Review of Terminologies

Oscillation, vibration

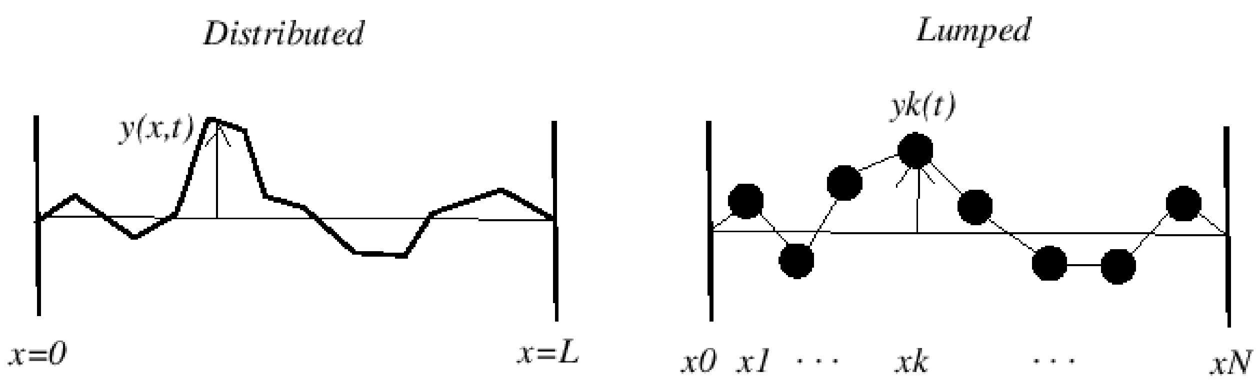
An oscillation is a periodic fluctuation between two ‘things’ – it’s a general term that can refer to anything from a person’s decision-making process, tides, or the pendulum of a clock.

Vibrations are oscillations of a mechanical or structural system about an equilibrium position.

Distributed model, lumped-parameter model; or Continuous model, discrete model

A lumped-parameter model is a system where all dependent variables are a function of time. This generally means solving a set of ordinary differential equations – you could also consider this to be a discrete model.

On the other hand, a distributed model is a system where the dependent variables are a function of time and one (or more) additional spatial variables – you could also consider this to be an analog model.



Degrees-of-freedom (DOFs)

The number of degrees of freedom for a system is the number of kinematically independent variables necessary to completely describe the motion of every particle in the system.

(Simple) Harmonic Motion, periodic motion

This is a ‘all squares are rectangles, but not all rectangles are squares’ scenario. Periodic motion is motion repeated in equal intervals of time – consider a rocking chair, a bounding ball, or a tuning fork.

The undamped motion of a SDOF system is known as simple harmonic motion – that is, simple harmonic motion is a special case of periodic motion where the restoring force on the moving object is directly proportional to the object’s displacement magnitude and acts towards to object’s equilibrium position, resulting in an oscillation that continues indefinitely (so long as it is uninhibited by friction or other means of dissipating energy)

Amplitude, period, circular frequency (or frequency)

The amplitude of a vibration is the maximum displacement from equilibrium. The period is the time required to execute one cycle – it’s usually measured in seconds.

The reciprocal of the period is the frequency and is the number of cycles executed in one second. The units for frequency are cycles/second, or more accurately the inverse of a second, which is known as

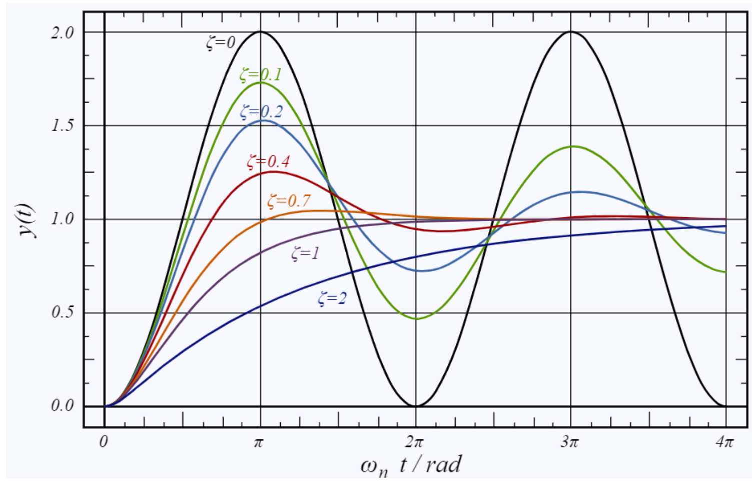
Hertz (Hz). Don't confuse this with the circular frequency, which is also referred to as frequency. The circular frequency (or angular frequency) is the rate at which an angle is changing and is measured in rad/s or revolutions per minute (rpm).

Damping coefficient, damping ratio, critical damping, logarithmic decrement

The damping coefficient of a system is a measure of how quickly it returns to rest as the frictional force dissipates its oscillation energy.

The damping ratio, ζ, describes how oscillations in a system decay after a disturbance – it's defined as the ratio of the damping coefficient of the system's differential equation to the critical damping coefficient. There are four different cases that are represented by the damping ratio:

$\zeta = 0$: undamped
$\zeta < 1$: underdamped
$\zeta = 1$: critically damped
$\zeta > 1$: overdamped



Critical damping exists between the overdamped and underdamped cases, where the system returns to equilibrium in the minimum amount of time – the system fails to overshoot and not a single oscillation is made.

Logarithmic decrement, δ, is defined for underdamped free vibrations as the natural logarithm of the ratio of the amplitudes of vibration on successive cycles. Which doesn't read very well, but if you have a vibration with decreasing amplitudes, it's the natural log of the ratio at which the amplitudes are decreasing.

Free vibration, forced vibration, self-excited vibration

If the vibrations are initiated by an initial energy present in the system and no other source is present, the resulting vibrations are called free vibrations. That is, it's a term that's generally used to indicate that the vibrations present in a system are only due to the initial conditions of the system, and not from external sources.

Conversely, if the vibrations are caused by external forces or motion, then the vibrations are called forced vibrations.

Self-excited vibration is a little more complicated – consider systems where the exciting force is a function of the motion variables (displacement, velocity, or acceleration) and thus varies with the motion it produces (this is called coupling) – this is known as self-excited vibration. This is a wordy definition, but consider examples like friction-induced vibration in vehicle clutches and brakes, or flow-induced vibration in circular saws and CDs.

Transient response, steady state response

The behavior of the system as time gets very large (read: infinite) is called the steady state response. It's independent of the initial position and velocity of the mass. The behavior of the system while it is approaching the steady state is called the transient response.

Time domain, frequency/spectrum domain

A time domain graph shows how a signal changes over time, whereas a frequency-domain graph shows how much of the signal lies within each given frequency band over a range of frequencies. The 'spectrum' of frequency components is the frequency-domain representation of the signal.

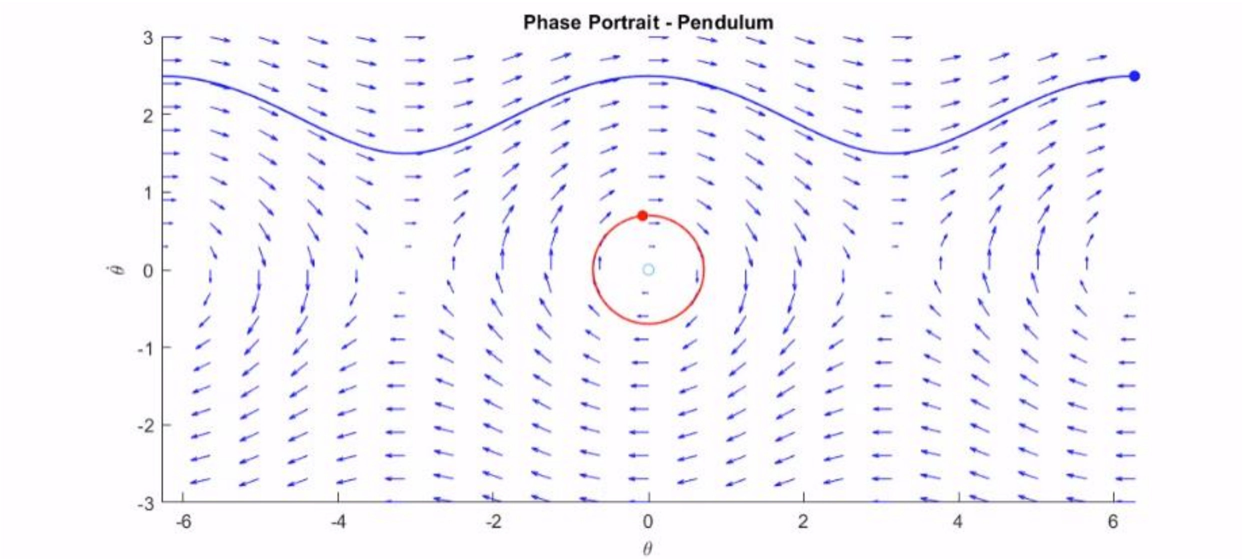
State, state variables, state space model

A state space model is a representation of the dynamics of an N^{th} order system as a first order differential equation in an N −vector. This N −vector is called the state, and the variables contained within the state space model are the state variables.

Phase Portrait

A phase portrait is a geometric representation of the trajectories of a dynamical system in the phase plane.

For example, consider this phase portrait for a pendulum – where the x-axis corresponds to the angle of the pendulum, and the y-axis corresponds to the angular velocity.



(Amplitude) Resonance

Resonance refers to the phenomenon when a quantity (or a state) becomes large.

In addition to amplitude resonance, there are also velocity resonance or phase resonance, and energy resonance, and so on.

FFT, Nyquist Frequency

Nyquist frequency is the minimum sampling frequency without introducing errors. It should be (at least) twice the highest frequency present in the signal.

Chapter 1 – Introduction

1.3 Generalized Coordinates

They are a set of coordinates $(q_1, q_2, q_3, \ldots)$ that describe the configuration (or positions) of a dynamic system.

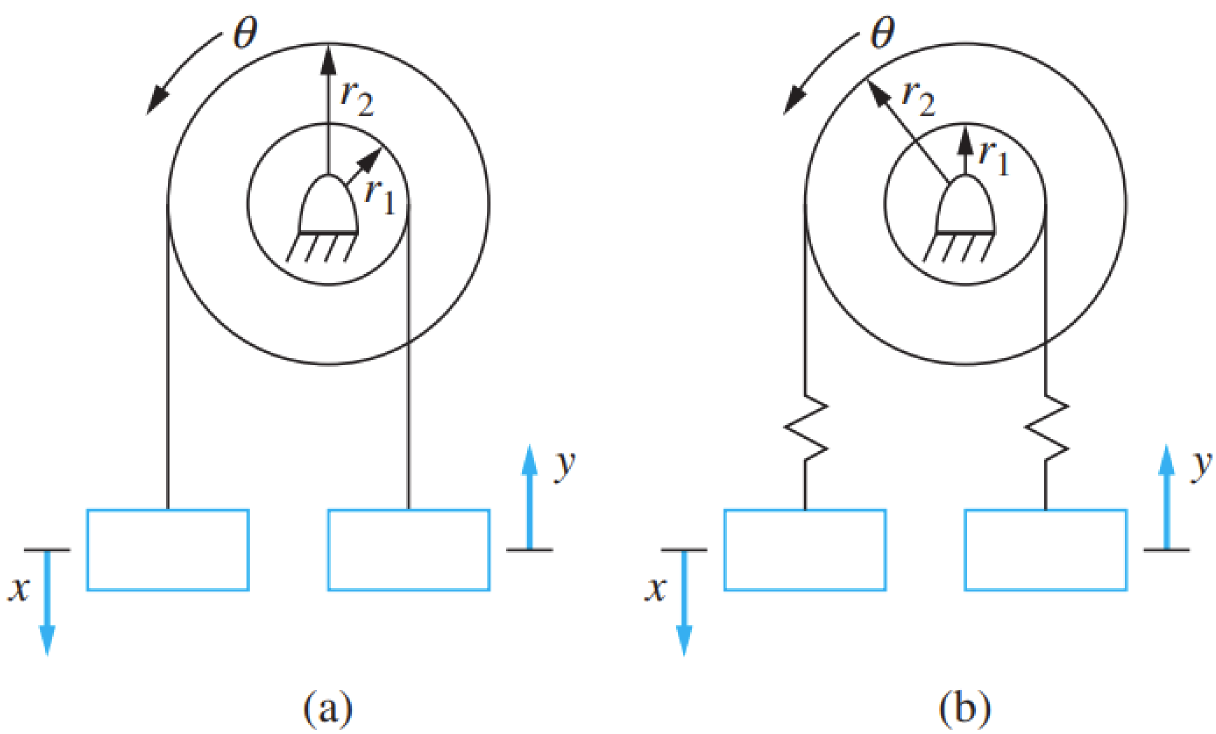
For any given system, the choice of generalized coordinates is not unique, but the number of independent coordinates is unique.

The number of independent coordinates equals the Degrees-of-Freedom (DOFs) needed/used to completely specify the configuration of the system.

When determining degrees of freedom, check if both ends of a 'device' would have the same displacement. If they don't, then it's a new degree of freedom – such as when slip exists or a spring.

Example 1.1
(TODO)

Figure 1.5



(1) Cables are inextensible and no slips between pulley and cables
(2) Cables are inextensible with slips between pulley and cables
(3) Cables are extensible (modeled as springs) and no slips
(4) Cables are extensible with slips

Parts (1) and (2) belong to (a)
Parts (3) and (4) belong to (b)

Determine DOF and q_i (coordinates) of the cases above.

(1) DOF = 1 ; x or y or θ
(2) DOF = 3; x, y, θ or u_1, u_2, θ
(3) DOF = 3; x, y, θ
(4) DOF = 5; x, y, θ, u_1, u_2 (where u_i is typically used for slip)

1.7 Review of Dynamics

1.7.1 Kinematics

Rigid bodies in general motion

1.7.2 Kinetics

Newton's 2nd law of motion

For a particle, eq. (1.32)

$$\sum F = ma$$

For a rigid body in general motion, eqs. (1.33), (1.34)

$$\sum F = m\bar{a} \quad | \quad \sum M_G = \bar{I}\alpha$$

For a rigid body in fixed-axis rotation, eqs. (1.33), (1.35)

$$\sum F = m\bar{a} \quad | \quad \sum M_o = I_o\alpha$$

The difference between (1.34) and (1.35):
$\sum M_G$ is for a rigid body undergoing planar motion – G is the mass center of the rigid body, and $\bar{I}$ is the mass moment of inertia about an axis parallel to the z-axis that passes through the mass center.

$\sum M_o$ is used when the axis of rotation is fixed, and I_o is the moment of inertia about the axis of rotation.

1.7.3 Principle of Work and Energy

- Kinetic Energy T:

For a rigid body, eq. (1.38); use the first term for a particle (there are 2 terms, one for translation and 1 for rotation)

$$T + \left(\frac{1}{2}\right) m\bar{v}^2 + \left(\frac{1}{2}\right)\bar{I}\omega^2$$

For a rigid body in fixed-axis rotation, eq. (1.39)

$$T = I_o\omega^2$$

- Work done by a force $U_{A\to B}$:

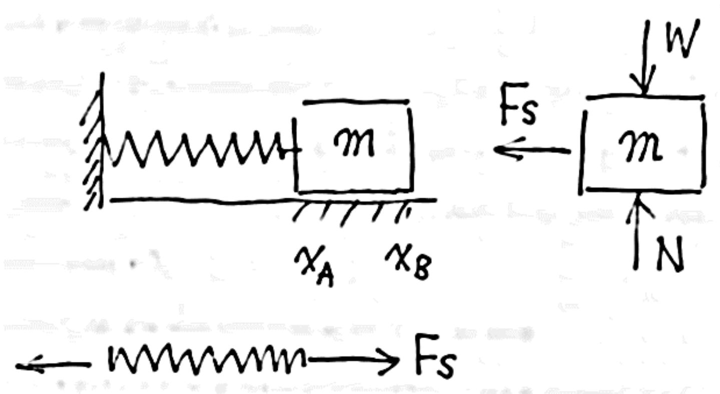
<u>On</u> a particle or a rigid body: eq. (1.40)

$$U_{A\to B} = \int_{\tau_A}^{\tau_B} F\, d\tau$$

- Work done by a moment $U_{A\to B}$:

On a rigid body: eq. (1.41)

$$U_{A\to B} = \int_{\theta_A}^{\theta_B} M\, d\theta$$

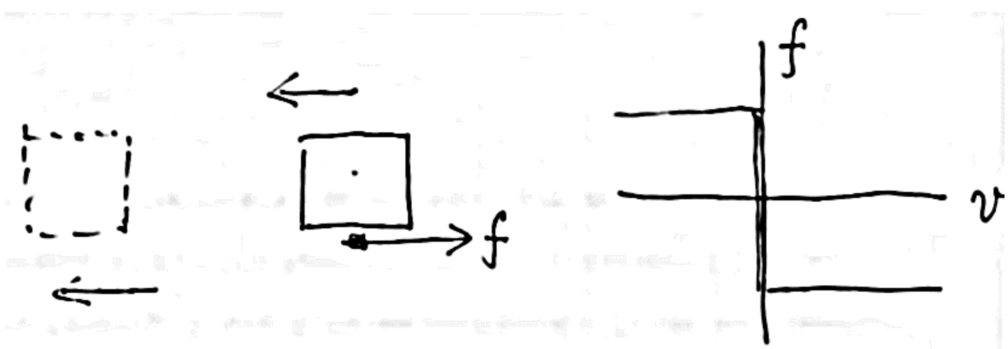


- Conservative Forces and Non-Conservative Forces:

Conservative forces store and release energy.

Typical conservative forces include spring forces (linear or nonlinear; but must be elastic), gravitational forces, and central forces.

The work done by such forces is independent of the path taken from A to B.



Non-conservative forces dissipate energy.
Samples include friction and air resistance.

The work done by such forces is dependent of the path taken from A to B.

Potential energy function V
V is related to conservative forces, and the work done by such forces.

For example:
The potential energy of a gravitational force is $V = mgh$, where h is positive if above the datum.

The potential energy of a linear spring is, $V = (1/2)kx^2$, where x is the elongation or compression from the natural length of the spring.

- Conservative force in terms of V:

$$\vec{F} = -\nabla V = -\left(\frac{\delta}{\delta x}\vec{\imath} + \frac{\delta}{\delta y}\vec{\jmath} + \frac{\delta}{\delta z}\vec{k}\right)V$$

- Work done by a conservative force:

$$U_{A\to B} = V_A - V_B$$

- The principle of energy conservation, eq. (1.45)

$$T_A + V_A = T_B + V_B$$

- The principle of work and energy, eq. (1.47)

$$T_A + V_A + U_{A\to B,NC} = T_B + V_B$$

Where $U_{A\to B,NC}$ is the work done by non-conservative forces from A to B.

In general, friction contributes to $U_{A\to B,NC}$. However, for cases of rolling without slip, friction does no work.

Examples 1.4, 1.5, and 1.6
(TODO)

1.7.4 Principle of Impulse and Momentum
Definition of linear impulse $I_{1\to 2}$ (a vector): eq. (1.48)

$$I_{A\to B} = \int_{t_1}^{t_2} F\,dt$$

Definition of angular impulse about O, $J_{O,1\to 2}$: eq. (1.49)

$$J_{o_{1\to 2}} = \int_{t_1}^{t_2} \sum M_o\,dt$$

Definition of linear momentum L (a vector): eq. (1.50)

$$L = m\bar{v}$$

Definition of angular momentum about G, H_G: eq. (1.51)

$$H_G = \bar{I}\omega$$

The principle of linear momentum: eq. (1.52)

$$L_1 + I_{1\to 2} = L_2$$

The principle of angular momentum: eq. (1.53)

$$H_{G_1} + J_{G_{1\to 2}} = H_{G_2}$$

1. 8 Two Benchmark Examples

Problem 1.18

(TODO)

Problem 1.20 (changed the applied force)

(TODO)

Problem 1.22

(TODO)

Chapter 2 Modeling of SDOF Systems

2.1 Introduction

Key components in a SDOF system:

Inertia element that has mass and stores kinetic energy;
Stiffness element that stores and releases potential energy;
Damping element that dissipates energy.
And source of work or energy (i.e., the excitation)

The chapter cover the principles behind the determination of the key components in a SDOF system, with the objective of modeling a SDOF system with equivalent mass, equivalent stiffness, equivalent damping, and equivalent excitation.

Principles reviewed in Ch. 1: Newton's laws; Work & energy; and Impulse & Momentum.

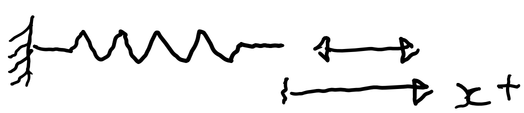
Topics covered:
2.2-2.4: stiffness
2.5-2.6: damping
2.7: inertia/mass
2.8: external sources
2.9: FBD method (or Newton's 2nd law)
2.12: equivalent system method
2.14: further examples

Additional topics:
2.10: static deflection & gravity
2.13: benchmark example
2.15: chapter summary

2.2 Springs

2.2.1 Introduction

<u>1. translational springs</u>
$F = f(x)$: spring force



Where x is the stretch or compression from the natural length.

For spring materials having the same properties in tension and under compression, $f(x)$ is an odd function.
Or: $f(-x) = -f(x)$

Taylor expansion of $f(x)$ about $x = 0$, then

$$F = f(x) \approx k_1 x + k_3 x^3 + k_5 x^5 + \cdots \tag{2.3}$$

re. Eq. (2.3):

i) All springs are inherently nonlinear; and
ii) Linear springs result from the assumption of small x

For linear springs,

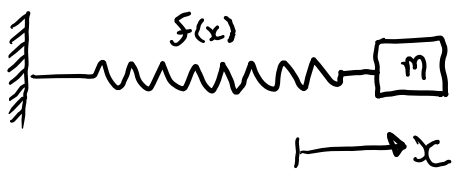
$$F = f(x) \approx k_1 x \tag{2.4}$$

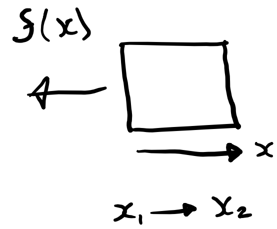
2. work and energy for translational springs

A particle is attached to a spring. The work done by the spring force, on the particle, when the particle moves from x_1 to x_2 (for the particle) is:

$$U_{1\to2} = \left(\frac{1}{2}\right) k x_2^2 - \left(\frac{1}{2}\right) k x_1^2 \tag{2.5}$$

Consider:





$$U_{1\to2} = \int_{x_1}^{x_2} -f(x)\, dx$$

If $f(x) = k_1 x + k_3 x^3$

If $f(x) = kx$, then $U_{1\to2}$ is by Eq. (2.5)

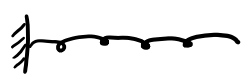
The potential function V (for the spring) is:

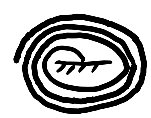
$$V(x) = \left(\frac{1}{2}\right) k x^2 \tag{2.6}$$

Eqs. (2.5) and (2.6): for linear translational springs only.

