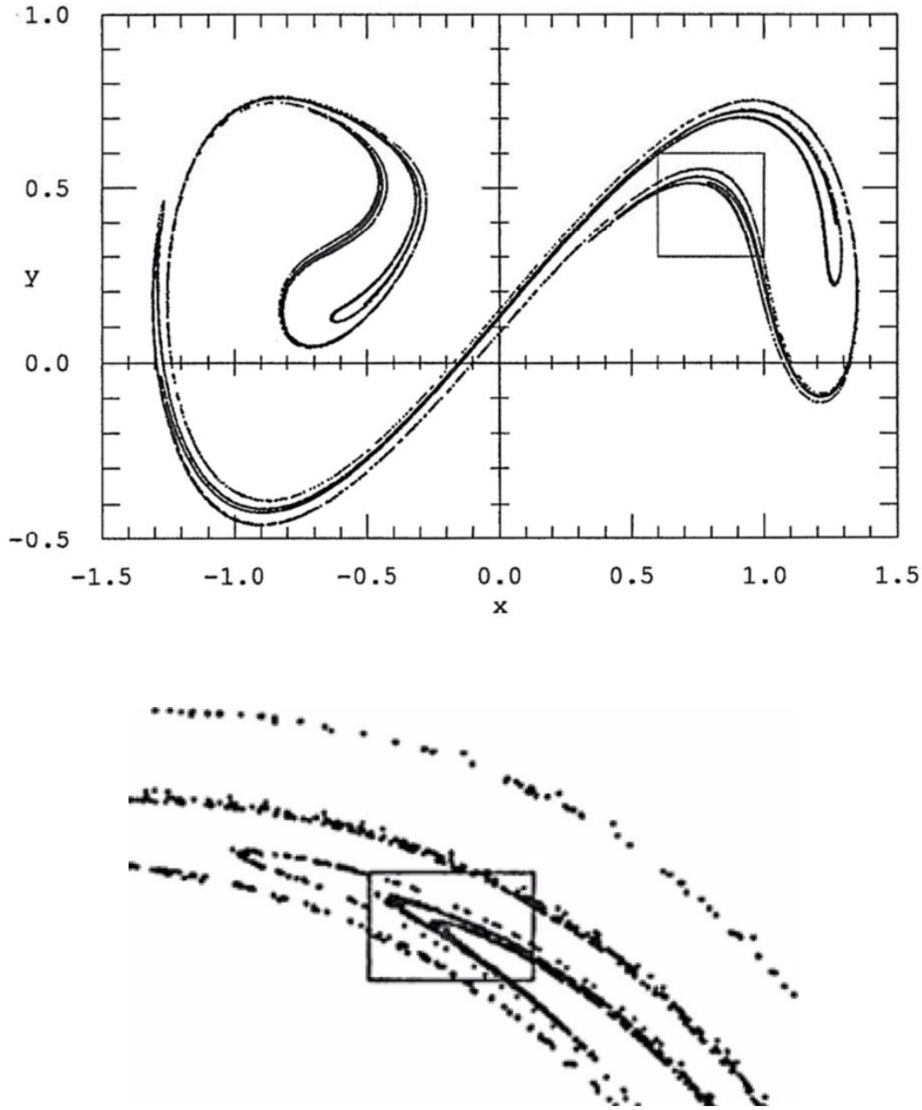
Multiple closed curves indicate a quasiperiodic solution



3rd subharmonic (left) and Fractal collection of points (right)



Many points indicating chaos (left) and another example of chaos (right)



Cantor set (infinitely embedding structure)