3. torsional springs

Eqs. (2.7) and (2.8): for linear torsional springs only.





$$M = k_t \theta \tag{2.7}$$

$$V = \frac{1}{2} k_t \theta^2 \tag{2.8}$$

2.2.2 Helical coil springs

Eq. (2.11) relates spring constant k to coil spring material and dimensions.

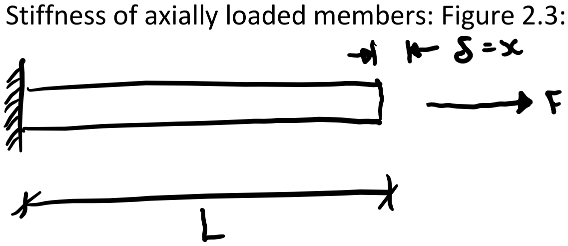
$$k = \frac{GD^4}{64Nr^3} \tag{2.11}$$

It is a formula commonly seen in machine design texts.

2.2.3 Elastic elements as springs

This sub-section deals with spring constant (or stiffness) when the spring is, for example, an axially loaded rod, a straight member in torsion, a simply supported beam, and so on.

Stiffness of axially loaded members: Figure 2.3:



$$\delta = \frac{FL}{EA} = x$$

$$k = \frac{F}{x} = \frac{EA}{L} \tag{2.16}$$

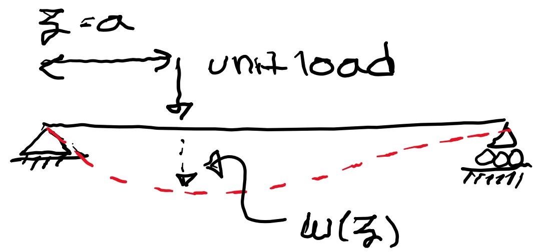
Stiffness of beams and frames:

i) $\mathrm{k} = \mathrm{F}/\Delta$: Δ must be the deflection in the same direction of F, and at the point of application of F;

ii) Generalized interpretation of $k = F/\Delta$: F can be replaced by moment M; Δ represents the angular deflection at the point of application of M and in the same sense of M;

iii) Eqs. (2.19) and (2.20): both involve $\omega(z)$ which is the deflection due to a unit load at z; and

iv) Table D. 2.: list of $\omega(z)$, given as $y(z)$ on Table D.2



where

$x = F \cdot \omega(z)$

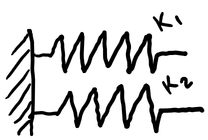
Stiffness of circular shafts:

i) See top portion of p. 62

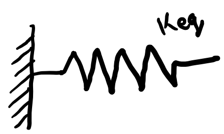
2.2.4 Static deflection

p. 61 Figure 2.5, Eq. (2.23)

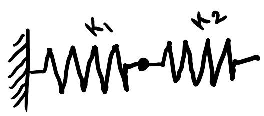
2.3.1 Parallel combination



$K_{eq} = k_1 + k_2$



2.3.2 Series combination



$\frac{1}{K_{eq}} = \frac{1}{k_1} + \frac{1}{k_2}$

<u>Notes on Table D.2</u>

Table D.2 (uniform beam, unit concentrated load at $z = a$)

$$y(z) = \frac{1}{EI}\left[\frac{1}{6}(z-a)^3 u(z-a) + \frac{1}{6}\sum_{i=1}^{n} R_i (z-z_i)^3\, u(z-z_i) + C_1\frac{z^3}{6} + C_2\frac{z^2}{2} + C_3 z + C_4\right]$$

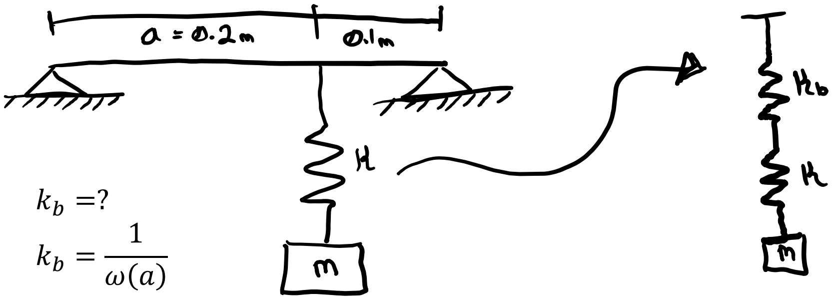
Referring to Figure D.1

z_i is the z —coordinate of the i —th support (excluding the first of the left support)

$u(z-a)$ is the unit step; i.e.

$$u(z-a) = \begin{cases} 1 & z > a \\ 0 & z \le a \end{cases}$$

Example 2.4



$k_b = ?$

$k_b = \dfrac{1}{\omega(a)}$

$E = 210\ GPa$

$I = 5 \cdot 10^{-4}\ m^4$

Using Table D.2

Case 6:

$a = 2$

$z_1 = 3$

And:

$R_1 = -\dfrac{2}{3}$

$C_1 = -\dfrac{1}{3}$

$C_2 = C_4 = 0$

$C_3 = -\left(1 - \dfrac{a}{z_1}\right)\dfrac{z_1^2}{6}\left[\left(1 - \dfrac{a}{z_1}\right)^2 \cdot u(z_1 - a) - 1\right]$

$\therefore u(3-2) = u(1) = 1$

$\therefore C_3 = \left(\dfrac{4}{9}\right)$

$$\therefore y(z) = \frac{1}{EI}\left[\frac{1}{6}(z-2)^3 u(z-2) + \frac{1}{6}\left(-\frac{2}{3}\right)(z-3)^2 u(z-3) + \left(-\frac{1}{3}\right)\frac{z^3}{6} + 0 + \left(\frac{4}{9}\right)z + 0\right]$$

Now $z = a = 2$ (terms highlighted turn to zero)

$$\therefore y(z) = \frac{1}{EI}\left[\frac{1}{6}(0)^3 u(0) + \frac{1}{6}\left(-\frac{2}{3}\right)(-1)^2 u(-1) + \left(-\frac{1}{3}\right)\frac{(2)^3}{6} + 0 + \left(\frac{4}{9}\right)(2) + 0\right]$$

$$\therefore \omega(z) = y(z) = \frac{1}{EI}\left[0 + 0 - \frac{4}{9} + 0 + \frac{8}{9} + 0\right]$$

$$\therefore \frac{4}{9EI}$$

$$k_b = \frac{1}{\omega(2)} = \frac{9EI}{4} = 2.3625 \cdot 10^8\ \left(\frac{N}{m}\right)$$

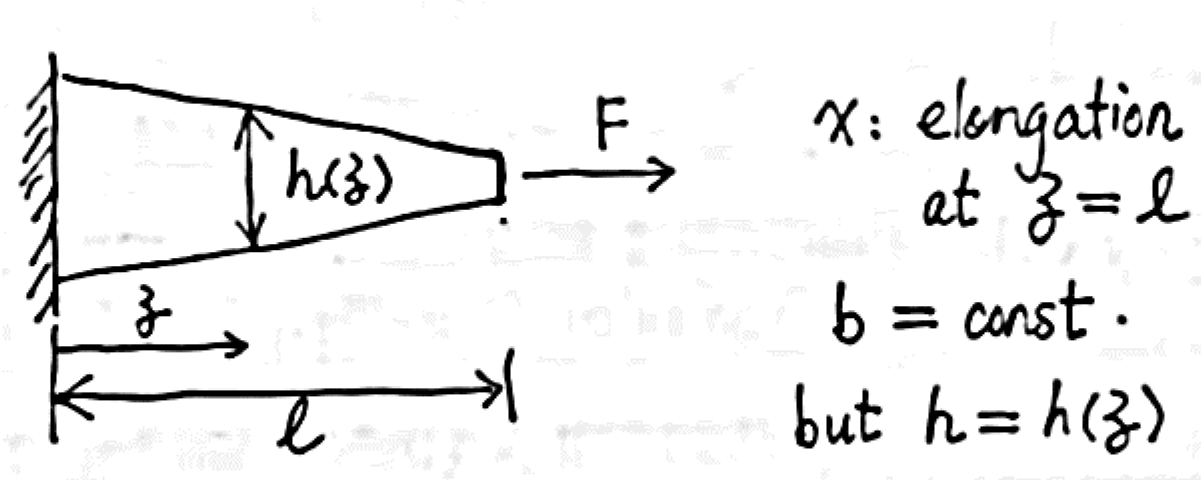
Choose

$$k = 1 \cdot 10^8\ \left(\frac{N}{m}\right)$$

Then

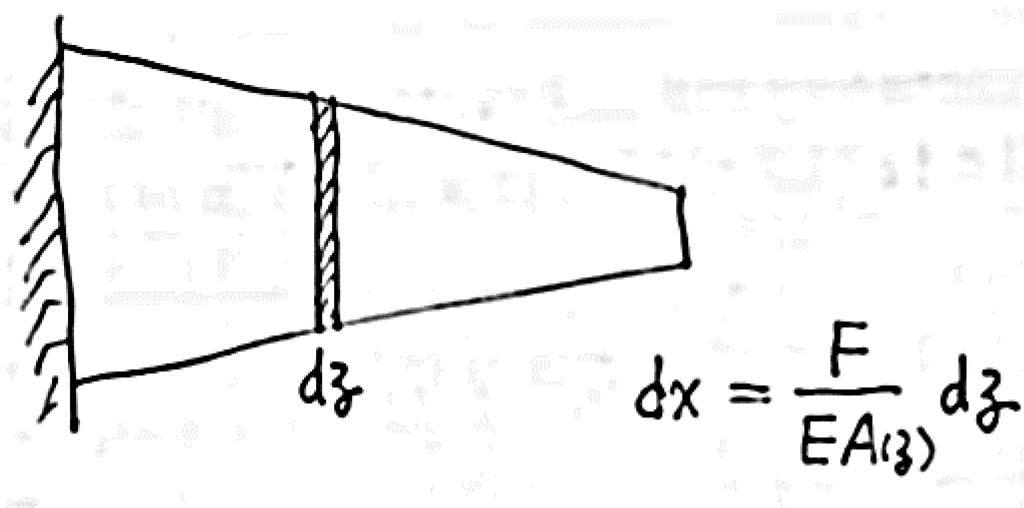
$$k_{eq} = \left(\frac{1}{k_b} + \frac{1}{k}\right)^{-1} = 0.7026 \cdot 10^8\ (N/m)$$

Example 2.4 (d)



$$x = \frac{FL}{EA}$$

This is no longer applicable (EA is no longer constant)



$$\therefore x = \int_0^l \frac{F}{EA(z)}\, dz$$

By integral approach:

$$k = 32.568 \cdot 10^6 \ (N/m)$$

Using A_{av}:

$$k = 30.7125 \cdot 10^6 \ (N/m)$$

2.3.3 General Combination

The system has various springs, translational and/or torsional. The i-th spring has potential energy $(1/2)k_i x_i^2$, where k_i and x_i should be interpreted in the general sense.

Total potential energy in the system is:

$$V = \sum(1/2)k_i x_i^2$$

For the equivalent spring k_{eq}, the generalized coordinate is x. Each x_i is assumed directly proportional to x. Then:

$$V = (1/2)k_{eq}x^2$$

Therefore

$$k_{eq}x^2 = \sum k_i x_i^2$$

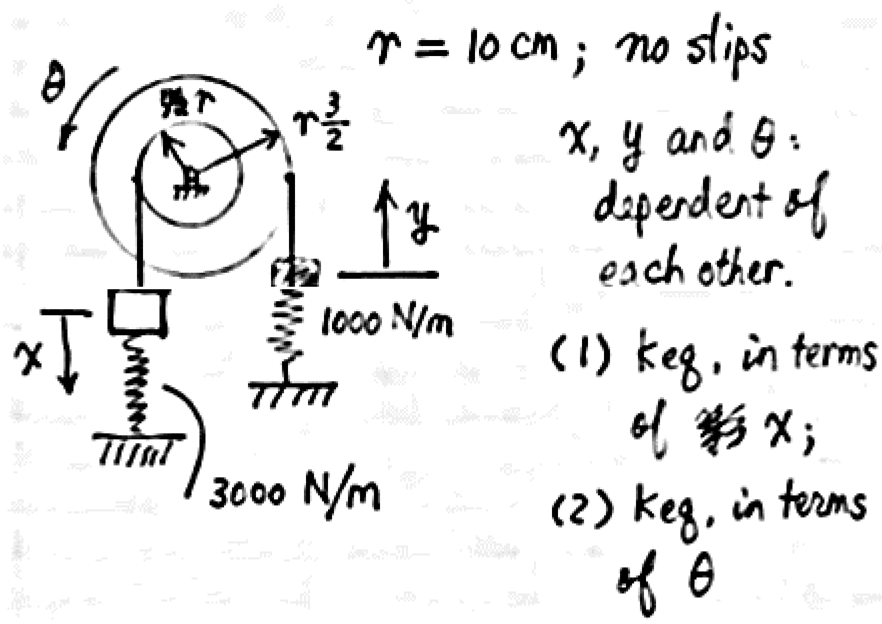
And

$$k_{eq} = \sum k_i \left(\frac{x_i}{x}\right)^2$$

Note: each x_i/x is a constant

Example 2.5

On a horizontal plane



Determine:
(1) k_{eq} in terms of x
(2) k_{eq} in terms of θ

Solution:

$\therefore no\ slips$
$\therefore x = r\theta$
$y = \left(\frac{3}{2}\right) r\theta$
$y = \left(\frac{3}{2}\right) x$

(1) k_{eq} in terms of x

$$V = \left(\frac{1}{2}\right)(3000)x^2 + \left(\frac{1}{2}\right)(1000)y^2$$
$$= \left(\frac{1}{2}\right)(3000)x^2 + \left(\frac{1}{2}\right)(1000)\left(\frac{9}{4}\right)x^2$$
$$= \left(\frac{1}{2}\right)\left(3000 + \frac{9}{4} \cdot 1000\right)x^2$$
$$\therefore k_{eq} = 5{,}250\ (N/m)$$

(2) k_{eq} in terms of θ

$$V = \left(\frac{1}{2}\right)(3000)x^2 + \left(\frac{1}{2}\right)(1000)y^2$$
$$= \left(\frac{1}{2}\right)k_{eq}\theta^2$$
$$k_{eq} = 52.5\ (N \cdot m/rad)$$

2.4 Other sources of potential energy

2.4.1 Gravity

It is a conservative force

V due to gravity is:

$$V = mgh$$

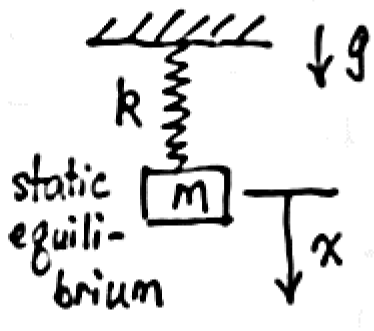
Where h is the positive if the particle relative to the datum. h is positive if the particle is positioned above the datum, and h is negative if positioned below.

Example 2.6 (A pendulum, 3 choices of datum)

V due to spring force as well as gravity:

$$V = V_{spring} + V_{gravity}$$

As an example, a mass (i.e. a particle) is suspended from a spring (Figure 2.15)



- If datum is located at the static equilibrium position:

$$V = \left(\frac{1}{2}\right)k(\Delta_{st} + x)^2 - mgx$$

- The work done (by spring force and gravity, on the particle) from 0 to x is:

$$U_{1\to 2} = V_1 - V_2$$

- The principle of energy conservation can be more conveniently expressed as:

$$T_1 + V_1 = T_2 + V_2$$
$$\rightarrow T_1 - T_2 = -U_{1\to 2} = \left(\frac{1}{2}\right)kx^2$$

As long as x is measured from the static equilibrium position.

Note: Static deflection is <u>not</u> always by $\Delta_{st} = mg/k$, see for example, Example 2.8 and Problem 2.18.

2.4.2 Buoyancy

If a floating or submerged object has constant cross-section, buoyancy functions very much like the linear translational spring.

ρ: mass density of fluid per unit volume, in kg/m^3
A: cross-sectional area of the object

Then spring constant is:

$$k = \rho g A$$

The work done by buoyancy and gravity on the object is:

$$U_{1\to 2} = -\left(\frac{1}{2}\right)kx^2$$

Where x is measured from the static equilibrium position.

Note: static deflection is <u>not</u> by $\Delta_{st} = mg/k$ when buoyancy is involved.

2.5 Viscous Damping

Viscous damping force has a magnitude that is directly proportional to the velocity.

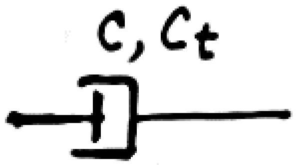
$F = cv$: c is the (translational) damping coefficient;
Figure 2.20; eq. (2.37)

Or,

$M = c_t\dot{\theta}$: c_t is the (torsional) damping coefficient;
Figure 2.21; eq. (2.42)

Direction of viscous damping force: opposite to v or $\dot{\theta}$

Schematic representation:



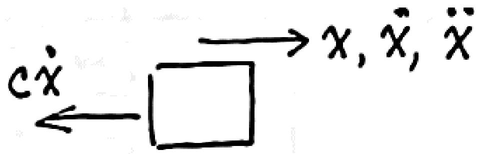
Devices to achieve viscous damping: the dashpot (Figure 2.19); the piston-cylinder damper (Figure 2.20); the torsional viscous damper (Figure 2.21).

Example 2.9

2.6 Energy Dissipated by Viscous Damping

Viscous damping force is non-conservative.

The dissipated energy is measured by work done, see eq. (2.44)



$$U_{1\to2} = \int_0^x -c\dot{x}\,dx$$

Energy dissipated by a system of dampers: eq. (2.45)

$$U_{1\to2} = \sum \int_0^{x_i} -c_i\dot{x}_i\,dx_i$$

Equivalent damping coefficient c_{eq} in terms of generalized coordinate x:

- $\dot{x}$ and $\dot{x}_\iota$ are directly proportional to each other (i.e., $\frac{\dot{x}_i}{\dot{x}} = constant = \gamma_y$)

$$\int_0^{x_i} -c_i\dot{x}_i\,dx_i$$

Consider:
$\dot{x}_i = \gamma_i \cdot \dot{x}$
$x_i = \gamma_i \cdot x$

$$= \int_0^x -c_i(\gamma_i \cdot \dot{x})d(\gamma_i \cdot x) = \int_0^x -c_i\gamma_i^2\dot{x}\,dx$$

- Energy dissipation

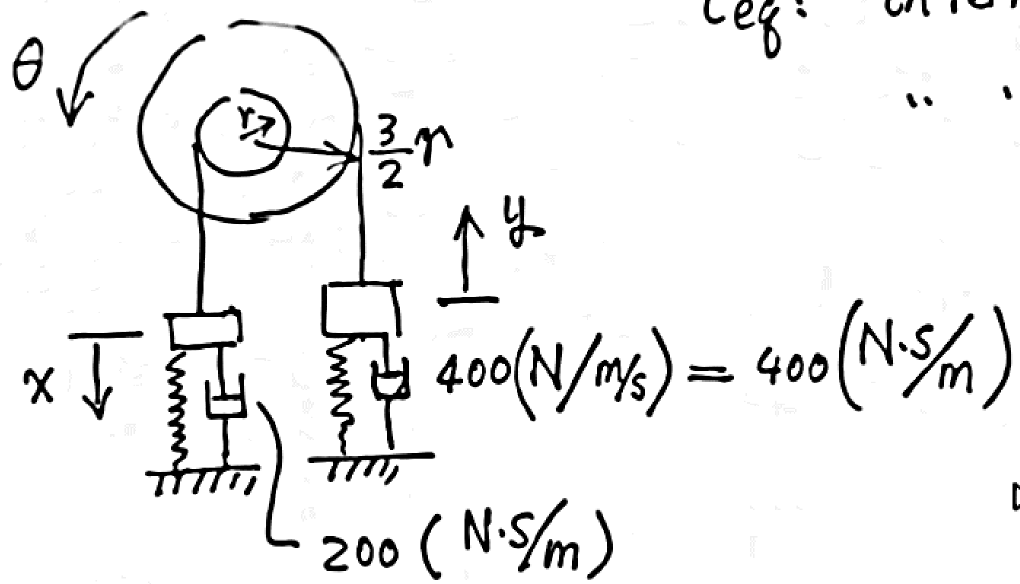
$$U_{1\to2} = \sum \int_0^{x_i} -c_i\,\dot{x}_i\,dx_i = \sum \int_0^x -c_i\gamma_i^2\,\dot{x}dx$$
$$U_{1\to2} = \int_0^x -c_{eq}\dot{x}\,dx$$

Example 2.10

on a horizontal plane

C_{eq}: in terms of x

" " " θ



Solution:

$x = r\theta, \quad y = \left(\frac{3}{2}\right) r\theta$

$\dot{x} = r\dot{\theta}, \quad \dot{y} = \left(\frac{3}{2}\right) r\dot{\theta}$

C_{eq} in terms of x:

$y = \left(\frac{3}{2}\right) x, \quad \dot{y} = \left(\frac{3}{2}\right) \dot{x}$

$$\therefore U_{1\to 2} = \int_0^x -(400)\left(\frac{3}{2}\right)\dot{x}\, d\left(\frac{3}{2}x\right) + \int_0^x -(200)\dot{x}\, dx$$

$$\therefore U_{1\to 2} = \int_0^x \underbrace{\left(-\frac{3600}{4} - 200\right)}_{-C_{eq}} \dot{x}\, dx$$

c_{eq} in terms of θ:

$$U_{1\to 2} = \int_0^\theta (-400)\left(\frac{3}{2}\right) r\dot{\theta}\, d\left(\frac{3}{2}r\theta\right) + \int_0^\theta (-200) r\dot{\theta}\, d(r\theta)$$

$$U_{1\to 2} = \int_0^\theta -\underbrace{\left(400 \cdot \frac{9}{4} r^2 + 200 r^2\right)}_{C_{eq}} \dot{\theta}\, d\theta$$

2.7 Inertia Elements

2.7.1 Equivalent mass

The kinetic energy of a system of rigid bodies is

$$T = \sum\left(\frac{1}{2}m_i v_i^2 + \frac{1}{2}I_i\omega_i^2\right)$$

Note:

(i) Table 2.1 for centroidal moments of inertia

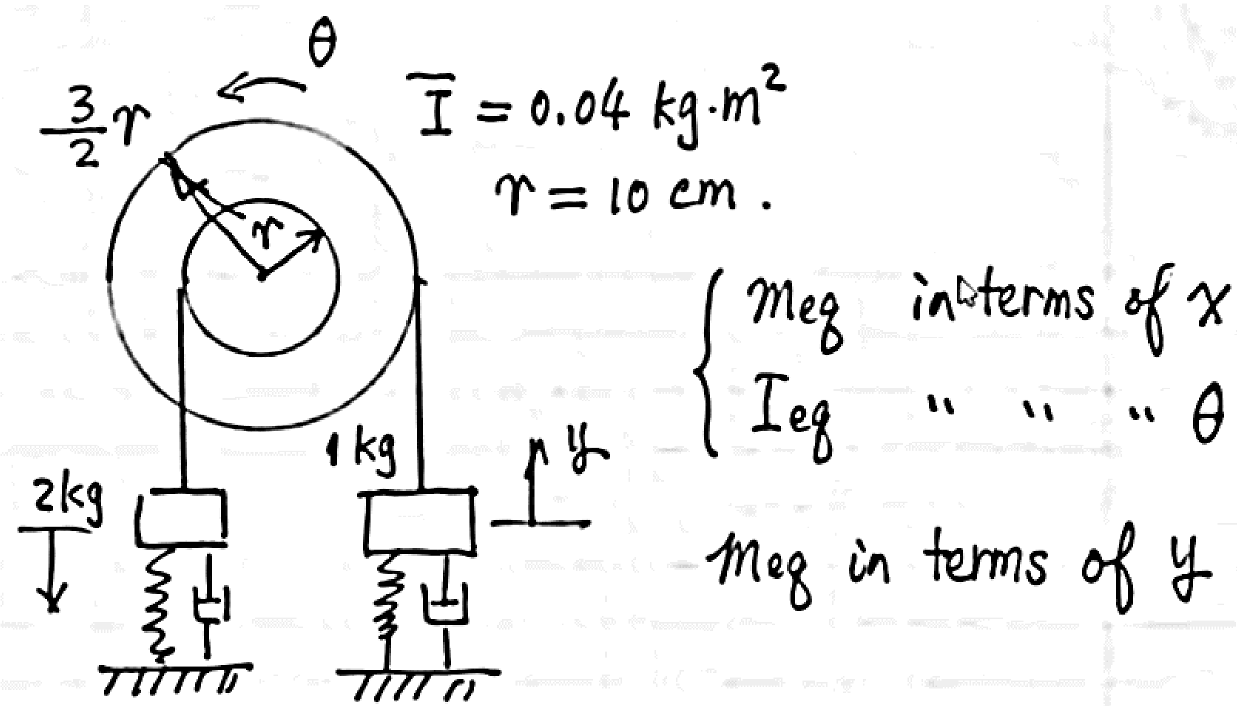
(ii) If v_i and ω_i are directly proportional to a generalized coordinate x, the kinetic energy is then, eq. (2.50)

$$T = \left(\frac{1}{2}\right)m_{eq}\dot{x}^2$$

(iii) Or eq. (2.51) if the generalized coordinate is θ.

$$T = \left(\frac{1}{2}\right)I_{eq}\dot{\theta}^2$$

Example 2.11



Solution:
x in terms of y
θ in terms of y

$$\therefore \dot{x} = \frac{2}{3}\dot{y} \quad ; \quad \dot{\theta} = \frac{2}{3r}\dot{y}$$

$$T = \left(\frac{1}{2}\right)(2)\dot{x}^2 + \left(\frac{1}{2}\right)(1)\dot{y}^2 + \left(\frac{1}{2}\right)(0.04)\dot{\theta}^2$$

$$T = \left(\frac{1}{2}\right)(2)\left(\frac{4}{9}\right)\dot{y}^2 + \left(\frac{1}{2}\right)(1)\dot{y}^2 + \left(\frac{1}{2}\right)(0.04)\left(\frac{4}{9r^2}\right)\dot{y}^2$$

$$T = \left(\frac{1}{2}\right)\underbrace{\left[\frac{8}{9} + 1 + \frac{0.16}{9(0.1)^2}\right]}_{m_{eq}}\dot{y}^2$$

And $m_{eq} = \frac{11}{3}\,(kg)$

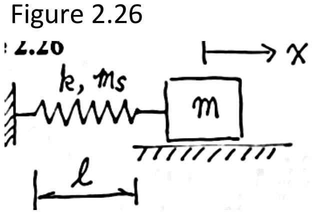
2.7.2 Inertia effects of springs

In reality, springs are structural components. They have mass.

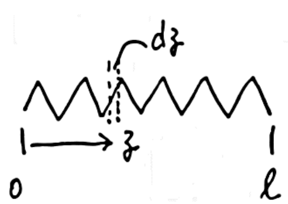
When the mass of a spring is small but not negligible, the mass of the spring is typically added to that of the particle or rigid body.

$$T = T_s + \frac{1}{2}mv^2 = \frac{1}{2}m_{eq}\dot{x}^2$$

Figure 2.26



ρ =mass density per unit length
$\rho \cdot l = m_s$



at z, displacement is:
$\frac{z}{l}x = u(z)$

$$\therefore dT_s = \frac{1}{2}(\rho\, dz)[\dot{u}(z)]^2$$

$$\therefore T_s = \int_0^l \frac{\rho}{2}\left(\frac{\dot{x}}{l}z\right)^2 dz$$

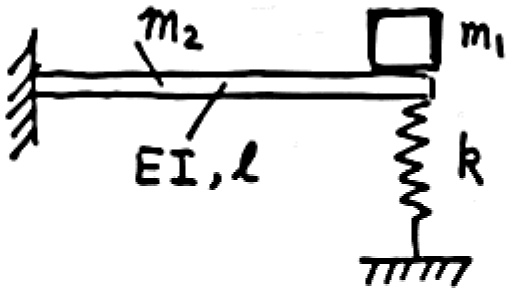
$$T_s = \frac{1}{2}\left(\frac{m_s}{3}\right)\cdot \dot{x}^2$$

$$\therefore T = \frac{1}{2}\left(\frac{m_s}{2}\right)\dot{x}^2 + \left(\frac{1}{2}\right)m\dot{x}^2$$

$$T = \frac{1}{2}\left(m + \frac{m_s}{3}\right)\dot{x}^2$$

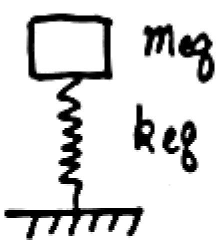
Example 2.13 (a beam as a spring)

Example: Evaluate the equivalent stiffness and equivalent mass of the system shown in Figure P2.20, where the beam has a mass of 75 kg.



$l = 3\ m;\ E = 210\ GPa$
$I = 8.2 \times 10^{-7}\ m^4$
$m_1 = 150\ kg$
$m_2 = 75\ kg$
$k = 5000\ N/m$

Solution:



k_b: stiffness of the beam

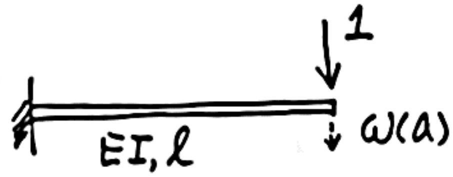


Table D.2, Case 1

$$C_1 = -1$$
$$C_2 = a = l = 3$$
$$C_3 = C_4 = 0$$
$$y(z) = \frac{1}{6EI}(9.z^2 - z^3)$$
$$\omega(a) = y(a) = y(3) = \frac{9}{EI}$$

$$\therefore k_b = \frac{EI}{9} = 19133\ (N/m)$$
$$and\ k_{eq} = k + k_b = 24133\ (N/m)$$

Now,
$m_{eq} = m_1 + m_b$

Where,
Assume ρ being the mass density per unit length, such that $m_2 = \rho \cdot l$

Dynamic deflection, in terms of (z, t)

$$X(z,t) = x(t) \cdot Y(z)$$

Where:
$x(t)$ is the response of the system
$Y(z)$ is chosen as the static deflection meeting the requirement of $Y(a) = 1$.

$\therefore$ scaling $y(z)$ such that at the tip, the static deflection is unity.

$\therefore Y(z) = k_b \cdot y(z) = \dfrac{1}{54}(9z^2 - z^3)$
$dz \to \rho dz$
[$\to$ at z, dm has velocity of $\frac{d}{dt}\, X(z,t) = \dot{x} \cdot Y(z)$]

$$\therefore T_b = \tfrac{1}{2}\textstyle\int_0^l \rho Y^2(z)\, \dot{x}^2\, dz$$
$$= \frac{\rho \dot{x}^2}{2} \int_0^l Y^2(z)\, dz$$
$$= \frac{1}{2}\underbrace{(0.23571 \cdot \rho \cdot l)}_{m_b = 0.23571\, m_z}\dot{x}^2$$

$\therefore m_b = 17.679\ (kg)$
$\therefore m_{eq} = 167.7\ (kg)$

2.7.3 Added mass

If the particle is submerged in an inviscid fluid, the movement of the particle causes movement of the surrounding fluid, see figure 2.29. The added mass is to include the inertia effect of the fluid.

Kinetic energy of the system is:

$$T = T_m + T_f$$

T_f, the kinetic energy of the fluid, is:

$$T_f = \frac{1}{2} m_a \dot{x}^2$$

Or,

$$T_f = \left(\frac{1}{2}\right) I_a \omega^2$$

See Table 2.2 or m_a, or Table 2.3 for I_a.

The equivalent mass is then $m_{eq} = m + m_a$ or $I_{eq} = I + I_a$.

2.8 External Sources

Excitation can be a force (or moment), or a motion input.

Work done by a force moment is, eq.(2.63)

$$U_{1\to 2} = \int_{x_1}^{x_2} F(t)\,dx = \int_{t_1}^{t_2} F(t)\dot{x}\,dt$$

Work done by a number of forces/moments is, eq. (2.64)

$$U_{1\to 2} = \sum \int_{t_1}^{t_2} F_i(t) \cdot \dot{x}_i\,dt$$

Assume $\dot{x}_i$ and $\dot{x}$ are directly proportional to each other $\frac{\dot{x}_i}{\dot{x}} = \gamma_i$

$$= \sum \int_{t_1}^{t_2} F_i(t) \cdot \gamma \cdot \dot{x}\,dt$$

The equivalent force is,

$$\sum \int_{t_1}^{t_2} F_i(t) \cdot \gamma_i \cdot \dot{x}\,dt$$

$$\int_{t_1}^{t_2} F_{eq}(t) \cdot \dot{x}\,dt$$

Examples 2.14 and 2.15

Summary of principles behind equivalent stiffness, damping, mass, and excitation:

Equivalent stiffness: potential energy of the original system = potential energy of the equivalent stiffness

Equivalent damping: energy dissipated in the original system = energy dissipated by the equivalent damper.

Equivalent mass: kinetic energy of the original system= kinetic energy of the equivalent mass

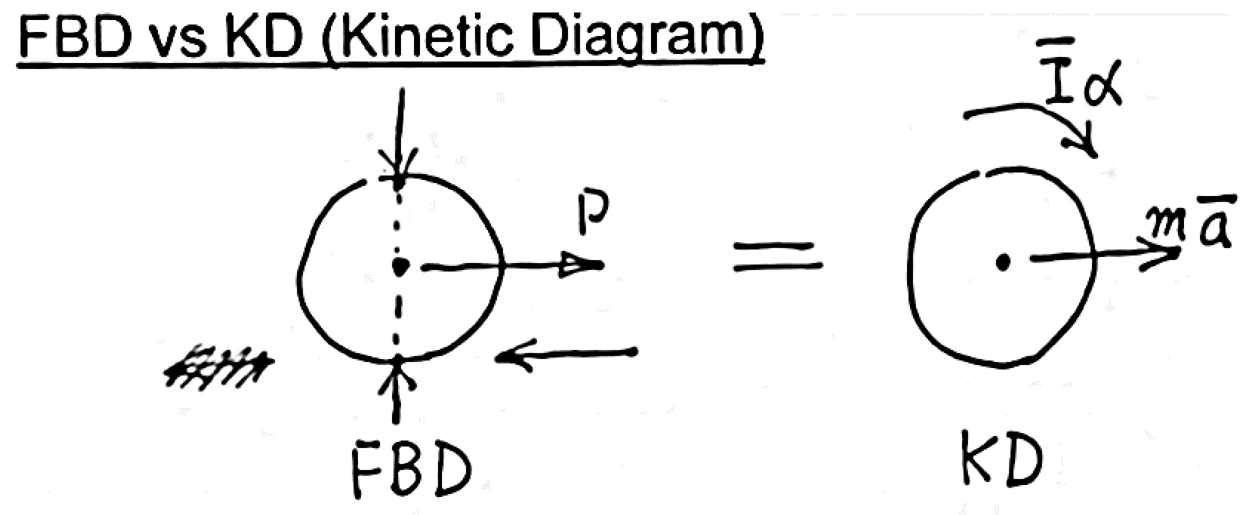
Equivalent excitation: work done by external forces (excitations) in the original system = work done by the equivalent excitation

2.9 FBD Method

This method combines FBD and Newton's Laws of motion. It is the fundamental way of arriving at the equation of motion, or the E.O.M.

Advantages: universal; able to deal with most engineering systems; able to determine reaction forces/moments in terms of the generalized coordinate.

Disadvantages: tedious, involving many equations when dealing with a system of rigid bodies.



2.12 Equivalent Systems Method

It is based on the general form of the principle of work and energy:

$$T_1 + V_1 + U_{1\to 2} = T_2 + V_2$$

Note that $U_{1\to 2,NC}$ is usually how it's written, where NC is non-conservative

Where $U_{1\to 2}$ includes the work done by the viscous damping forces and the excitation forces.

State 1: pertaining to a specific or known configuration, for example, the static equilibrium configuration. That is,

$$T_1 = const. \ ; \quad V_1 = const.$$

State 2: pertaining to an arbitrary configuration. That is,

$$T_2 = T(t) \ ; \quad V_2 = V(t)$$

Then,

$$T_1 + V_1 + U_{1\to 2} = T + V$$

And

$$\frac{d}{dt}(T + V) = \frac{d}{dt}(U_{1\to 2})$$

$$T = \frac{1}{2} m_{eq} \dot{x}^2$$
$$\frac{dT}{dt} = \frac{1}{2} m_{eq} 2\dot{x}\ddot{x}$$

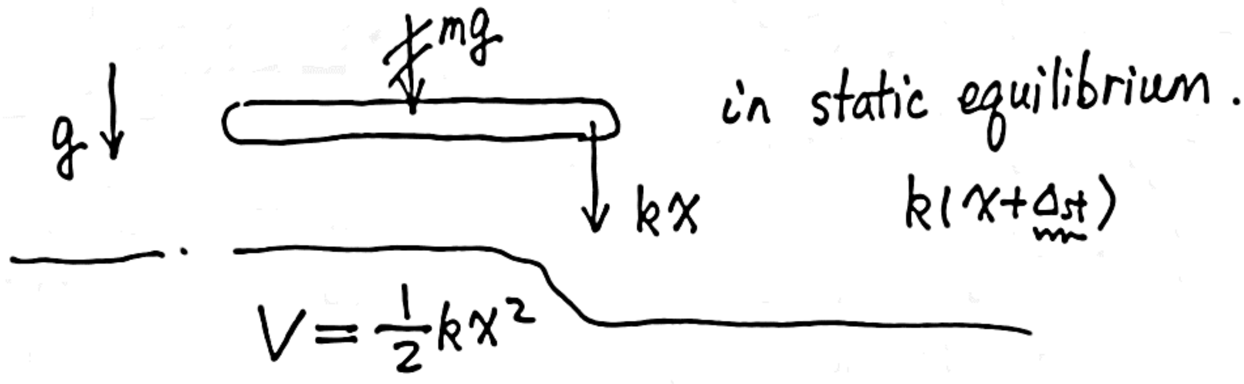
Following the few steps shown in eqs. (2.78) through (2.83), eq. (2.84) which is the equation of motion, is then obtained:

$$m_{eq}\ddot{x} + c_{eq}\dot{x} + kx = F_{eq}(t)$$

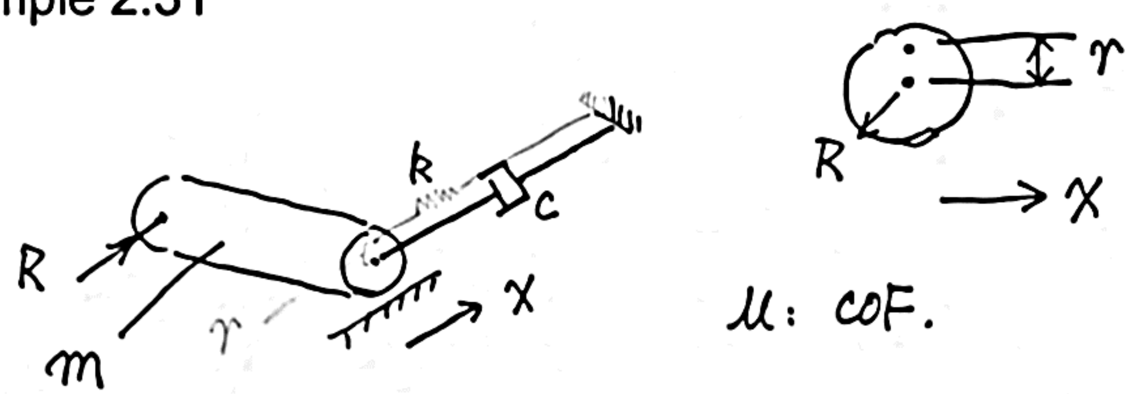
Or Eq. (2.85) if the generalized coordinate is an angle.

Examples 2.25~2.29
(TODO)

<u>Static deflection and gravity</u> (for Section 2.9 and Section 2.12)



Example 2.31



(a) Rolling without slip

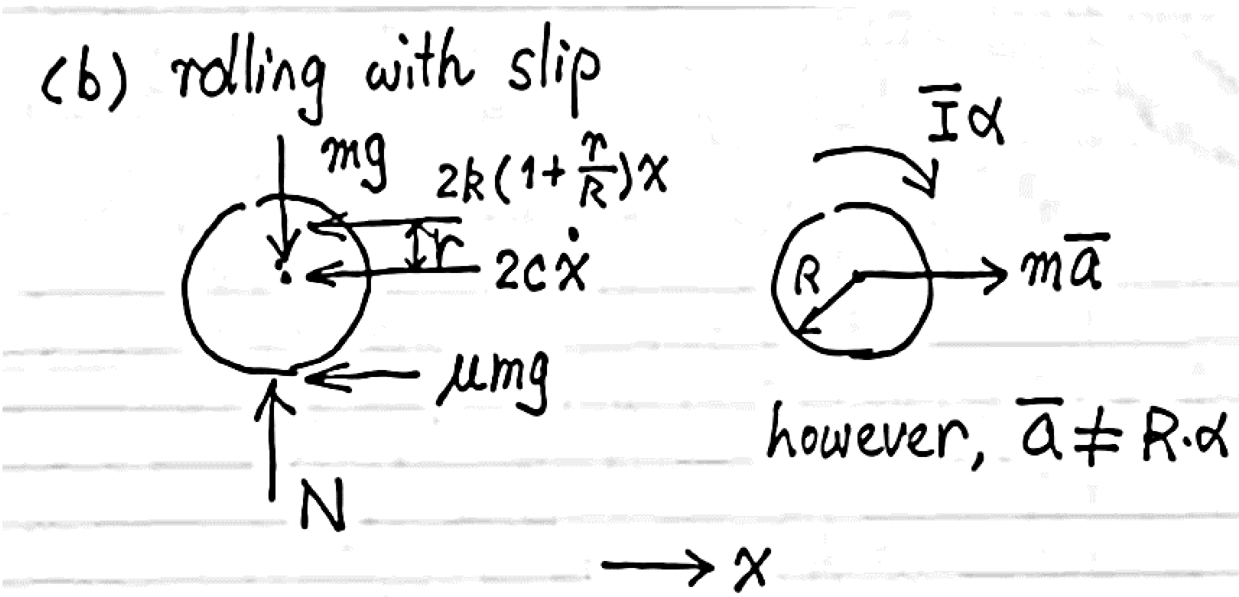
2.12 m_{eq}: need T

$$T = \frac{1}{2} m\dot{x}^2 + \frac{1}{2}\bar{I}\dot{\theta}^2$$

$$T = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}\bar{I}\left(\frac{1}{2}mR^2\right)\left(\frac{\dot{x}}{R}\right)^2$$
$$T = \frac{1}{2}\left(\frac{3}{2}m\right)\dot{x}^2$$

$\therefore m_{eq} = \frac{3}{2}m$

$\therefore m_{eq}\ddot{x} + c_{eq}\dot{x} + k_{eq}x = 0$



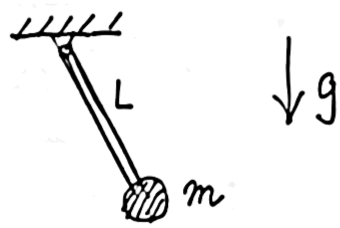
2.11 Small Angle or Displacement Assumption

Structural element (shaft, rod, beam, etc.) as a spring:

If the material is metal, the linear spring assumption is valid.

What if the material is, say, plastic or composites, or others? The linear assumption needs to be thoroughly explained. For example, plastic is nonlinear even when deformation is small.