Poincaré Maps for Autonomous DEs

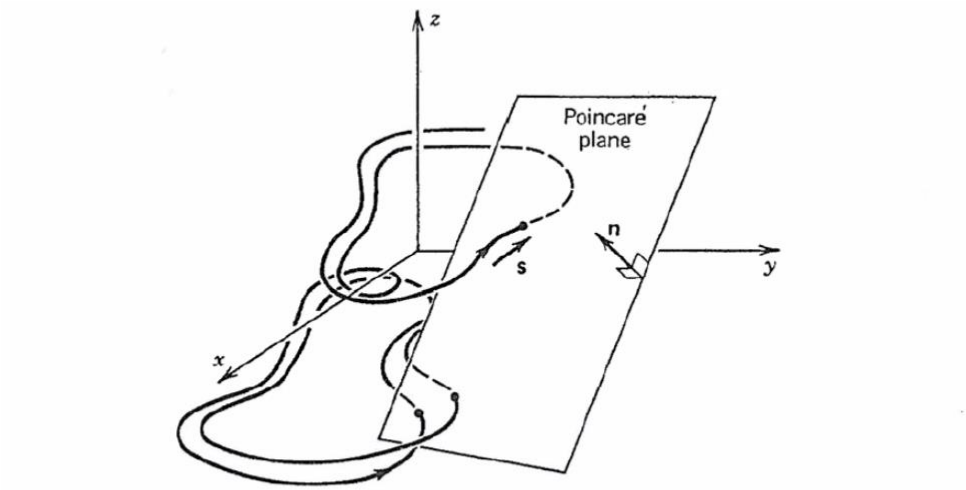


Figure 2-14 Sketch of time evolution trajectories of a third-order system of equations and a typical Poincaré plane.

As an example, consider the cases where the state variables are $x(t)$, $y(t)$, and $z(t)$. In this state space, a Poincaré section is a plane defined by:

$$n_1x + n_2y + n_3z - c = 0$$

Where c is a constant. The normal to the plane is

$$\boldsymbol{n} = \pm \begin{Bmatrix} n_1 \\ n_2 \\ n_3 \end{Bmatrix}$$

$\boldsymbol{n}$ will take either the positive or negative sign.

The Poincaré map will then consist of points (x_k, y_k, z_k) $(k = 1, 2, 3, \ldots)$ that meet eq. (1). In addition, if $\boldsymbol{s_k}$ represents the tangential vector to the trajectory at (x_k, y_k, z_k) $\boldsymbol{s_k} \cdot \boldsymbol{n}$ must have the same sign.

The classification of Poincaré maps of autonomous DEs follows that for non-autonomous DEs. That is,

- A fixed point on the Poincaré map indicates a period-1 solution (periodic solution, only 1 point!)
- K distinct points on the map, indicates a $K - th$ subharmonic
- One or more closed curves on the map indicate a quasi-periodic solution.
- Fractal collection of points, strange attractor, or Cantor set indicate chaos.

Example: The Lorenz attractor is

$$\dot{x} = \sigma(y - x)$$
$$\dot{y} = \rho x - y - xz$$
$$\dot{z} = xy - \beta z$$

Chaos is observed when $\sigma = 10$, and $\rho = 28, \beta = 8/3$.

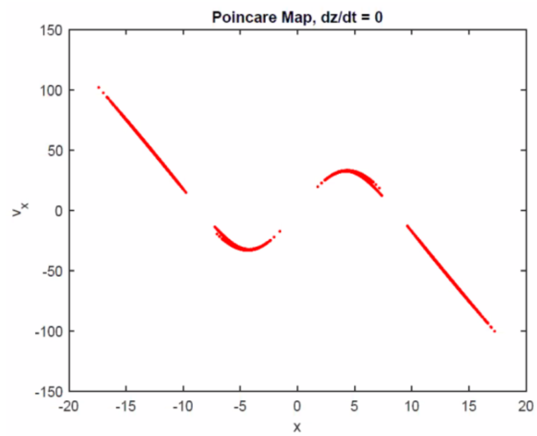
Dimension of phase space: $n = 6$;

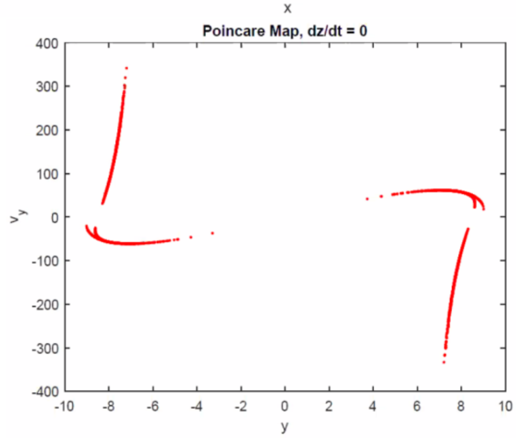
The Poincaré section: $z = 0$;

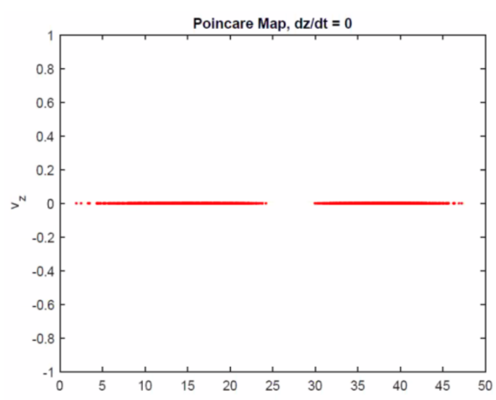
Collected “points”: 2,699 (from negative to positive; down from 20,001 for time history computation);

Poincaré maps are 5-dimensional plots.

For example,

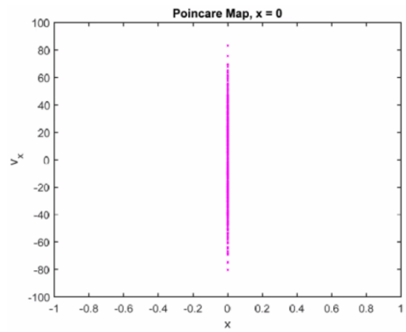




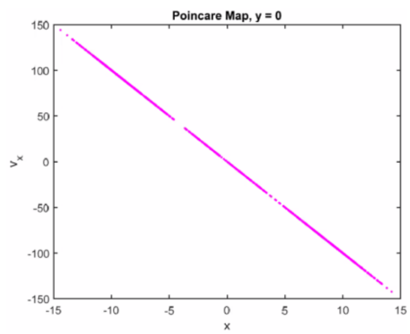


More Examples

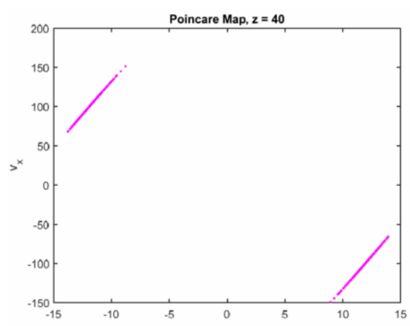
If $x = 0$ (negative to positive), 567 points



If $y = 0$ (negative to positive), 987 points



If $z = 40$ (less than to greater than), 369 points



Lyapunov Exponents and Fractal Dimensions

Organization of the Wolf et al.'s paper

The paper shows that the Lyapunov exponents can be determined from a set of first-order ODEs or from a time series. Recorded experimental data are a time series, for example.

Sec. 1: Introduction
Sec. 2: Definition
Sec. 3: Computational Aspect
Sec. 4 – 7: Time Series (Experimental Data)
Sec. 8: Results
Sec. 9: Conclusions
Appendix A: Fortran Code (Lyapunov Spectrum from computation)
Appendix B: Fortran Code (Lyapunov Spectrum from time series)

Lyapunov Exponents, the Theory

Considering a chaotic oscillator. Due to its sensitivity on initial conditions, one can image that any two trajectories close to another in the n −dimensional phase space will move exponentially away from (or towards) each other.

Let d_0 be a measure of the distance between the two trajectories at t_0. At t, a later time (keeping in mind $t - t_0$ should be small), the distance will grow (or shrink) following a base-2 exponential relation.

$$d(t) = d_0 2^{\lambda(t-t_0)}$$

The constant λ is called the Lyapunov exponent.

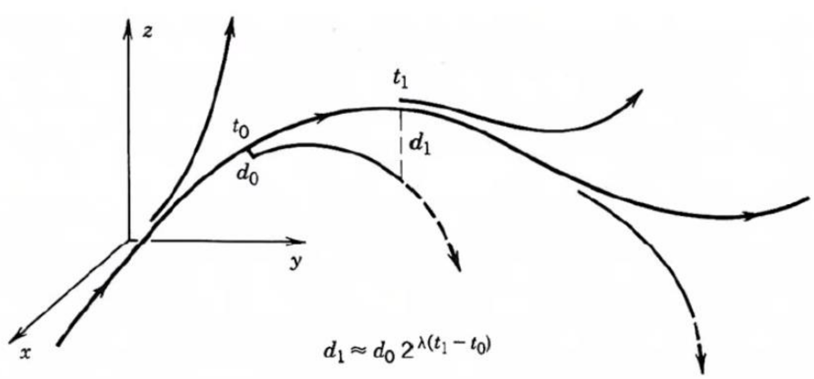


Figure 5-26 Sketch of the change in distance between two nearby orbits used to define the largest Lyapunov exponent.