Example 2.23
(a pendulum)



Without any assumption, the E.O.M. is:

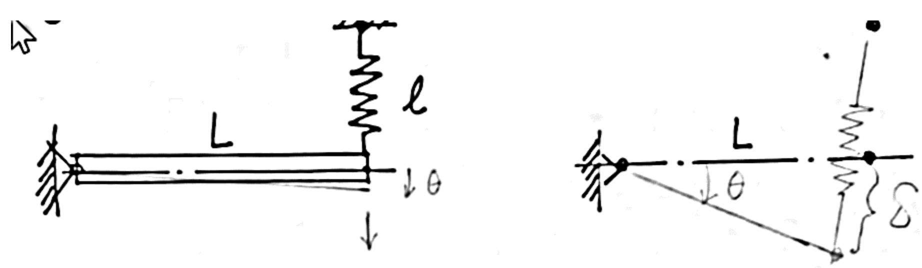
$$\ddot{\theta} = \frac{g}{L}\sin\theta = 0$$

Which is nonlinear. The Taylor expansion on $\sin\theta$ is:

$$\sin\theta = \theta - \frac{1}{6}\theta^3 + \frac{1}{120}\theta^5 \dots$$

If the first two terms are kept, the result is the softened Duffing oscillator.

Figure 2.42



L: length of the bar
l: natural length of the spring

Under small angle assumption, the deformation of the spring is $\delta = L\theta$.

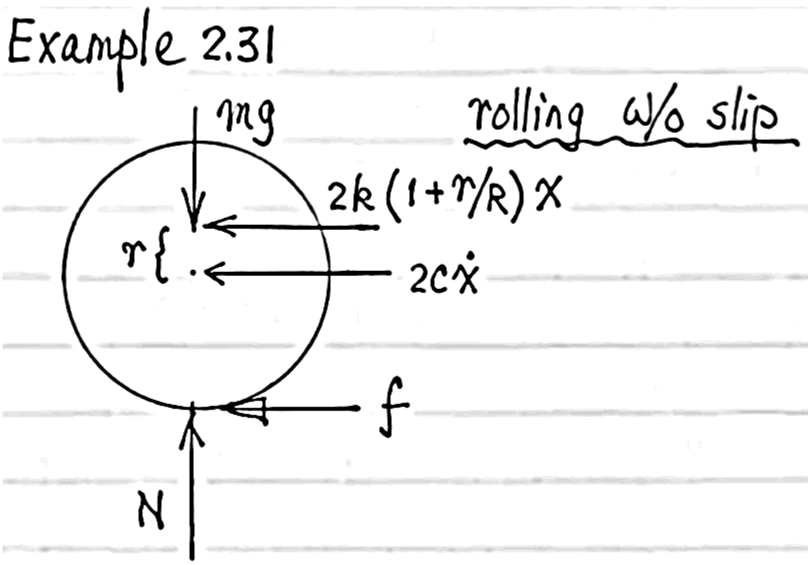
If θ is large, then the deformation will be:

$$\delta = \sqrt{(L - \mathrm{L}\cos\theta)^2 + (l + Lsin\theta)^2} - l \qquad (2.76)$$

Textbook Corrections (for this Chapter)

A2
Table D.2, case 5

$$C_2 = \frac{z_1}{2}\left(1 - \frac{a}{z_1}\right)\left[1 - \left(1 - \frac{a}{z_1}\right)^2 u(z_1 - a)\right]$$



$E.O.M$ is:

$$\left(\frac{I}{R^2} + m\right)\ddot{x} + 3c\dot{x} + 2k\left(1 + \frac{r}{R}\right)^2 x = 0$$

(c) is correct

<u>Rolling w/ slip</u>

Spring force is:

$$2k(x + r\theta)$$

And:

$$\theta \neq x/R$$

Friction is $f = \pm\mu mg$

$\therefore$ it's a 2 DOF system; DOFs are x and θ

$$\begin{cases} m\ddot{x} + 2c\dot{x} + 2kx + 2kr\theta = \pm\mu mg \\ \frac{1}{2}mR^2\ddot{\theta} + 2k(x + r\theta)r = \pm\mu mgR \end{cases}$$

Rolling w/ slip but keeping SDOF.

Rolling w/ slip but keeping SDOF

Spring force is:

$$2k\left(1+\frac{r}{R}\right)x$$

$$\therefore m\ddot{x}+2c\dot{x}+2kx+2k\left(1+\frac{r}{rR}\right)x=\pm\mu mgR$$

Then:

$$\frac{1}{2}mR^2\ddot{\theta}=-2kr\left(1+\frac{r}{R}\right)x\pm\mu mgR$$

(g) should read:

$$35\ddot{x}+2000\dot{x}+3.422(10^5)x$$
$$=\begin{cases}-85.84 & ; \quad \dot{x}>0\\ +85.84 & ; \quad \dot{x}<0\end{cases}$$

The following presentation is based on *Chaotic vibrations An Introduction for Applied Scientists and Engineers*, F.C. Moon, Wiley, 2004.

Definition of chaos

Chaos is one of the scientific concepts that enter the popular culture.

In the non-scientific world, chaos means without pattern, out of control.

In the scientific world, there is no universally agreed definition of chaos.

However, a widely accepted working definition is:

> *Chaos is the aperiodic time-asymptotic behavior in a deterministic system which exhibits sensitive dependence on initial conditions.*

Misconceptions

- *Is chaotic vibration a random vibration?*

Random vibration means that the true values of input forces and/or systems parameters are unknown. In other words, probability and its distribution are needed for solving random motion.

Example: earthquakes, environmental sounds

Chaotic vibration is a deterministic phenomenon. The key characteristics is the sensitivities.

- *What is the necessary condition for chaotic motions?*

Nonlinearity in the system.

However, not all nonlinear systems will be chaotic.

- *Is chaotic motion associated with high-dimension, and/or high-order differential equations (DEs)?*

Not necessarily.

For example, the three well-studied chaotic systems are,

Duffing oscillator:

$$\ddot{x} + 2\gamma\dot{x} + \alpha x + \beta x^3 = f(x)$$

van der Pol (VDP) oscillator:

$$\ddot{x} - \gamma\dot{x}(1 - \beta x^2) + \alpha x = f(t)$$

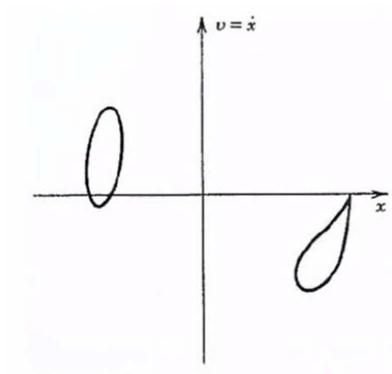
Lorenz attractor:

$$\begin{aligned} \dot{x} &= \sigma(y - x) \\ \dot{y} &= \rho x - y - xz \\ \dot{z} &= xy - \beta z \end{aligned}$$

These well-known systems will be examined later during the course of this course (EMEC 5671 FC).

Three main elements in the working definition of chaos

1. *Aperiodic time-asymptotic behavior.* This implies the existence of phase-space trajectories that do not settle down to fixed points or periodic orbits.



Courtesy of Chaotic vibrations An Introduction for Applied Scientists and Engineers, F.C. Moon (Figure 2.10)

2. *Deterministic*. This implies that the equations of motion of the system possess no random inputs or parameters. As a result, the irregular behavior of the system arises from non-linear dynamics.

3. *Sensitive dependence on initial conditions.* This implies that nearby trajectories in phase-space separate exponentially fast in time: *i.e.*, that the system has a positive Lyapunov exponent.

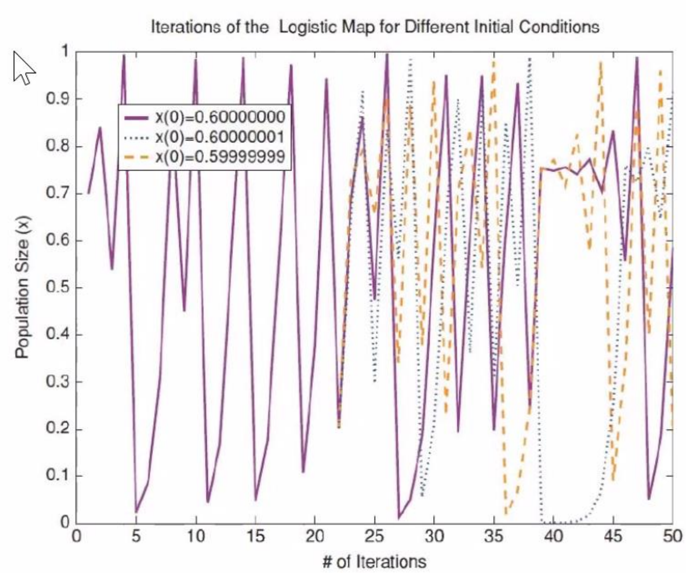


Fig. 2 Sensitivity to initial conditions in the logistic map

Courtesy of Chaos and Its Computing Paradigm, D. Kuo, IEEE Potentials, April/May 2005, pp 13-15

There are systems whose dynamic responses are sensitive to *parametric changes in system parameters.*

Bifurcation is a means to investigate the effects of parametric changes on a system's dynamics, and if parametric changes lead to chaos.

What is bifurcation? The definition will be given later.

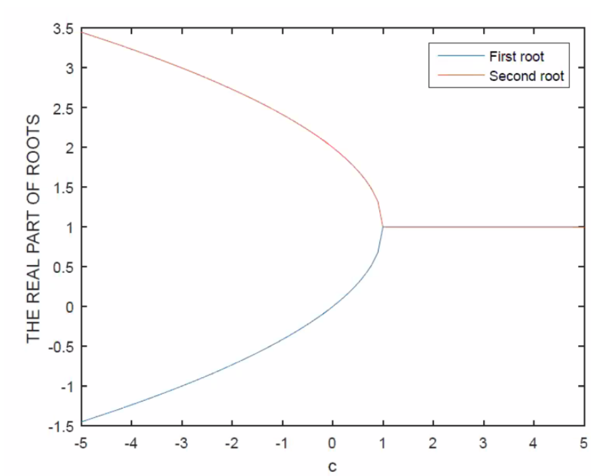
But as a simple example, lets consider the roots of a quadratic equation $ax^2 + bx + c = 0$

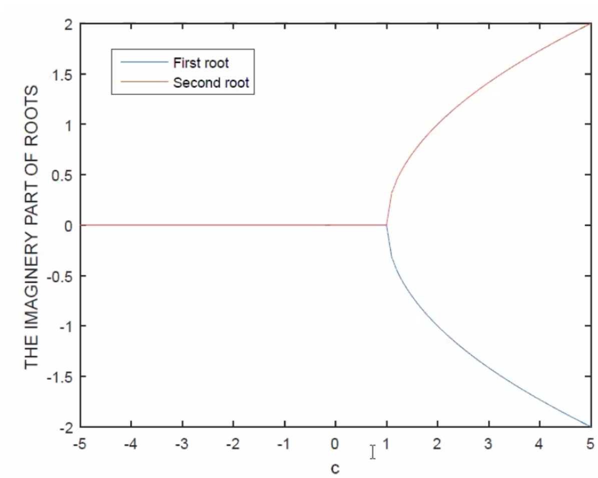
$$(where\ a \neq 0)$$

(1) Two identical roots, if $\sqrt{b^2 - 4ac} = 0$
(2) Two distinct real roots, if $\sqrt{b^2 - 4ac} > 0$
 and
(3) Two complex roots, if $\sqrt{b^2 - 4ac} < 0$

Let $a = 1, b = 2, c = [-5, 5]$
$c < 1$, two distinct real roots
$c = 1$, two identical real roots
$c > 1$, two complex roots

Bifurcation diagram:





Cascade bifurcation:

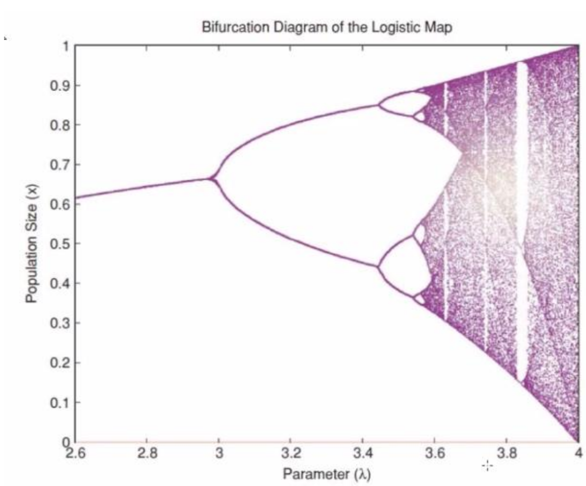


Fig. 3 Bifurcation diagram for the logistic map

Courtesy of Chaos and Its computing paradigm, D. Kuo, IEEE Potentials, April/May 2005, pp. 13-15

Why should engineers study chaos?
To know the source of chaos. Chaos can arise in low-order deterministic nonlinear systems.

To learn the tools within the chaos theory to (1) detect chaotic vibrations in physical systems, and (2) to quantify the chaos.

The know the flipped side of chaos suppression. Chaos can be suppressed, but some feedback control forces are known to cause chaos.

To incorporate chaos in design. Engineers have been using factor of safety to account for unknowns in engineering design. The unknowns can be caused by noises which in turn can lead to long term unpredictability.

Why the title *nonlinear vibrations and chaos?*
The necessary condition for chaotic vibration is nonlinearity in the system.

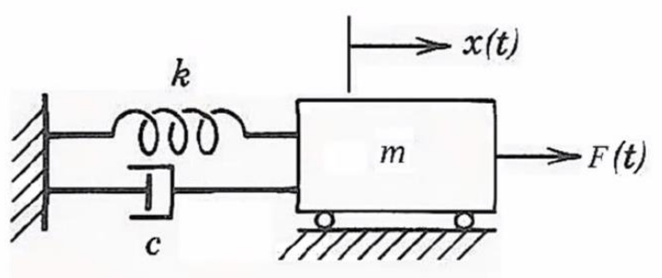
However, not all nonlinear vibration is chaotic.

Why linear vibrations first?
Linear vibrations are the foundation for nonlinear vibrations.

The emphasis has been on the modeling of SDOF linear vibration systems; i.e., the mass, spring, damper, and excitation. Another emphasis was the use of energy (kinetic, potential, …) and work done (by non-conservative forces in particular) in problem-solving.

It is also important to understand the simplifications or assumptions made to obtain linear systems.

Examples of Nonlinear Vibrations
1. Duffing Oscillator



Revised from Chaotic vibrations An introduction for Applied Scientists and Engineers, Moon

Starting with the classical mass-spring damper oscillator subject to a periodic force, the equation of motion is, *after normalization*,

$$\ddot{x} + \delta\ddot{x} + \alpha x = f(t)$$

Now considering a cubic (hence nonlinear) spring,

$$\ddot{x} + \delta\dot{x} + \alpha x + \beta x^3 = f(t)$$

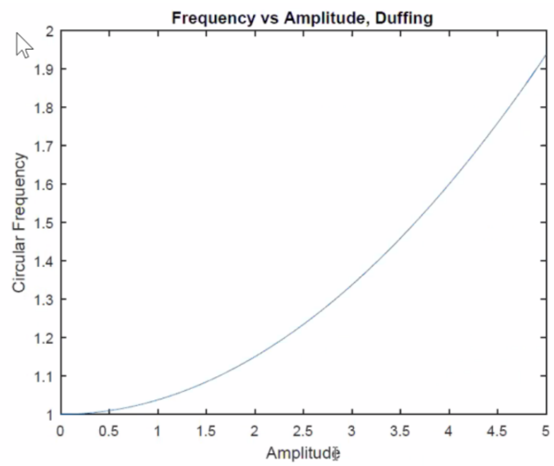
It then becomes the Duffing oscillator, and the E.O.M. is known as the Duffing equation.

Duffing equation is often used in structural analysis involving nonlinear restoring forces.

1.1 Unforced and Undamped Duffing Oscillator

$$\ddot{x} + \alpha x + \beta x^3 = 0$$

- Frequency of vibration depends on amplitude of vibration. (This is true for other nonlinear oscillators.)
- Approximate solutions of period (or frequency) are available.
- Exact solutions of period (or frequency) is only available for a few special cases, typically in the form of elliptic integrals.



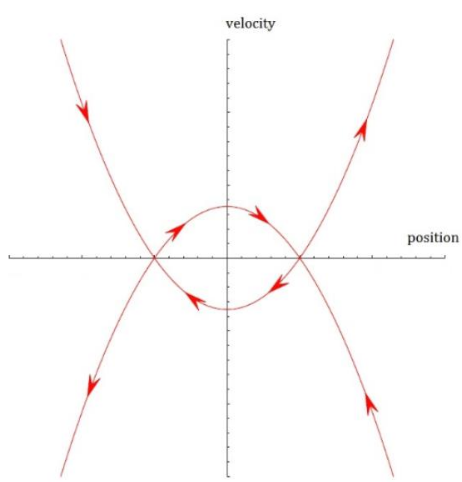
1.1.1 $\beta > 0$ or hardening:

- Phase portraits show continuous, closed curves surrounding the origin O.
- O is a center, or a stable equilibrium point.

1.1.2 $\beta < 0$ or softening:

Two situations may arise depending on the amplitude of vibration:

- Saddle points (or nodes) and separatrices:



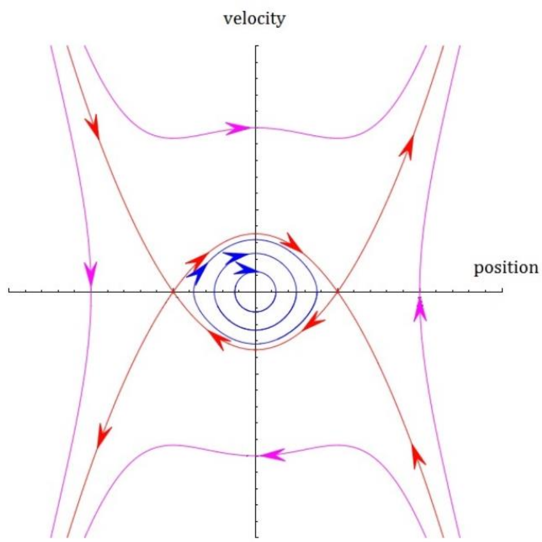
Courtesy of Alfred Clark, Jr., Professor Emeritus of Mechanical Engineering, Mathematics, and Biomedical Engineering, University of Rochester

Saddle points (or nodes) are unstable equilibrium points.

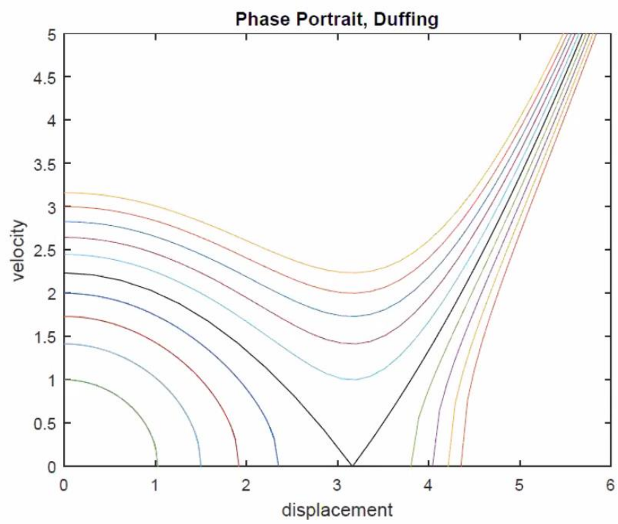
Separatrix refers to the boundary separating different modes of vibration.

- The two situations:

1) Continuous, closed curves inside the separatrices; or
2) Curves “running off” to infinity outside the separatrices.



Courtesy of Alfred Clark, Jr., Professor Emeritus of Mechanical Engineering, Mathematics, and Biomedical Engineering, University of Rochester



1.2 Forced Duffing Oscillator

$$\ddot{x} + \delta\dot{x} + \alpha x + \beta x^3 = f_0 \cos(\omega t)$$

In addition to having nonlinear restoring forces, the forces Duffing oscillators are often used when the system demonstrates hysteresis (or the state variables’ dependence of history)

The amplitude-frequency relation if $f(t) = f_0 \cos(\omega t)$:

- The jump phenomenon
- The upsweep and downsweep paths

1.3 Typical Analytical Approaches for Duffing Oscillator

Perturbation methods (straightforward expansion, Lindstedt-Poincare method, …)
Harmonic balance method
Averaging method
…

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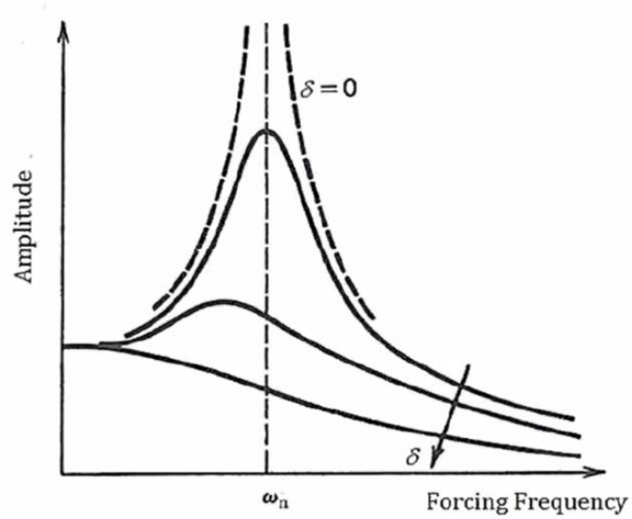
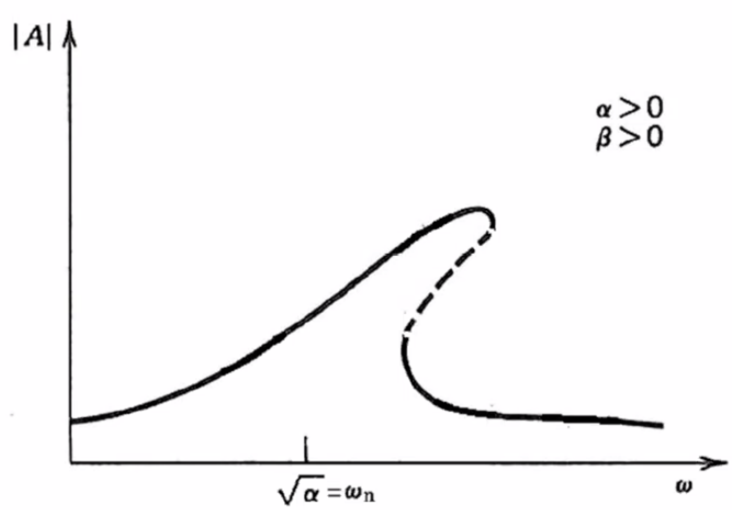
Linear Resonance Curves



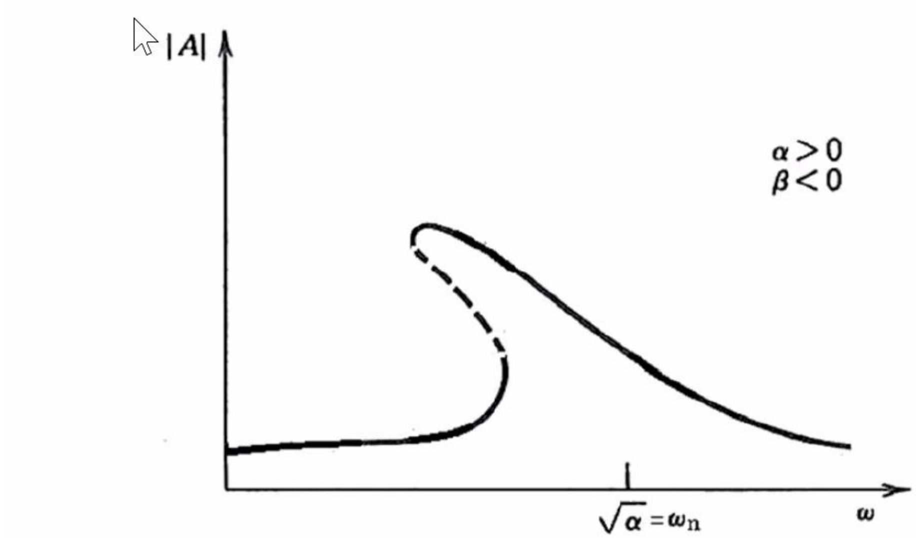
Revised from *Chaotic Vibrations An Introduction for Applied Scientists and Engineers,* Moon

Nonlinear Resonance Curve for a Hardened Duffing Oscillator



Revised from *Chaotic Vibrations An Introduction for Applied Scientists and Engineers,* Moon

Nonlinear Resonance Curve for a Softened Duffing Oscillator



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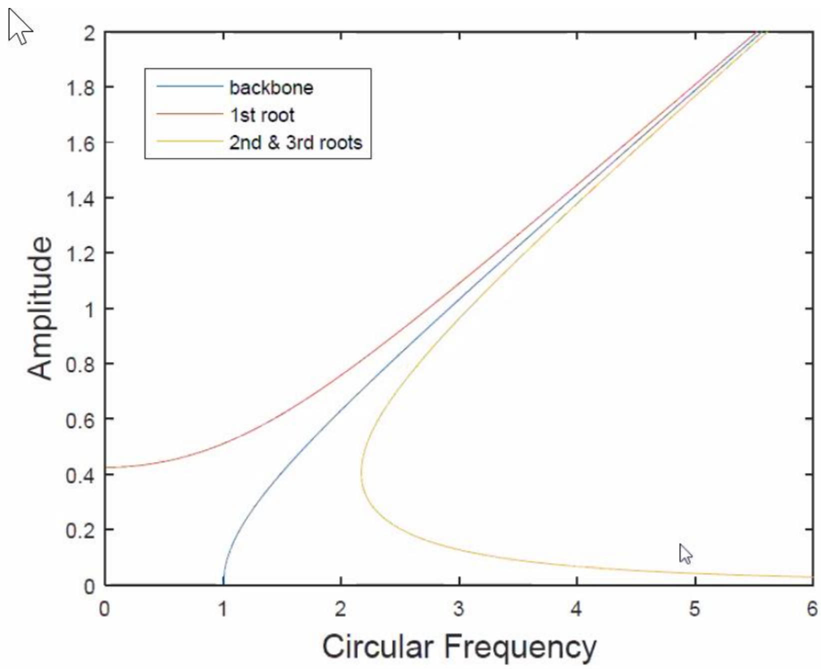
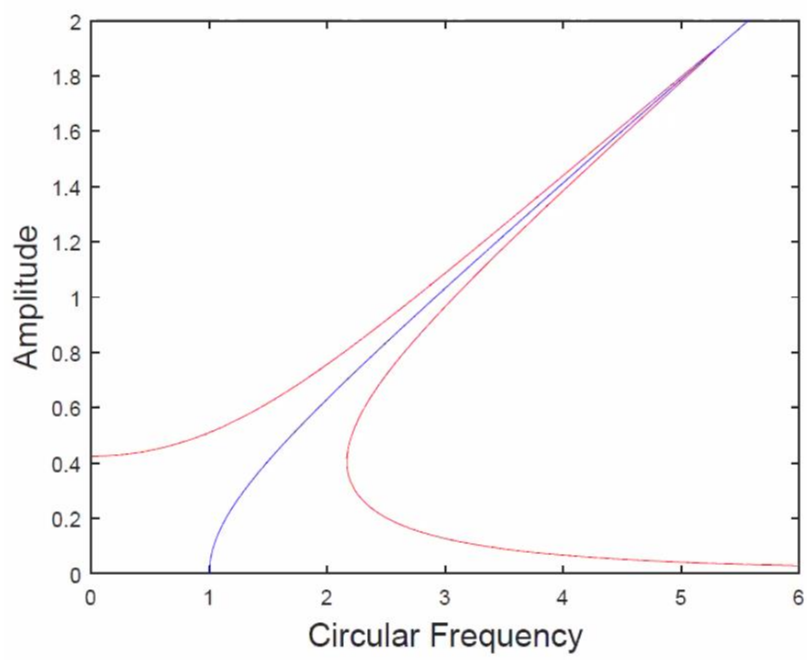
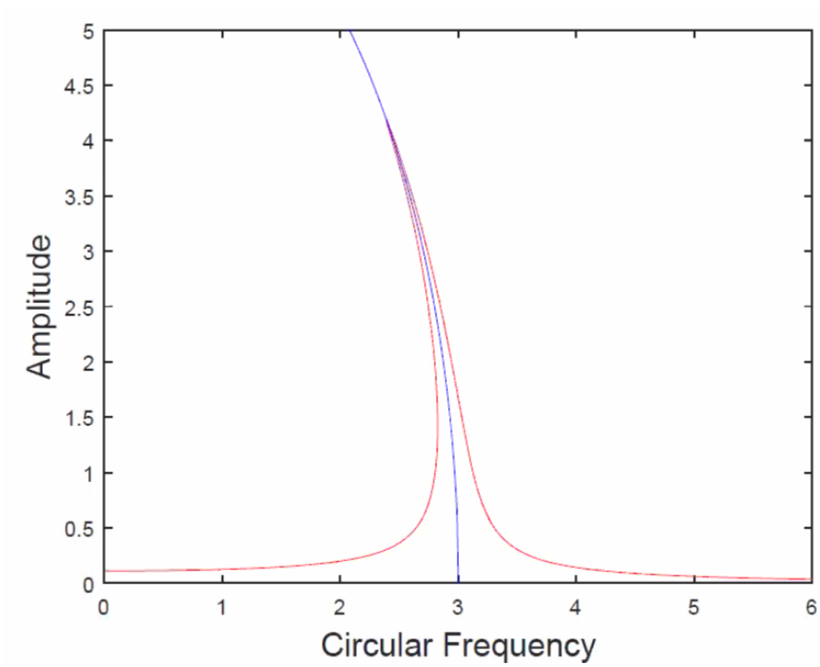


Plot of duffing oscillator with excitation, but damping is zero. Backbone can be drawn from the natural frequency and seems to become the asymptote.



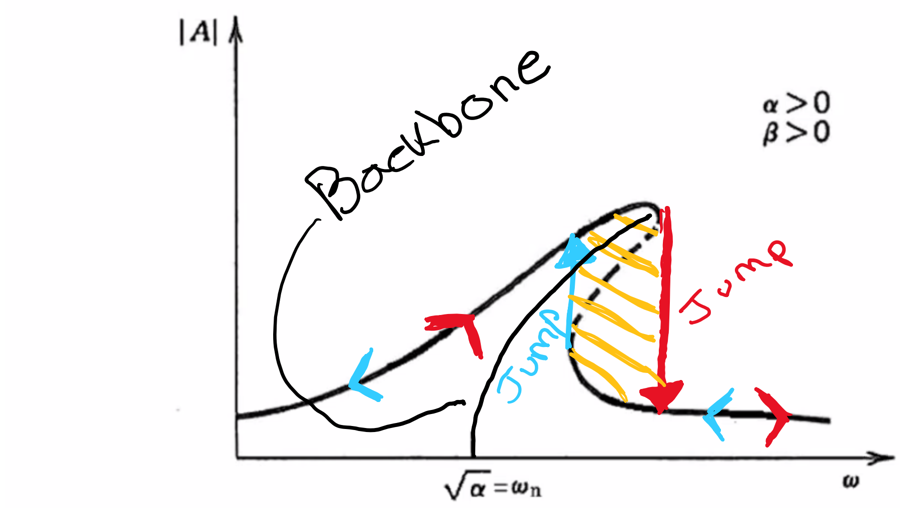
Plot of duffing oscillator where we have damping. The curve intersects the backbone at some point, then turns back.



1.3 Typical Analytic Approaches for Duffing Oscillator

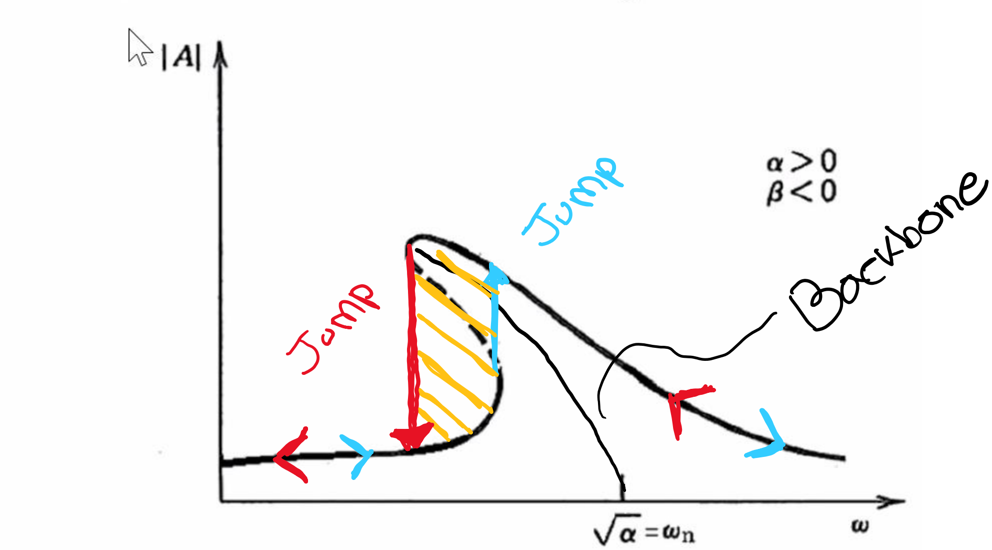
Perturbation methods (straightforward expansion, Lindstedt-Poincare method, …)
Harmonic balance method
Averaging method
…

Nonlinear Resonance Curve for a Hardened Duffing Oscillator



The red arrows indicate the upsweep path, and the blue arrows indicate the downsweep path. Notice the 'Jump,' and thus physically the dashed portion does not exist – however it is needed computationally. The hatched area indicates the hysteresis (the entire area enclosed by the curves).

Nonlinear Resonance Curve for a Softened Duffing Oscillator



Again, the red arrow indicates the upsweep path, and the blue arrow indicates the downsweep path. The dashed portion doesn't exist, and the enclosed hatched area is the hysteresis.

Equilibrium points, and Energy Curves

$$\ddot{x} + \alpha x + \beta x^3 = 0$$

$$\ddot{x} - f(x) = 0$$

$\dot{x} = y$

Kinetic energy

potential energy

Total energy $E(x,y) = H(x,y)$

Where H is the Hamiltonian

$$= \frac{1}{2}y^2 + \left(\frac{1}{2}\alpha x^2 + \frac{1}{4}\beta x^4\right)$$

Then:

$$\dot{x} = \frac{\partial H}{\partial y} = y$$

$$\dot{y} = -\frac{\partial H}{\partial x} = -\alpha x - \beta x^3$$

Equilibrium points:

$$\begin{cases} \dot{x} = 0 \to y = 0 \\ \dot{y} = 0 \to \alpha x + \beta x^3 = 0 \end{cases}$$

H values at equilibrium points:

$$H(0,0) = 0$$
$$\vdots$$

Separatrices:

$$\int \frac{\partial H}{\partial y} dy + \frac{\partial H}{\partial x} dx = H$$

$$\uparrow \ at\ equilibirum\ points \downarrow$$

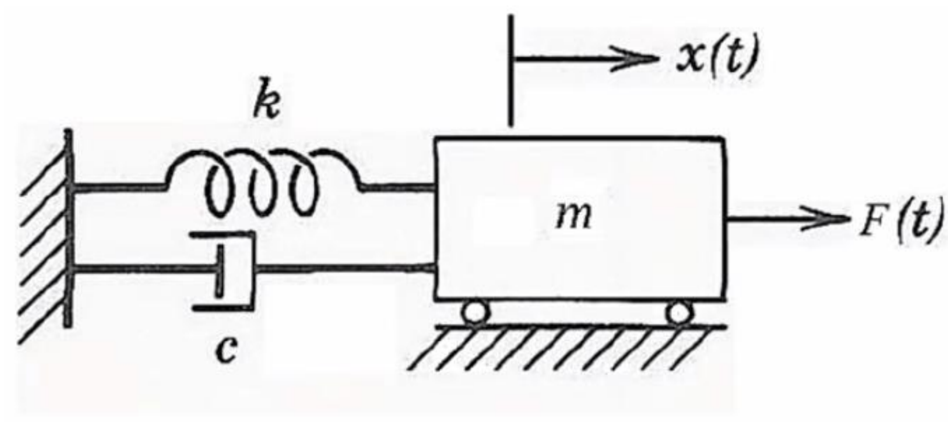
$$F(x,y) = H$$

Other energy curves:

$$F(x,y) = C \leftarrow energy\ level$$

4.2 van der Pol Oscillator

Back to the linear oscillator,



This time the damping is non-linear. This results in the so-called van der Pol oscillator:

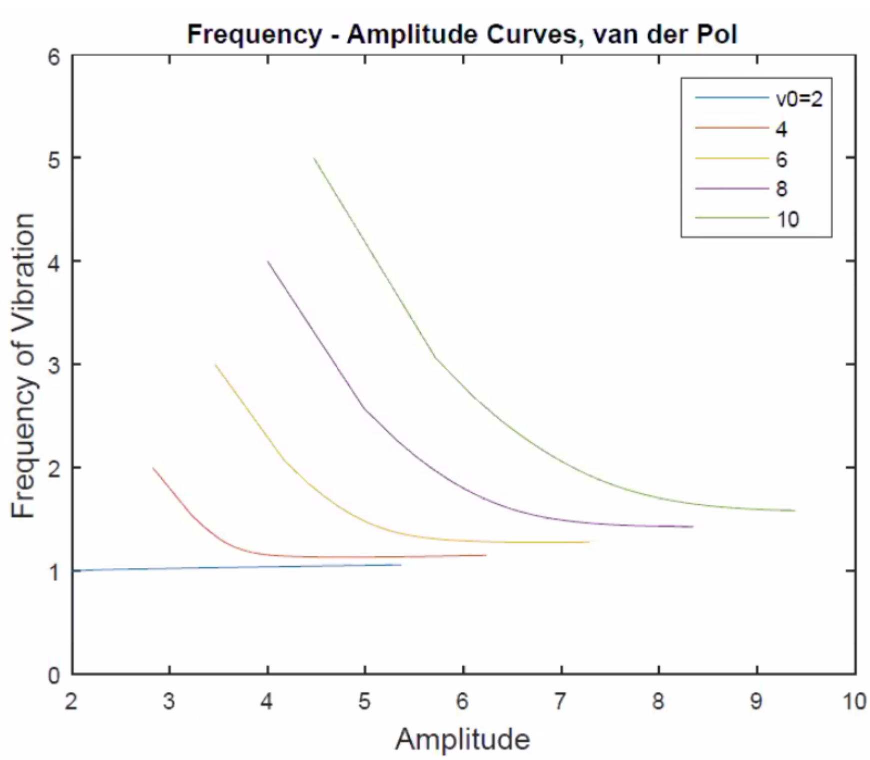
$$\ddot{x} - \delta\dot{x}(1 - \gamma x^2) + \alpha x = f(t)$$

Van der Pol oscillators are used when the so-called stick-slip phenomena, aero-elastic flutter, and biological phenomena are involved.

4.2.1 Unforced van der Pol Oscillator

$$\ddot{x} - \delta\dot{x}(1 - \gamma x^2) + \alpha x = 0$$

Similar to Duffing oscillator, frequency of vibration depends on amplitude of vibration.



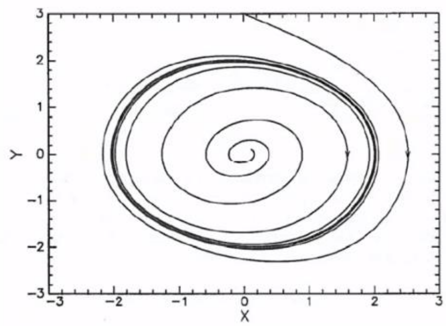
Two situations to focus on: limit cycle and relaxation oscillator.

They occur as long as one initial condition (the initial displacement or initial velocity) is not zero.

They are periodic response of the oscillator, signifying sustained vibration.

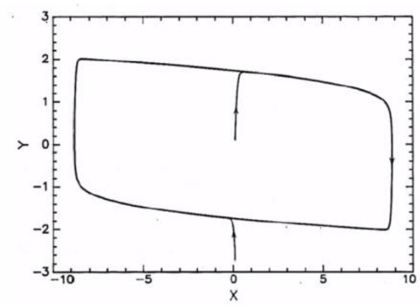
Differences?

- Limit cycle takes place when $\delta/\sqrt{\alpha}$ is small
- Relaxation oscillation occurs when $\delta/\sqrt{\alpha}$ is large



Limit Cycle Solution for the van der Pol Oscillator

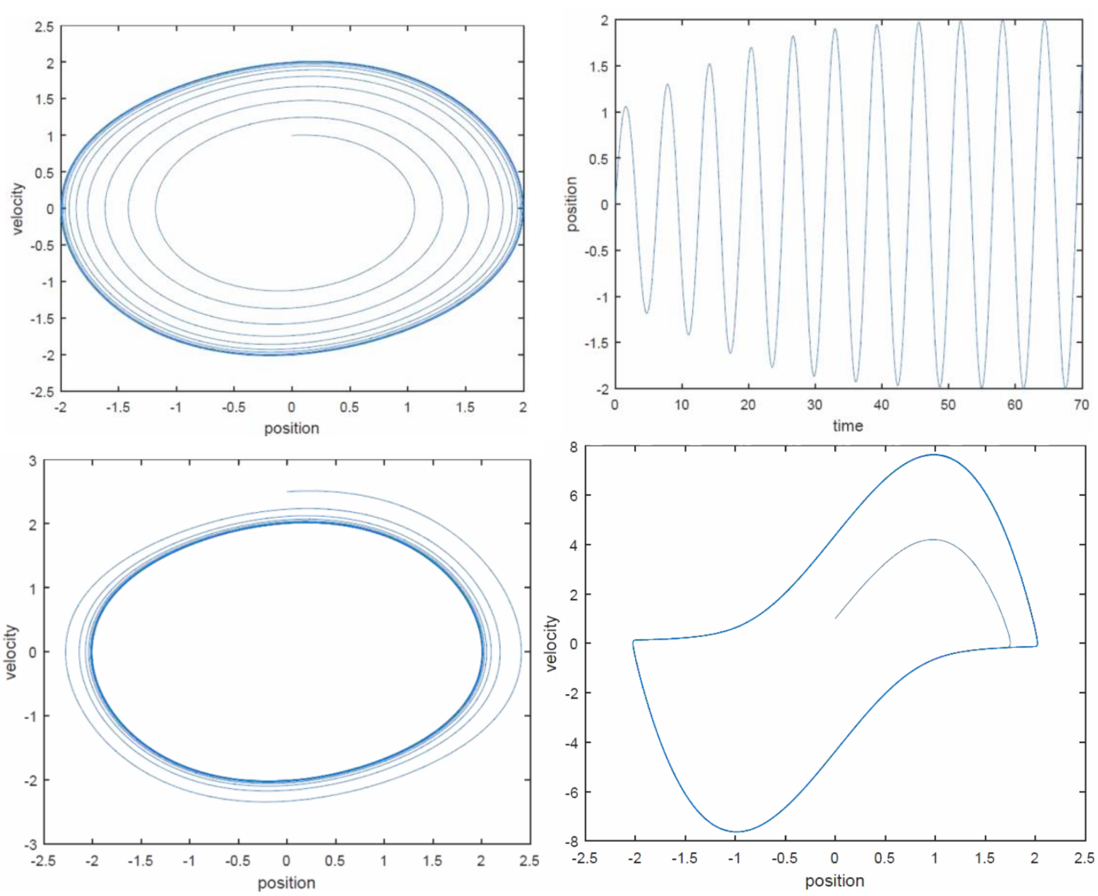
Courtesy of *Chaotic vibrations An Introduction for Applied Scientists and Engineers*, Moon