Courtesy of *Chaotic vibrations An Introduction for Applied Scientists and Engineers*, Moon, 2004

Similarly, areas, spheres, and hyper-spheres in the phase space may stretch or shrink. As a result, there are respective Lyapunov exponents to measure the extents to which the principal axes of the areas, spheres, and hyper-spheres, are stretched/shrunk.

The set of Lyapunov exponents $\{\lambda_1, \lambda_2, \dots, \lambda_n\}$, where $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ is called the Lyapunov spectrum. Note that n is the dimension of the dynamic system.

A positive Lyapunov exponent is an indicator of chaos.

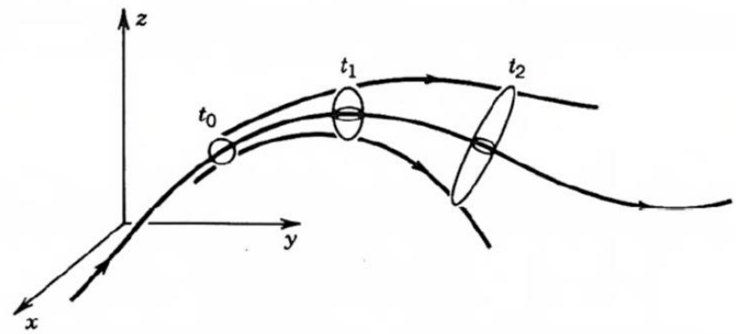


Figure 5-30 Sketch showing the divergence of orbits from a small sphere of initial conditions for a chaotic motion.

Courtesy of *Chaotic vibrations An Introduction for Applied Scientists and Engineers*, Moon, 2004

Lyapunov Exponents, the Computation
How to determine the Lyapunov exponents from a set of first-order ODEs is explained in Sec. 3 of Wolf et al's paper.

<u>One Lyapunov Exponent</u>
Assuming the vector $\boldsymbol{X}^*(t)$ represents solutions of the first-order ODEs. They are known as the reference trajectory. To compute the Lyapunov exponents, small variations from the trajectory are given, and how the variations grow or shrink is computed.

The small variations are denoted at $\boldsymbol{\delta}(t)$ (a vector), and the rate of growth or shrinkage of $\boldsymbol{\delta}$ is determined by:

$$\dot{\boldsymbol{\delta}} \approx \boldsymbol{J}(\boldsymbol{X}^*)\boldsymbol{\delta}$$

Where $\boldsymbol{J}(\boldsymbol{X}^*)$ is the Jacobian evaluated at the reference trajectory.

Numerically, $\boldsymbol{X}^*$ and $\boldsymbol{\delta}$ are solved by ODE solver at the same time. That is, in addition to the first-order ODEs, the Jacobian needs to be coded and included in the "function".

One $\boldsymbol{\delta}(t)$ is determined, the Lyapunov exponent is by:

$$\lambda(t_M) = \frac{1}{t_M - t_0} \sum_{k=1}^{M} \log_2 \frac{|\boldsymbol{\delta}(t_{k+1})|}{|\boldsymbol{\delta}(t_k)|}$$

The above equation has two key messages. (1) λ is a function of time, as M gets larger and larger; (2) Averaging over a long period of time, or over large expanse of the phase space will give rise to a relatively stable λ.

<u>Lyapunov Spectrum</u>

To find the Lyapunov spectrum, n sets of $\delta(t)$ will be needed. They are orthonormal to each other. Or they are the base vectors of an n −dimensional linear space.

GSR (Gram-Schmidt Reorthonormalization) is employed to ensure that the n sets of δ(t) remain orthonormal to each other, as time evolves. Without the GSR, the λ_i's will become indistinguishable as time evolves. In other words, only the largest Lyapunov exponent (LLE) will be meaningfully computed.

GSR can be performed at a certain fixed time interval, say, every time-step, or every 10 time-steps.

Additionally, the n small variation vectors are normalized (to unit vectors) at the beginning of each time step, to avoid overflow during the computation.

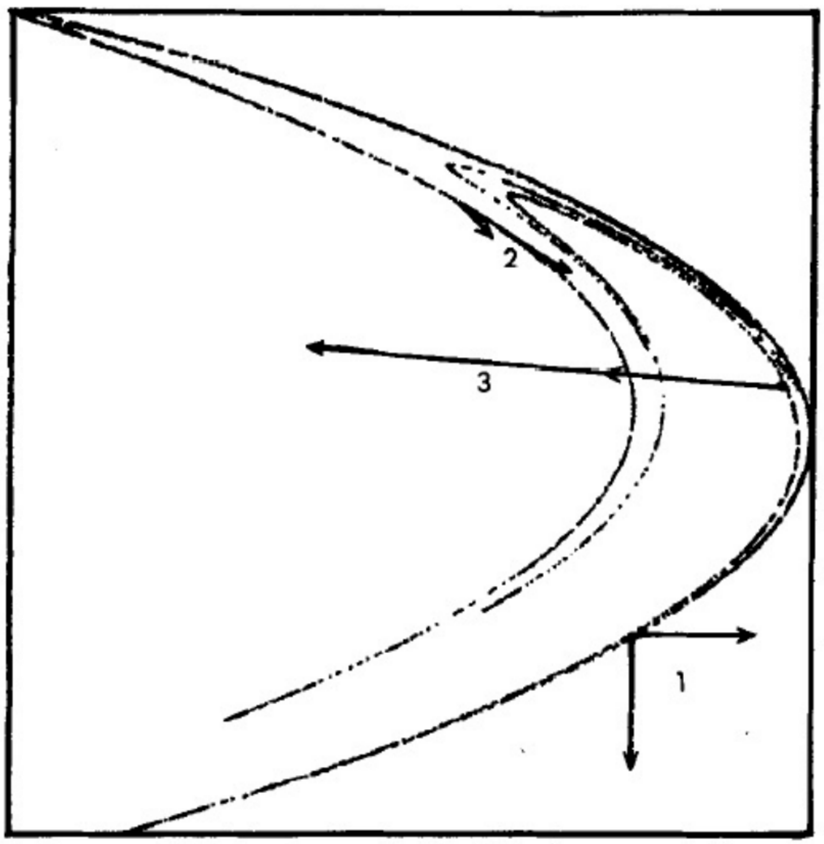


Fig. 2. The action of the product Jacobian on an initially orthonormal vector frame is illustrated for the Hénon map: (1) initial frame; (2) first iterate; and (3) second iterate. By the second iteration the divergence in vector magnitude and the angular collapse of the frame are quite apparent. Initial conditions were chosen so that the angular collapse of the vectors was uncommonly *slow*.

Courtesy of *Determining Lyapunov exponents from a time series*, Wolf et al., Physica D, vol.16, pp. 285-317, 1985.

ODEs and Jacobians, Examples

The Rossler attractor:

$$\dot{x} = -(y+z)$$
$$\dot{y} = x + ay$$
$$\dot{z} = b + z(x-c)$$

The Jacobian is:

$$J = \begin{bmatrix} 0 & -1 & -1 \\ 1 & a & 0 \\ z & 0 & x-c \end{bmatrix}$$

Next, the Mathieu's equation.

$$\ddot{x} + [a - 2qcos(2\omega t)]x = 0$$

Where a and q are constants.

Set

$$x_1 = x$$
$$x_2 = \dot{x}$$
$$x_3 = \omega t$$

The first-order ODEs are:

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = [2qcos(2x_3)x - a]x_1$$
$$\dot{x}_3 = \omega$$

The Jacobian is:

$$J = \begin{bmatrix} 0 & 1 & 0 \\ 2qcos(2x_3) - a & 0 & -4qsin(2x_3) \\ 0 & 0 & 0 \end{bmatrix}$$

What are the ODEs needed for computing the Lyapunov spectrum?