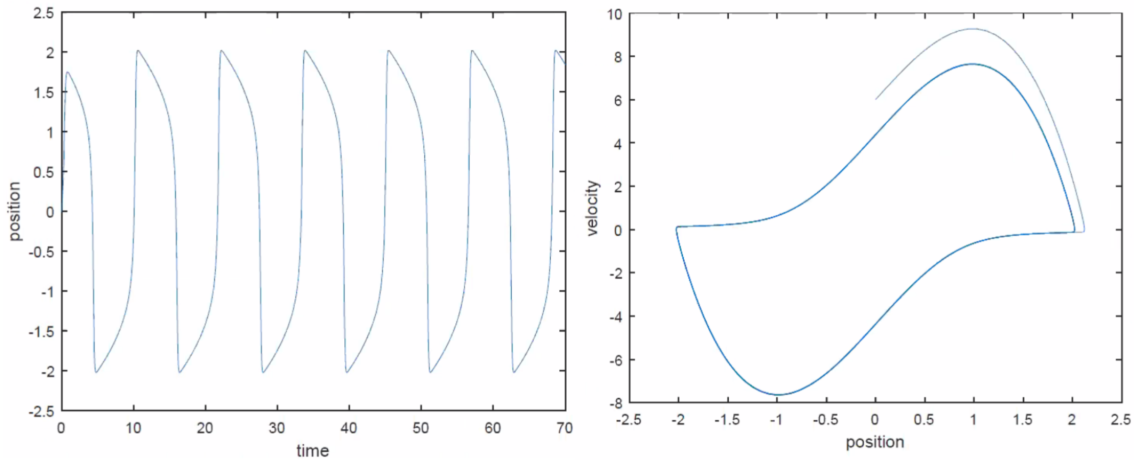


Relaxation Oscillation for the van der Pol Oscillator

Courtesy of *Chaotic vibrations An Introduction for Applied Scientists and Engineers*, Moon

Simulations from Dr. M. Liu (for reference):

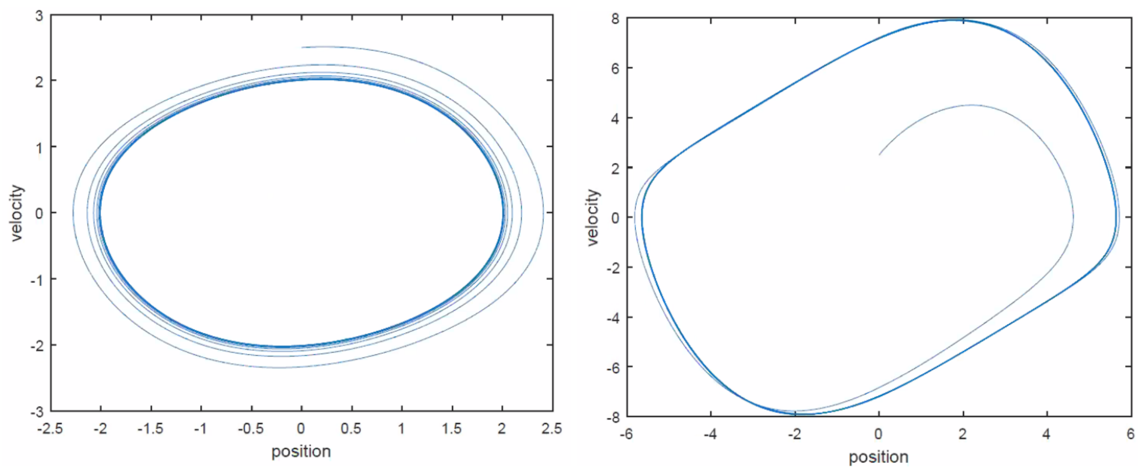




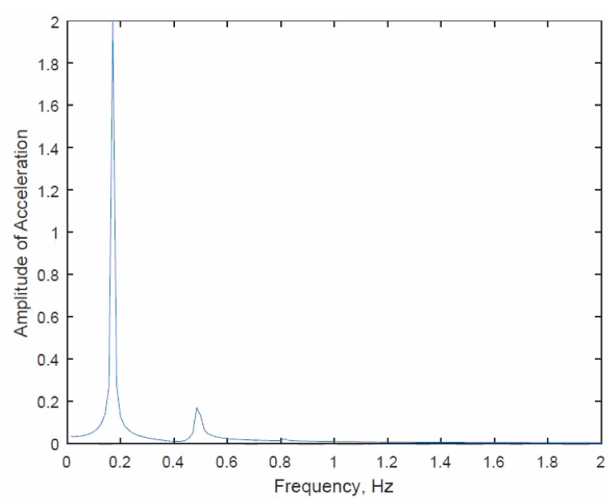
4.2.2 Forced van der Pol Oscillator

$$\ddot{x} - \delta\dot{x}(1 - \gamma x^2) + \alpha x = f_0 \cos(\omega t)$$

If the forcing is strong, and the forcing frequency and limit cycle are close to each other, the excitation entrains or enslaves the limit cycle. This phenomenon is known as entrainment.



Free vibration showing the limit cycle (left) and Forced vibration showing entrainment (right)



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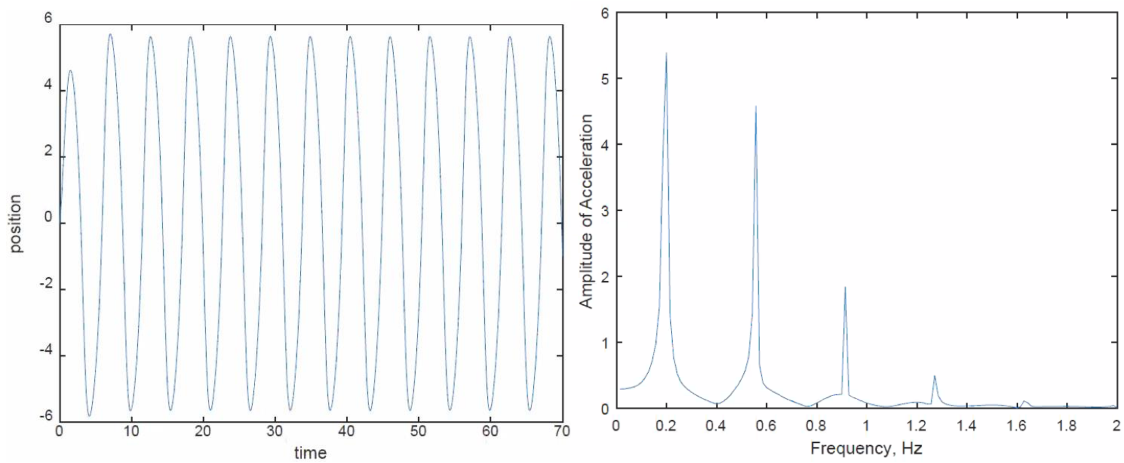


4.2.3 Typical analytic Approaches for van der Pol Oscillator:

The method of averaging

Perturbation methods (Lindstedt-Poincare methods, multiple time scales, …)

Harmonic balance method

…

Checklist for Identifying Chaotic Vibrations

- Identify nonlinear elements in the system
- Check for (so as to rule out) random inputs
- Observe the time history of a system variable
- Examine the phase portraits
- Examine the Fourier spectrum of system variables
- Examine the Poincare map of a state variable

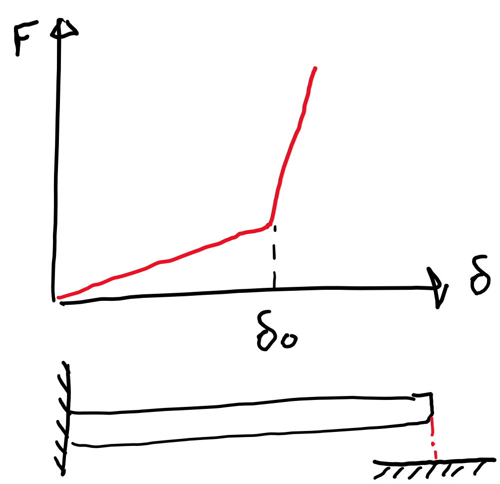
Just a single item from the checklist is not sufficient for conclusively identifying chaos. Typically, a combination is required.

There are more advanced techniques or measures, such as bifurcation diagram, Lyapunov exponents and fractal dimension.

Nonlinear elements

- Nonlinear elasticity, such a nonlinear springs, contact between elastic objects, and so on
- Nonlinear damping, such as stick-slip (dry friction)
- Most systems with fluids
- Nonlinear boundary conditions
- Backlash, play, or bilinear springs

What's a bilinear spring?



non-linear

$\neq k$, in essence it's piecewise

$$\begin{cases} k_1 < \delta_0 \\ k_2 > \delta_0 \end{cases}$$

Where have chaotic vibrations been observed?

- Vibrations of buckles elastic structures
- Mechanical systems with play or backlash
- Aeroelastic problems
- Wheel-rail dynamics
- Large-amplitude vibrations of structures such as beams, plates, and shells
- Systems with sliding friction
- Rotation and gyroscopic force
- Feedback control devices

The common thread is strong non-linearity. Other factors include electric magnetic and fluid related forces, and nonlinear boundary conditions.

Random inputs
In experiments and real-life: noise is always present

Numerical simulation: numerical noise exists

Checking for random inputs means to make sure that large non-periodic response does not arise from small input noise.

Random Inputs
In experiments and real-life: noise is always present

Numerical simulation: numerical noise exists

Checking for random inputs means to make sure that large non-periodic response does not arise from small input noise.

Time History
Usually the first clue of chaotic vibrations is that the $x(t) - t$ plot (or the $v(t) - t$ plot) shows no visible pattern or periodicity.

Time history test is not foolproof.

The system may exhibit transient chaos or intermittent chaos.

Figure 2-20 Sketch of the time history for intermittent-type chaos.
Courtesy of Chaotic vibrations An Introduction for Applied Scientists and Engineers, Moon, 2004.

Phase Portrait
Trajectory: the curve traded out by points $x(t), v(t)$.

Periodic vibrations: the trajectory is a closed curve.

Chaotic vibrations: the trajectory never closes or repeats, eventually filling up a section of the plane.

Again, the phase portrait alone is not foolproof.

In fact, it is believed that the Poincare map, considered by some the modified phase portrait, should be used instead of the phase portrait, as the Poincare map yields more relevant information.

Pseudo-phase Portrait
In physical experiments or real-life observations, there are times when only one measurement (i.e., one signal) s available. A time-delayed pseudo-phase portrait can be used as an alternative.

For a SDOF system with the signal $x(t)$, one plots the following pairs, $x(t)$, $x(t + \sigma)$, where σ is a fixed time constant.

Typically, for the same system, the phase portrait and the pseudo-phase portrait will reveal the same characteristics. Specifically, if the system has a periodic vibration, then both portraits will show closed trajectories. If the system's motion is chaotic, the portraits will show trajectories that do not close or repeat.

The choice of σ is not crucial, expect to avoid the natural or forcing period.

When state variables are more than three, high dimensional pseudo-phase portrait can be constructed using multiple delays. For example, points such as $x(t)$, $x(t+2\sigma)$ may be plotted in a 3D pseudo-phase portrait.

Fourier Spectrum

One clue of chaotic vibrations is the appearance of broad spectrum of frequencies when the input is of a single frequency.

This broad band spectrum characteristic is more prevalent in low dimension system with degrees of freedom of up to three.

Fourier spectrum is also useful in detecting subharmonics which are a precursor to chaos.

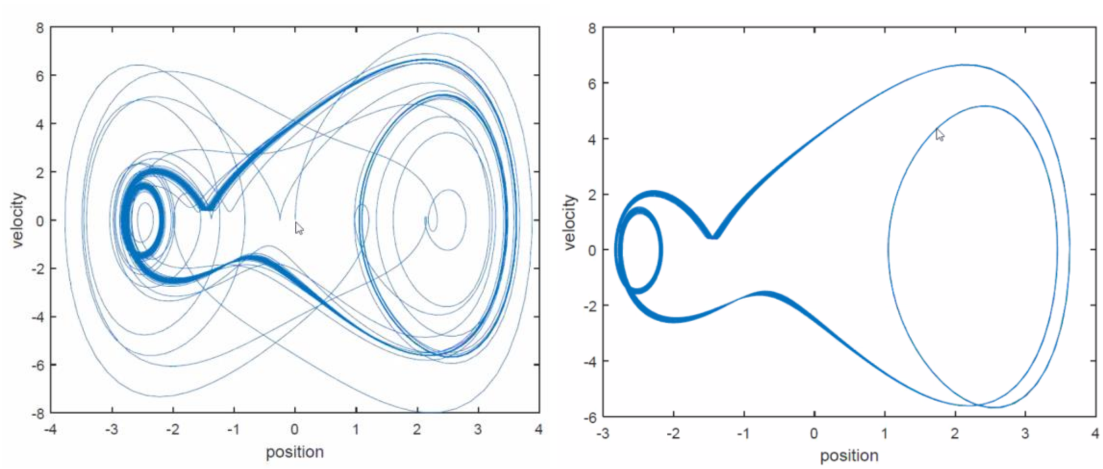
Subharmonic: if ω is the dominant frequency, then ω/n (n is an integer) is a subharmonic.

For example, Duffing oscillator

$$x + \delta x + \alpha x + \beta x^3 = f_0 \cos(\omega t)$$

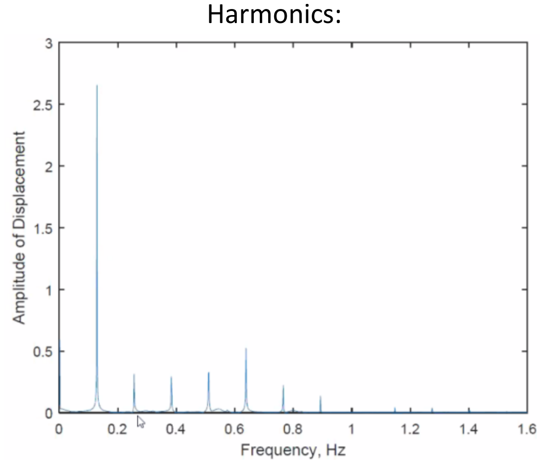
With $\delta = 0.18$, $\alpha = \beta = 1$, $\omega = 0.8$, and $f_0 = 19, 22$ and 22.5

Note that the forcing frequency is $0.127\ Hz$.



Phase portrait (left) and phase portrait with transient removed (right)

Harmonics:



Differential Equations (DEs) and Maps

Flows refer to the differential equations (Des). They are the responses of a dynamic system in continuous time, as represented by the trajectories in phase space.

Maps are algebraic rules for completing the next state of a dynamic system in discrete time.

For example, the ode45 solver in MATLAB is a set of algebraic rules: an explicit Runge-Kutta (4,5) formula with the Dormand-Prince pair.

Therefore, phase portraits that are based on computational responses and plotted as points instead of curves/lines, are in fact maps, or solution maps, to be precise.

Example: The logistic DE is:

$$\dot{x} = x(1-x)$$

Using the Euler's method to solve the DE, one has:

$$x_{n+1} = x_n + hx_n(1-x_n) = \lambda x_n(1-x_n)$$

With h being the time-step size, and $\lambda = 1 + h$. The logistic map is then,

$$x_{n+1} = \lambda x_n(1-x_n)$$

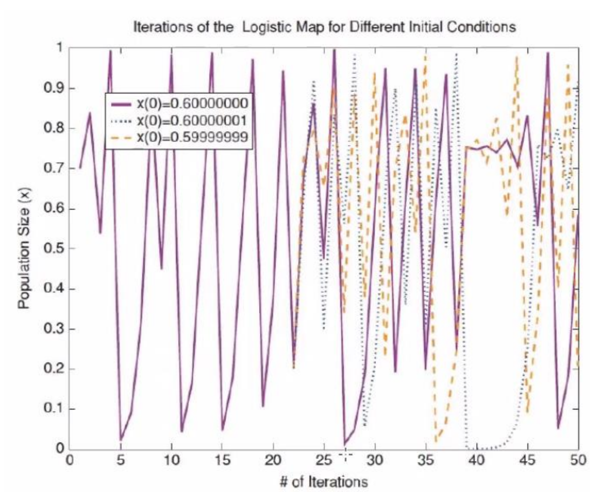


Fig. 2 Sensitivity to initial conditions in the logistic map

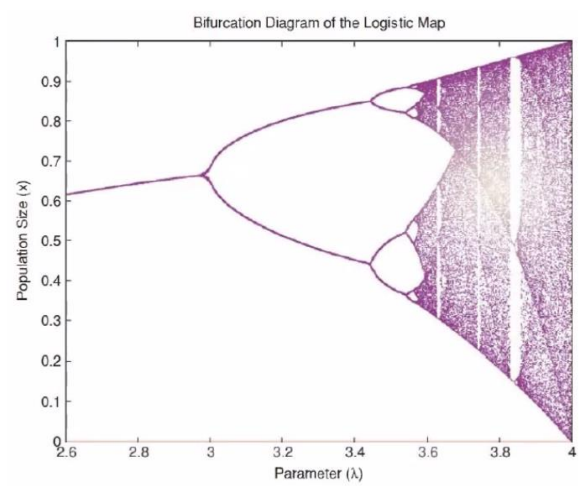


Fig. 3 Bifurcation diagram for the logistic map

Next-Return Maps; Next-Amplitude Maps

The rules for next-return maps are, for example,

- Position is zero
- Velocity is zero
- Position of x is zero and velocity of y is zero, or
- When the excitation has a phase angle of, say, $\omega t = 2n\pi + \pi/2$, or 90°, $450°$, $810°$, …

Poincare map is a next-return map.

Next-amplitude maps employ rules such as a state variable reaches maximum (or minimum for that matter).

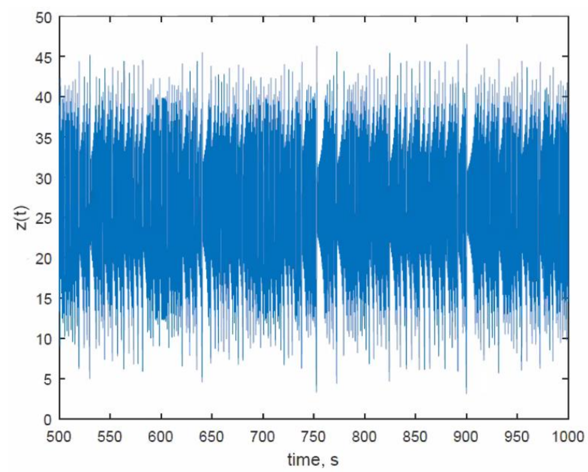
Next-amplitude maps were owing to Lorenz (of the Loren attractor fame). He intuitively constructed the first next-amplitude map, it is believed.

Example: The Lorenz attractor is:

$$\dot{x} = \sigma(y - x)$$
$$\dot{y} = \rho x - y - xz$$
$$\dot{z} = xy - \beta z$$

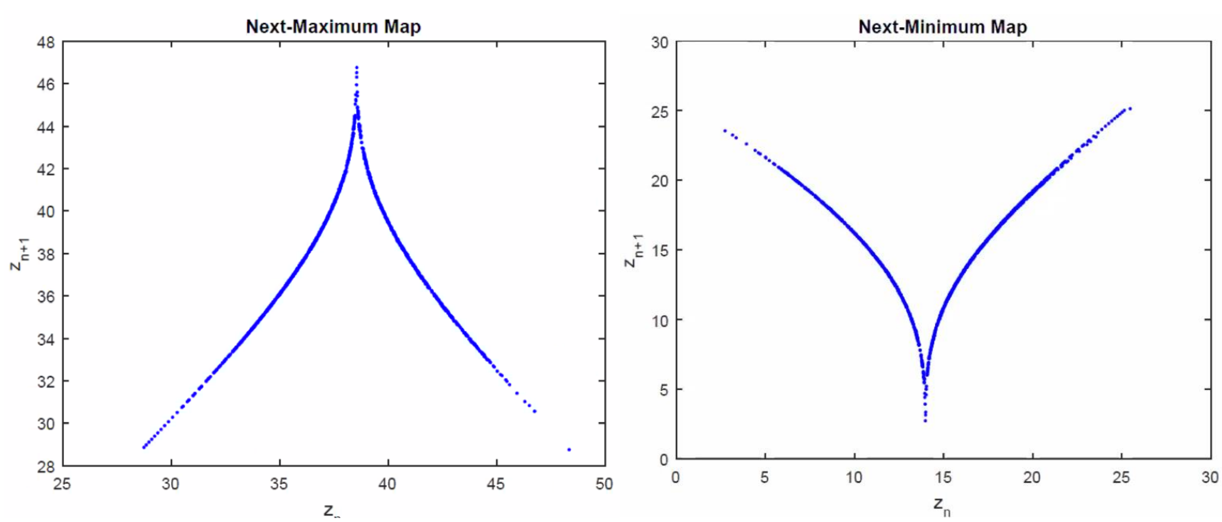
Chaos is observed when $\sigma = 10$, $\rho = 28$, $\beta = 8/3$.

Time History (run it for a while, remove transient – which could just be the beginning of the signal):

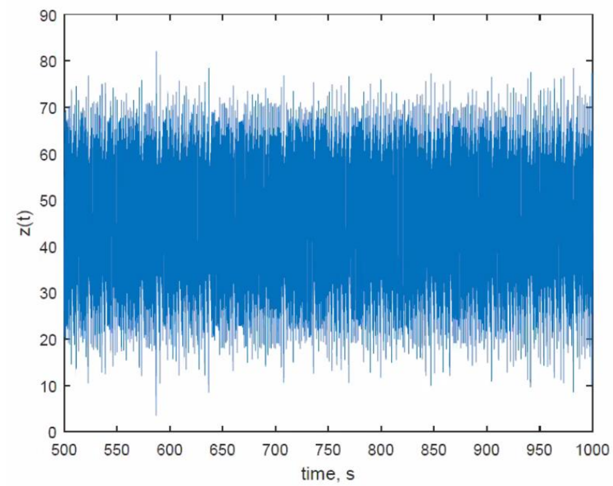


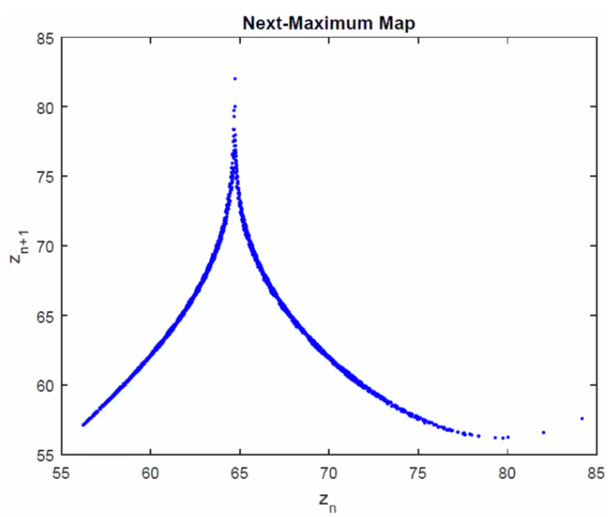
Strong indication of chaos because there is no discernable pattern by inspection.