If n is the number of first-order ODEs, the total number of ODEs for the spectrum is, $n(n+1)$.

For the systems above, $n = 3$, so 12 ODEs are needed, or 3 sets of ODEs with 3 ODEs in each set.

For example, the first-order ODEs (for the Mathieu's equation)

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = [2qcos(2x_3) - a]x_1$$
$$\dot{x}_3 = \omega$$

Are to compute the reference trajectory $\boldsymbol{X}^*(t)$. The other n sets are to determine the three $\boldsymbol{\delta}$'s by:

$$\dot{\boldsymbol{\delta}} = \boldsymbol{J}(\boldsymbol{X}^*)\boldsymbol{\delta}$$

With

$$\delta = \begin{bmatrix} x_4 & x_7 & x_{10} \\ x_5 & x_8 & x_{11} \\ x_6 & x_9 & x_{12} \end{bmatrix}$$

The initial conditions for the $\delta's$ are:

$$\delta(0) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Fractal Dimensions

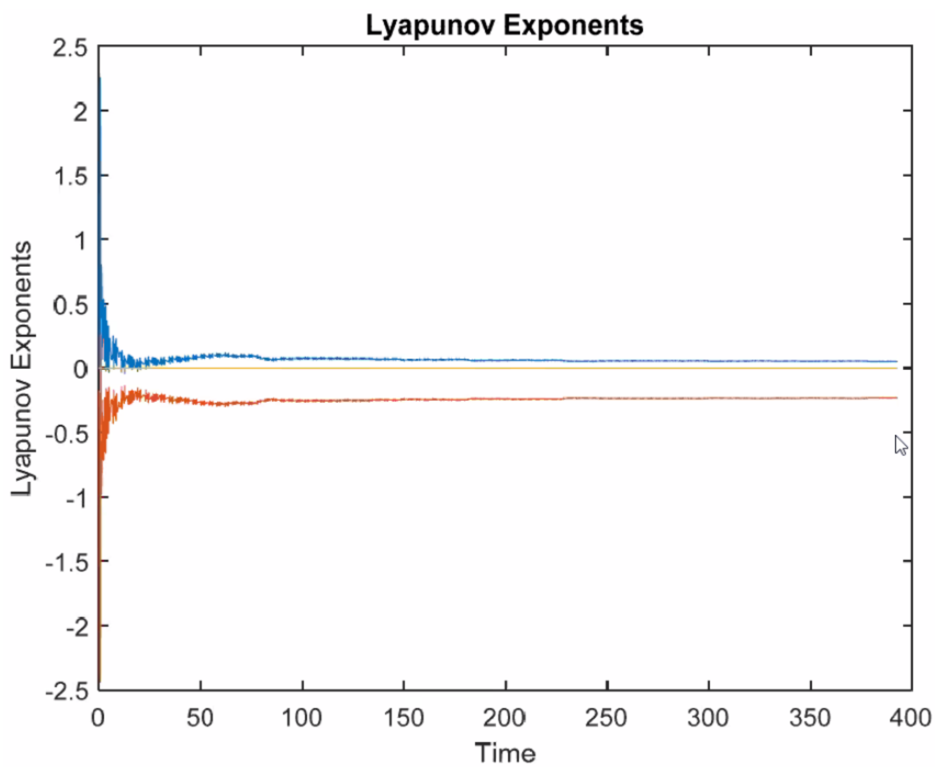
Fractal dimension refers to a non-integer dimension. The idea originates from the observations that chaotic oscillators occupy regions of the phase space.

Fractal dimension is used to measure the extent to which trajectories fill up a certain subspace.

A non-integer dimension is a hallmark of a strange attractor and implies the existence of chaos.

There are several definitions of fractal dimensions, for example, pointwise dimension, capacity dimension, correlation dimension, information dimension, and so on.

Information dimension can be easily determined once the Lyapunov spectrum is known, see Eqs. (2) and (3) in the paper by Wolf et al.



Information $d_I = 2.2229$

$$\ddot{x} + \delta\dot{x} + \alpha x + \beta x^3 = f_0 \cos(\omega t)$$

$\delta = 0.18$, $\alpha = \beta = 1$, $\omega = 0.8$, and $f_0 = 22.5$.