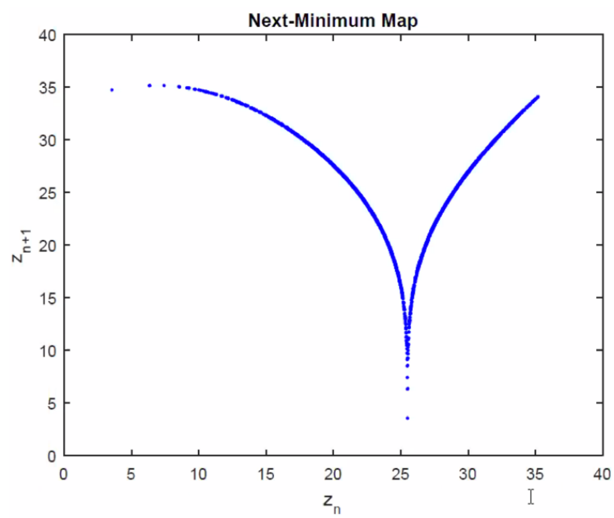
Next-Maximum and Next-Minimum Map



Another set of parameters – Chaos is observed when $\sigma = 28$, $\rho = 46.92$, and $\beta = 4$.







Autonomous DEs and Non-autonomous DEs

- An autonomous differential equation (DE) is one such that $\ddot{x} = f(x, \dot{x})$.
- A non-autonomous DE is one such that $\ddot{x} = f(x, \dot{x}, t)$

Basically, whether the function depends on time is the difference between autonomous and non-autonomous DEs.

NOTE:

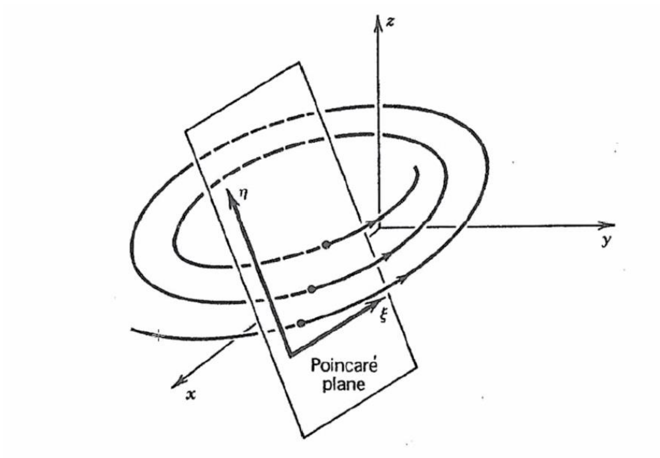
(1) Although a non-autonomous DE indicated forced oscillation, an autonomous DE does not suggest free oscillation, because the excitation may be a constant force, for example.
(2) For non-autonomous DEs, t is a state variable, in addition to position and velocity.
(3) Therefore, the dimension of the state space, n, of $\ddot{x} = f(x, \dot{x})$ is 2; but it is 3 for $\ddot{x} = f(x, \dot{x}, t)$, where x is a scalar.

Poincare Maps

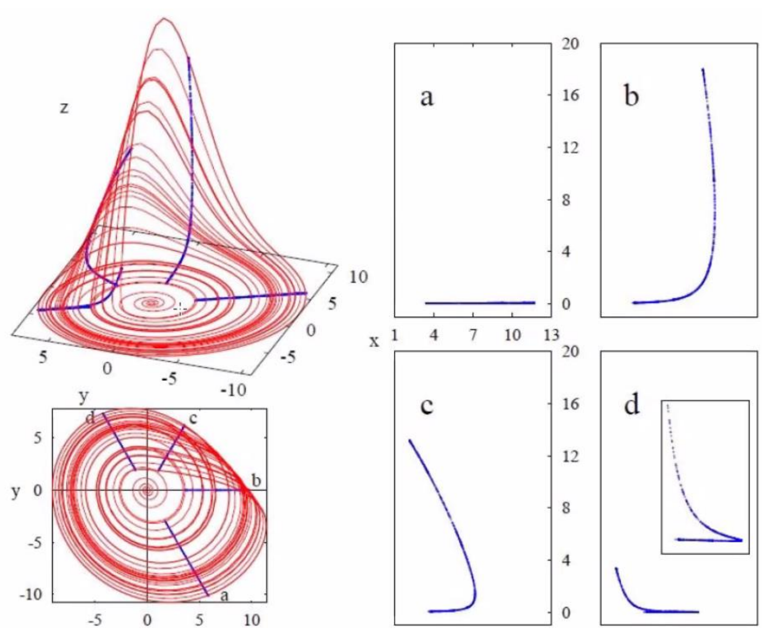
Poincare map is a classical technique for analyzing dynamic systems, conceived by Poincare. When an n −dimensional continuous-time system is replaced with an $(n-1)$ − dimensional discrete-time map, the result is the so-called Poincare map.

In other words, the continuous-time response is sampled according to certain rules. The rules are such that the dimension of the map is one less the dimension of the DE.

The rules can be reflected upon by the Poincare section, the choice of which is different for autonomous and non-autonomous DEs.

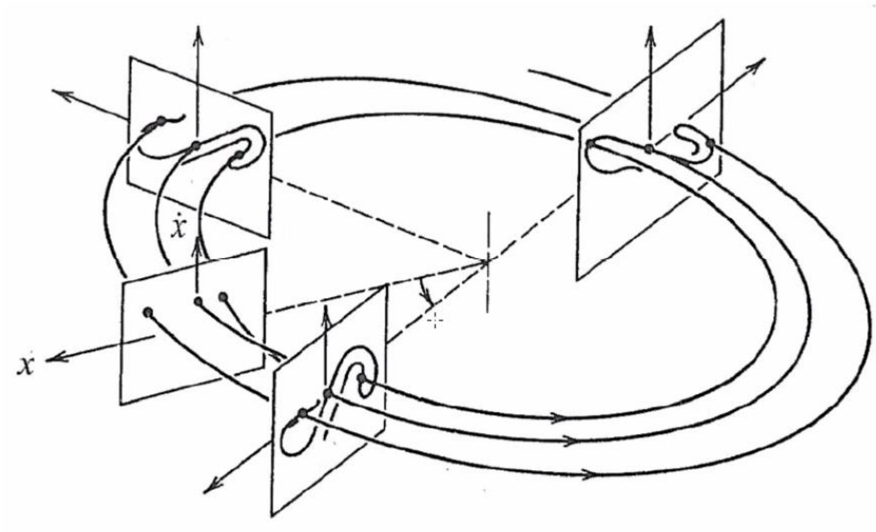


Courtesy of *Chaotic vibrations An Introduction for Applied Scientists and Engineers*, Moon, 2004.



Poincare Maps for Non-Autonomous DEs
Since t is a dimension, a convenient chouce of Poincare section would be $t = kT + t_0$, where T is the forcing/driving period, and $t_0 < T$ is an arbitrary time.

In other words, of every driving period T, one discrete point is sampled and plotted. The collection of such discrete points forms the Poincare map of the oscillator represented by non-autonomous DEs.



Courtesy of *Chaotic vibrations An Introduction for Applied Scientists and Engineers*, Moon, 2004.

Consider the forced vibration of an oscillator given by the following ODEs, where $y = \dot{x}$

$$\dot{x} = y$$
$$\dot{y} = F(x, y) + f_0 \cos(\omega t + \Phi_0)$$

By introducing a variable $z = \omega t + \Phi_0$, the first-order ODEs for the non-autonomous system is,

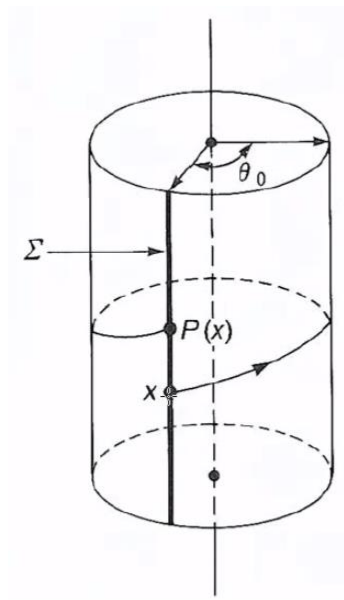
$$\dot{x} = y$$
$$\dot{y} = F(x, y) + f_0 \cos(z)$$
$$\dot{z} = \omega$$

The samples for the Poincare map may then be collected at $z = 0$, or Φ_0, or any angle within $360°$.

What can be revealed by Poincaré Maps?
Some math/definitions/explanations first.

Mathematically speaking, a Poincaré Map is a mapping $x_{k+1} = P(x_k), x_k, x_{k+1} \in P, k = 1, 2, 3, \ldots$

1) x is a fixed point is $x = P(x)$.



Courtesy of Practical Numerical Algorithms for Chaotic Systems, Parker & Chua, 1989

2) The set of sampled points $\{x_1, x_2, \ldots, x_k\}$ is a period$-K$ closed orbit, if $x_{k+1} = P(x_k)$ for $k = 1, 2, \ldots, K-1$ and $x_1 - P(x_K)$

3) Quasi-periodic solution is the sum of finite numbers of periodic solutions, each having a frequency that is an integer combination of frequencies taken from a finite base set.

   For example, $x(t) = Acos(2t) + Bcos(\sqrt{2}t)$ is quasiperiodic. So is $x(t) = Acos(t) + Bcos(3\sqrt{2}t)$, as ω_1/ω_2 is an irrational number. Quasi-periodic is not periodic.

4) Attractor is a set of values that a dynamic system tends to evolve toward. Examples of attractor include, but not limited to, a fixed point, a finite number of points, and a limit cycle.

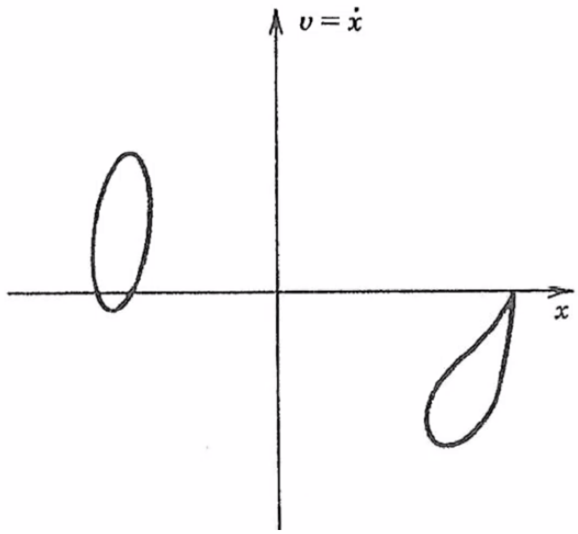
5) Strange attractor is a set of values showing fractal structure.

6) Cantor set refers to the embedding structure within structure, usually appearing at finer and finer scales.

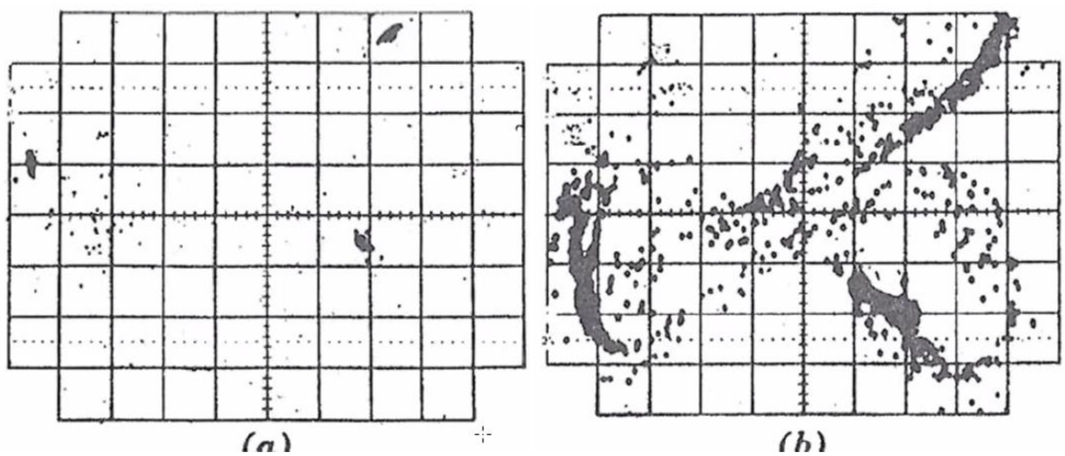
Classification of Poincaré Maps (non-autonomous DEs):

- A fixed point on the Poincaré map indicates a period-1 solution (periodic solution, only 1 point!)
- K distinct points on the map, indicates a $K - th$ subharmonic
- One or more closed curves on the map indicate a quasi-periodic solution.
- Fractal collection of points, strange attractor, or Cantor set indicate chaos.

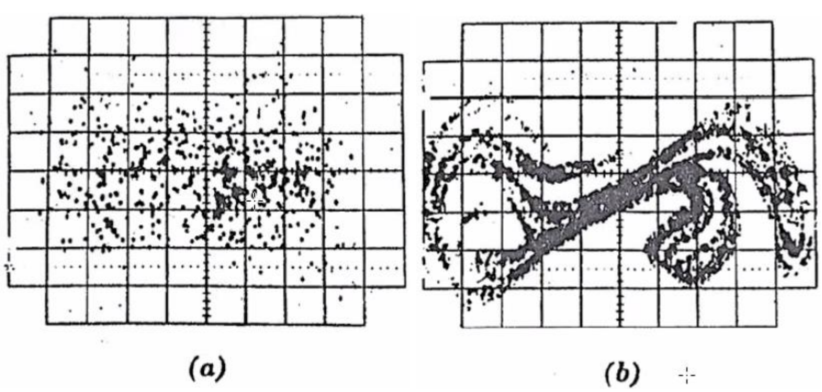
Poincaré Map Examples



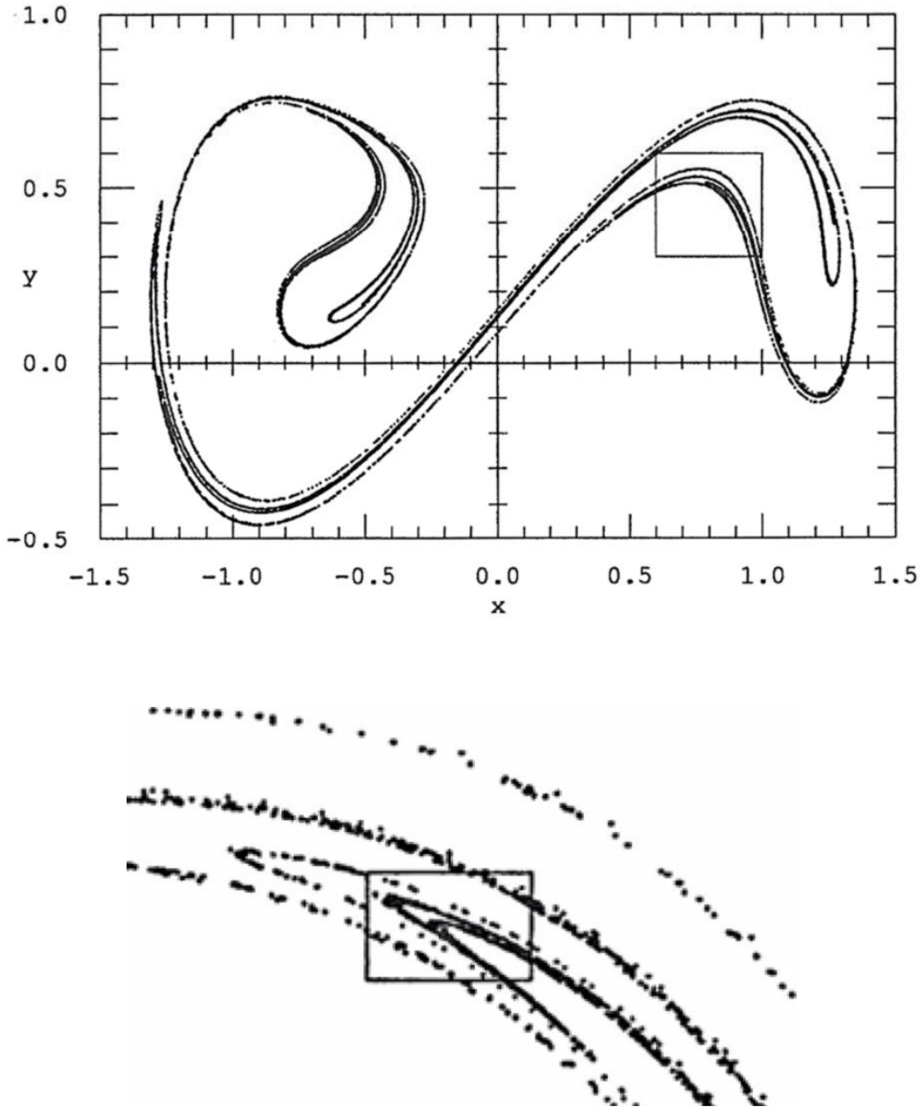
Multiple closed curves indicate a quasiperiodic solution



3rd subharmonic (left) and Fractal collection of points (right)



Many points indicating chaos (left) and another example of chaos (right)



Cantor set (infinitely embedding structure)

Poincaré Maps for Autonomous DEs

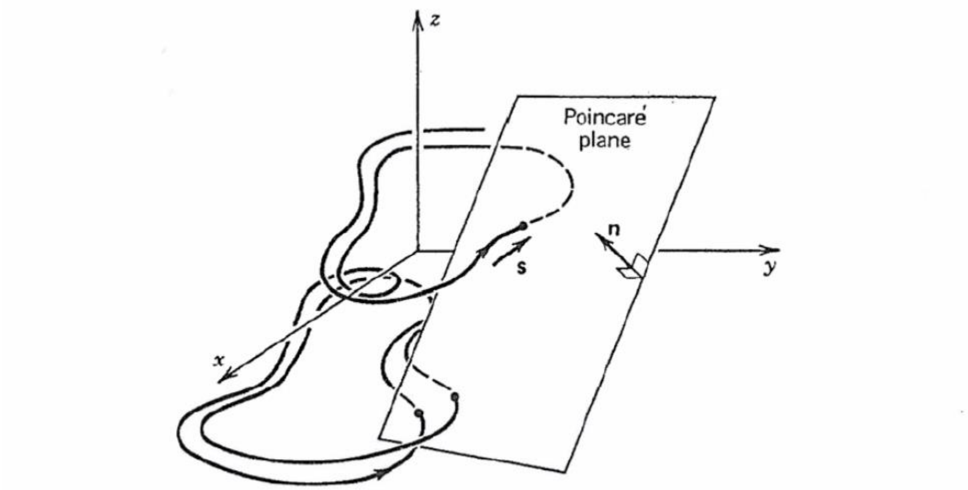


Figure 2-14 Sketch of time evolution trajectories of a third-order system of equations and a typical Poincaré plane.

As an example, consider the cases where the state variables are $x(t)$, $y(t)$, and $z(t)$. In this state space, a Poincaré section is a plane defined by:

$$n_1x + n_2y + n_3z - c = 0$$

Where c is a constant. The normal to the plane is

$$\boldsymbol{n} = \pm \begin{Bmatrix} n_1 \\ n_2 \\ n_3 \end{Bmatrix}$$

$\boldsymbol{n}$ will take either the positive or negative sign.

The Poincaré map will then consist of points (x_k, y_k, z_k) $(k = 1, 2, 3, \ldots)$ that meet eq. (1). In addition, if $\boldsymbol{s_k}$ represents the tangential vector to the trajectory at (x_k, y_k, z_k) $\boldsymbol{s_k} \cdot \boldsymbol{n}$ must have the same sign.

The classification of Poincaré maps of autonomous DEs follows that for non-autonomous DEs. That is,

- A fixed point on the Poincaré map indicates a period-1 solution (periodic solution, only 1 point!)
- K distinct points on the map, indicates a $K - th$ subharmonic
- One or more closed curves on the map indicate a quasi-periodic solution.
- Fractal collection of points, strange attractor, or Cantor set indicate chaos.

Example: The Lorenz attractor is

$$\begin{gathered} \dot{x} = \sigma(y - x) \\ \dot{y} = \rho x - y - xz \\ \dot{z} = xy - \beta z \end{gathered}$$

Chaos is observed when $\sigma = 10$, and $\rho = 28, \beta = 8/3$.

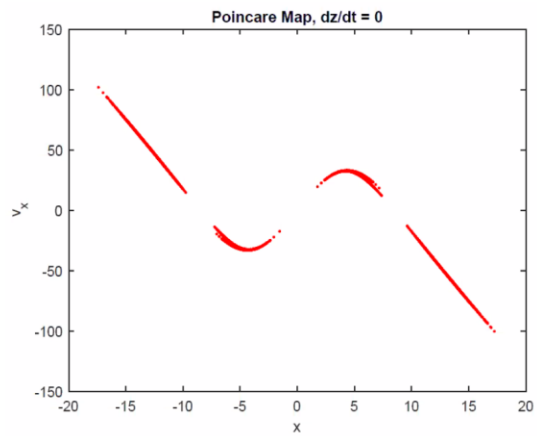
Dimension of phase space: $n = 6$;

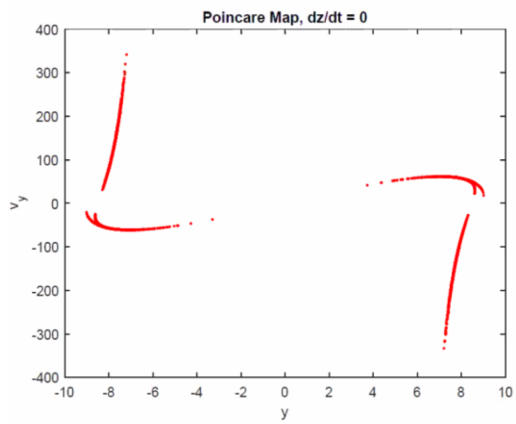
The Poincaré section: $z = 0$;

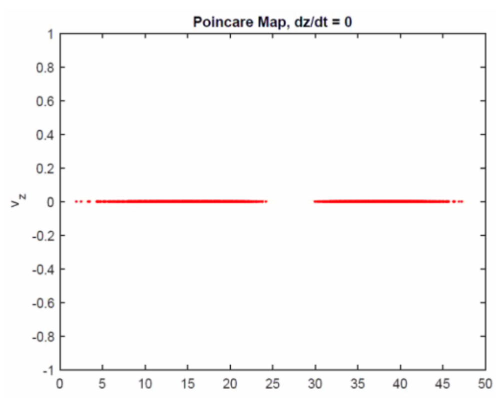
Collected “points”: 2,699 (from negative to positive; down from 20,001 for time history computation);

Poincaré maps are 5-dimensional plots.

For example,

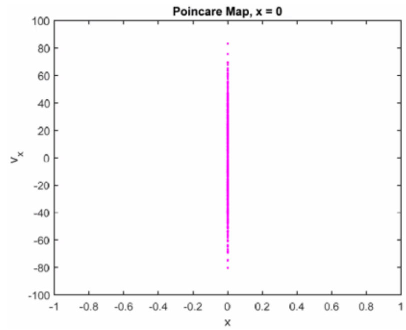




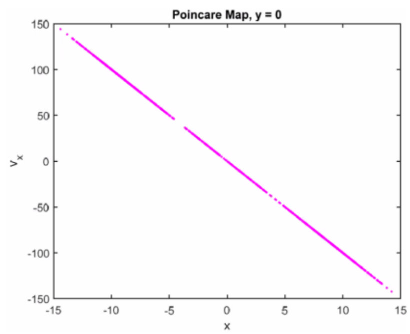


More Examples

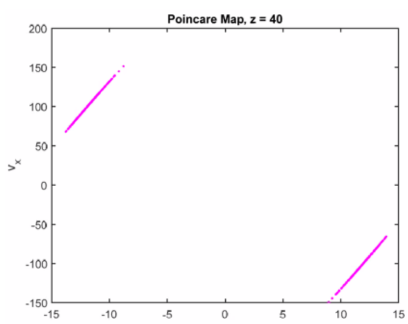
If $x = 0$ (negative to positive), 567 points



If $y = 0$ (negative to positive), 987 points



If $z = 40$ (less than to greater than), 369 points



Lyapunov Exponents and Fractal Dimensions

Organization of the Wolf et al.'s paper

The paper shows that the Lyapunov exponents can be determined from a set of first-order ODEs or from a time series. Recorded experimental data are a time series, for example.

Sec. 1: Introduction
Sec. 2: Definition
Sec. 3: Computational Aspect
Sec. 4 – 7: Time Series (Experimental Data)
Sec. 8: Results
Sec. 9: Conclusions
Appendix A: Fortran Code (Lyapunov Spectrum from computation)
Appendix B: Fortran Code (Lyapunov Spectrum from time series)

Lyapunov Exponents, the Theory

Considering a chaotic oscillator. Due to its sensitivity on initial conditions, one can image that any two trajectories close to another in the n −dimensional phase space will move exponentially away from (or towards) each other.

Let d_0 be a measure of the distance between the two trajectories at t_0. At t, a later time (keeping in mind $t - t_0$ should be small), the distance will grow (or shrink) following a base-2 exponential relation.

$$d(t) = d_0 2^{\lambda(t-t_0)}$$

The constant λ is called the Lyapunov exponent.

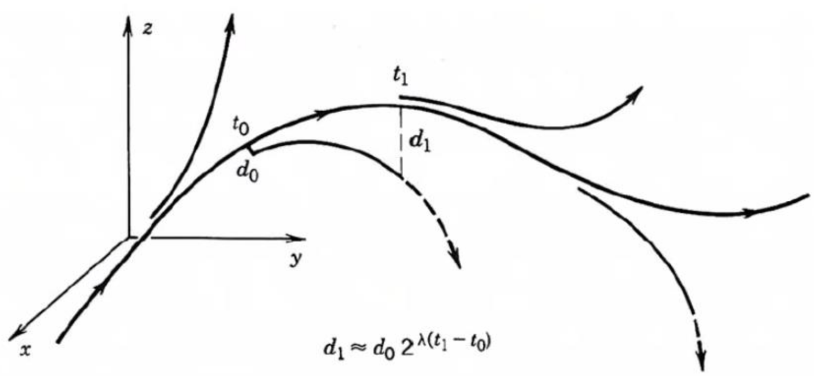


Figure 5-26 Sketch of the change in distance between two nearby orbits used to define the largest Lyapunov exponent.

Courtesy of *Chaotic vibrations An Introduction for Applied Scientists and Engineers*, Moon, 2004

Similarly, areas, spheres, and hyper-spheres in the phase space may stretch or shrink. As a result, there are respective Lyapunov exponents to measure the extents to which the principal axes of the areas, spheres, and hyper-spheres, are stretched/shrunk.

The set of Lyapunov exponents $\{\lambda_1, \lambda_2, \dots, \lambda_n\}$, where $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ is called the Lyapunov spectrum. Note that n is the dimension of the dynamic system.

A positive Lyapunov exponent is an indicator of chaos.

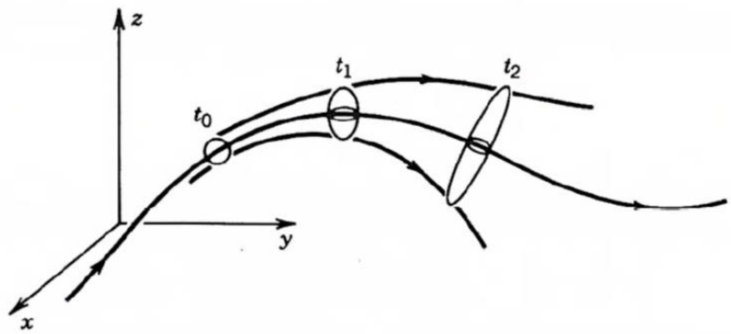


Figure 5-30 Sketch showing the divergence of orbits from a small sphere of initial conditions for a chaotic motion.

Courtesy of *Chaotic vibrations An Introduction for Applied Scientists and Engineers*, Moon, 2004

Lyapunov Exponents, the Computation
How to determine the Lyapunov exponents from a set of first-order ODEs is explained in Sec. 3 of Wolf et al's paper.

<u>One Lyapunov Exponent</u>
Assuming the vector $\boldsymbol{X}^*(t)$ represents solutions of the first-order ODEs. They are known as the reference trajectory. To compute the Lyapunov exponents, small variations from the trajectory are given, and how the variations grow or shrink is computed.

The small variations are denoted at $\boldsymbol{\delta}(t)$ (a vector), and the rate of growth or shrinkage of $\boldsymbol{\delta}$ is determined by:

$$\dot{\boldsymbol{\delta}} \approx \boldsymbol{J}(\boldsymbol{X}^*)\boldsymbol{\delta}$$

Where $\boldsymbol{J}(\boldsymbol{X}^*)$ is the Jacobian evaluated at the reference trajectory.

Numerically, $\boldsymbol{X}^*$ and $\boldsymbol{\delta}$ are solved by ODE solver at the same time. That is, in addition to the first-order ODEs, the Jacobian needs to be coded and included in the "function".

One $\boldsymbol{\delta}(t)$ is determined, the Lyapunov exponent is by:

$$\lambda(t_M) = \frac{1}{t_M - t_0} \sum_{k=1}^{M} \log_2 \frac{|\boldsymbol{\delta}(t_{k+1})|}{|\boldsymbol{\delta}(t_k)|}$$

The above equation has two key messages. (1) λ is a function of time, as M gets larger and larger; (2) Averaging over a long period of time, or over large expanse of the phase space will give rise to a relatively stable λ.

<u>Lyapunov Spectrum</u>

To find the Lyapunov spectrum, n sets of $\delta(t)$ will be needed. They are orthonormal to each other. Or they are the base vectors of an n −dimensional linear space.

GSR (Gram-Schmidt Reorthonormalization) is employed to ensure that the n sets of δ(t) remain orthonormal to each other, as time evolves. Without the GSR, the λ_i's will become indistinguishable as time evolves. In other words, only the largest Lyapunov exponent (LLE) will be meaningfully computed.

GSR can be performed at a certain fixed time interval, say, every time-step, or every 10 time-steps.

Additionally, the n small variation vectors are normalized (to unit vectors) at the beginning of each time step, to avoid overflow during the computation.

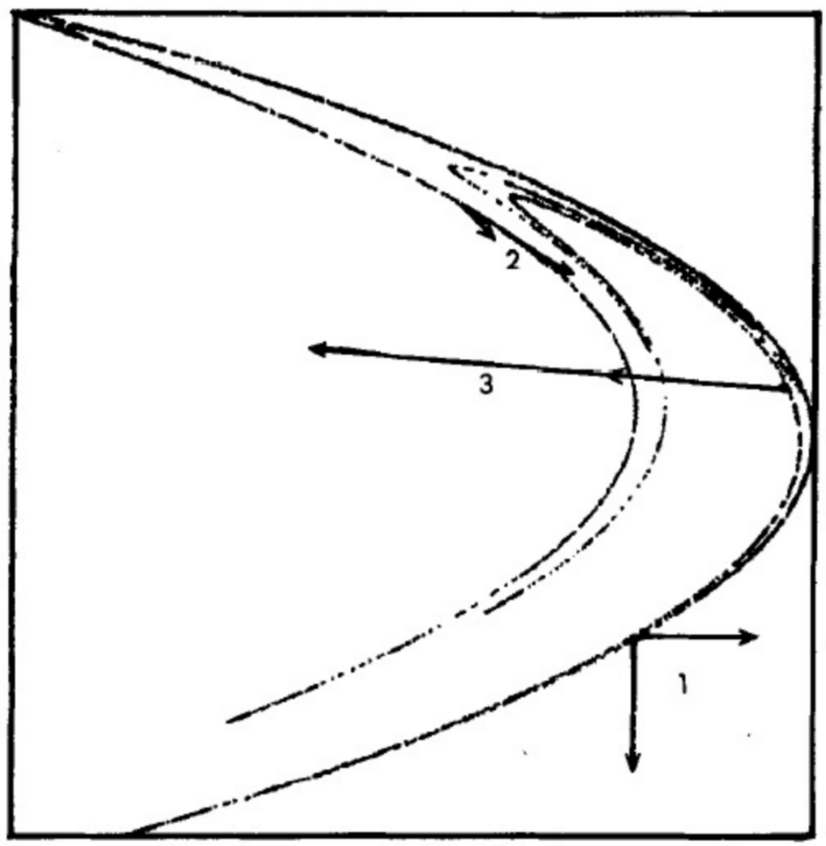


Fig. 2. The action of the product Jacobian on an initially orthonormal vector frame is illustrated for the Hénon map: (1) initial frame; (2) first iterate; and (3) second iterate. By the second iteration the divergence in vector magnitude and the angular collapse of the frame are quite apparent. Initial conditions were chosen so that the angular collapse of the vectors was uncommonly *slow*.

Courtesy of *Determining Lyapunov exponents from a time series*, Wolf et al., Physica D, vol.16, pp. 285-317, 1985.

ODEs and Jacobians, Examples

The Rossler attractor:

$$\dot{x} = -(y + z)$$
$$\dot{y} = x + ay$$
$$\dot{z} = b + z(x - c)$$

The Jacobian is:

$$J = \begin{bmatrix} 0 & -1 & -1 \\ 1 & a & 0 \\ z & 0 & x - c \end{bmatrix}$$

Next, the Mathieu's equation.

$$\ddot{x} + [a - 2qcos(2\omega t)]x = 0$$

Where a and q are constants.

Set

$$x_1 = x$$
$$x_2 = \dot{x}$$
$$x_3 = \omega t$$

The first-order ODEs are:

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = [2qcos(2x_3)x - a]x_1$$
$$\dot{x}_3 = \omega$$

The Jacobian is:

$$J = \begin{bmatrix} 0 & 1 & 0 \\ 2qcos(2x_3) - a & 0 & -4qsin(2x_3) \\ 0 & 0 & 0 \end{bmatrix}$$

What are the ODEs needed for computing the Lyapunov spectrum?

If n is the number of first-order ODEs, the total number of ODEs for the spectrum is, $n(n + 1)$.

For the systems above, $n = 3$, so 12 ODEs are needed, or 3 sets of ODEs with 3 ODEs in each set.

For example, the first-order ODEs (for the Mathieu's equation)

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = [2qcos(2x_3) - a]x_1$$
$$\dot{x}_3 = \omega$$

Are to compute the reference trajectory $\boldsymbol{X}^*(t)$. The other n sets are to determine the three $\boldsymbol{\delta}$'s by:

$$\dot{\boldsymbol{\delta}} = \boldsymbol{J}(\boldsymbol{X}^*)\boldsymbol{\delta}$$

With

$$\delta = \begin{bmatrix} x_4 & x_7 & x_{10} \\ x_5 & x_8 & x_{11} \\ x_6 & x_9 & x_{12} \end{bmatrix}$$

The initial conditions for the $\delta's$ are:

$$\delta(0) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Fractal Dimensions

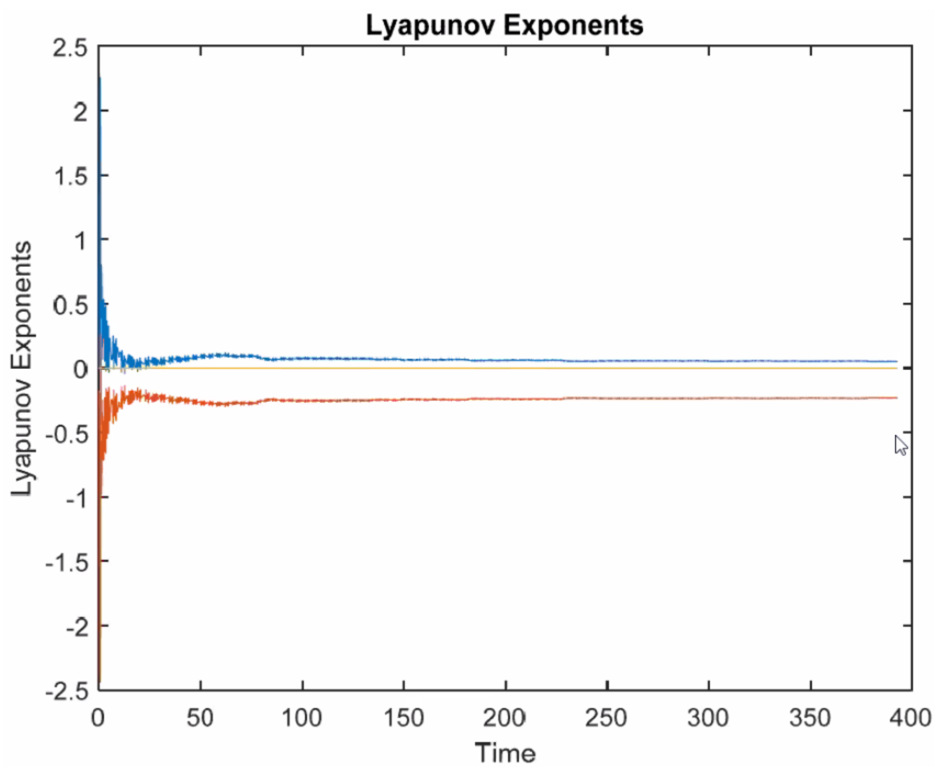
Fractal dimension refers to a non-integer dimension. The idea originates from the observations that chaotic oscillators occupy regions of the phase space.

Fractal dimension is used to measure the extent to which trajectories fill up a certain subspace.

A non-integer dimension is a hallmark of a strange attractor and implies the existence of chaos.

There are several definitions of fractal dimensions, for example, pointwise dimension, capacity dimension, correlation dimension, information dimension, and so on.

Information dimension can be easily determined once the Lyapunov spectrum is known, see Eqs. (2) and (3) in the paper by Wolf et al.



Information $d_I = 2.2229$

$$\ddot{x} + \delta\dot{x} + \alpha x + \beta x^3 = f_0 \cos(\omega t)$$

$\delta = 0.18$, $\alpha = \beta = 1$, $\omega = 0.8$, and $f_0 = 22.5$.

Bifurcation and Bifurcation Diagram

What is bifurcation?

For a dynamic system defined by $\dot{\boldsymbol{x}} = f(\boldsymbol{x}; c)$, the equilibrium points are those that meet the condition of $\dot{\boldsymbol{x}} = \boldsymbol{0}$, or $f(\boldsymbol{x_e}) = \boldsymbol{0}$, with $\boldsymbol{x_e}$ denoting the equilibrium points (basically you set the LHS equal to zero to solve for $\boldsymbol{x}$).

Equilibrium points are classified as:

- Centers or stable equilibrium points.
- Saddle points/nodes or unstable equilibrium points.

As the system's parameter c changes, the number of equilibrium points and the stability of such points can change as well.

The phenomenon that the number of equilibrium points and the stability of such points can change as the system's parameter changes is known as the bifurcation.

Specifically, it is about the change in the type of *long-term* behaviors of the system when parameters when parameters are varied.

Theory of bifurcation is the study of these changes is nonlinear systems.

What is a bifurcation diagram?

It is a widely used technique for examining the pre- or post-chaotic changes in a dynamic system under parameter variations.

Typically, some measure of the response of the system is plotted against a system parameter.

The measure may be, for example, the maximum amplitude, the local maxima or minima, or data sampled using a Poincaré map.

When the bifurcation diagram loses continuity, it means either quasi-periodic motion or chaotic motion. Bifurcating does not mean being chaotic, it simply means it is bifurcating. Other approaches to identify chaos should be jointly used.

Bifurcation diagram can be drawn through analytical ways or by computation.

Bifurcation diagram by computation

The following outline of computation assumes that the local maxima in position $x(t)$ are the measure to be plotted.

Steps are:

Loop over a parameter range (say, $f_0 = 20$ to 25, at an increment of 0.001).

> Run an ode solver for the response. Make sure that is covers a long period of time, say, ≥ 100 forcing period if excitation is present
>
> Use the second half of the computed time history of position, for plotting the bifurcation diagram. That is, the measure is $y(t)$.
>
> Loop over the length of $y(t)$ to find and store the local maxima
>
> > Find $y(t_{i-1}) < y(t_i)$ and $y(t_i) > y(t_{i+1})$
> >
> > Use the three points (three t −values and three y −values to evaluate the local maximum)
> >
> > Store the local maximum in an array (as a vector)
>
> End of loop
>
> Plot the local maxima vector against the specific parameter value

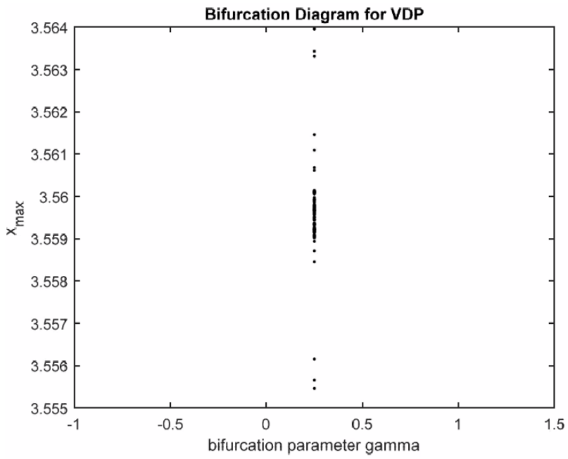
End of loop

Example: Consider the following van der Pol oscillator:

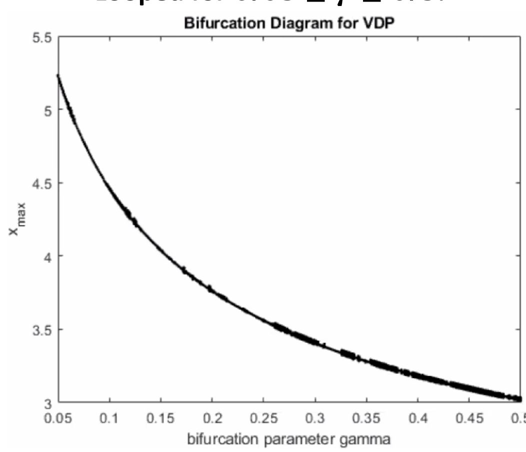
$$\ddot{x} - \gamma\dot{x}(1 - x^2) + \alpha x = f_0\cos(\omega t)$$

Where $\alpha = 1$, $\gamma = 0.25$, $f_0 = 3$, $\omega = 1.2$, $x(0) = 1$, and $\dot{x}(0) = 0$

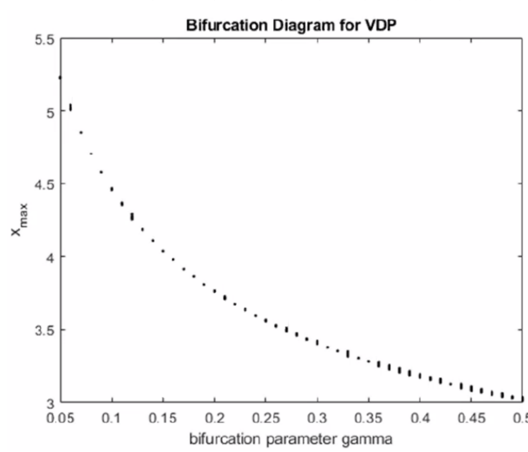
Plot when $\gamma = 0.25$ (plot of 100 points):



Looped for $0.05 \le \gamma \le 0.5$:



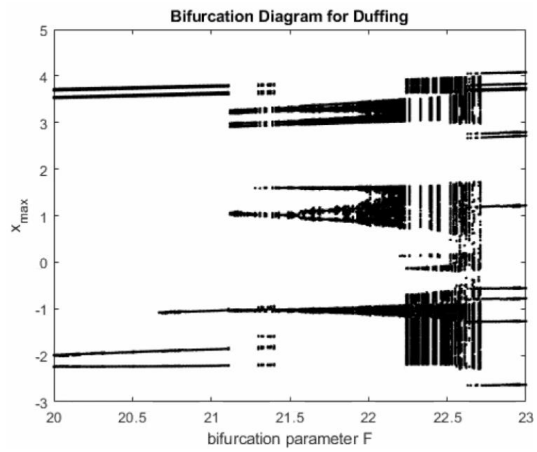
With 10x the increment as above:



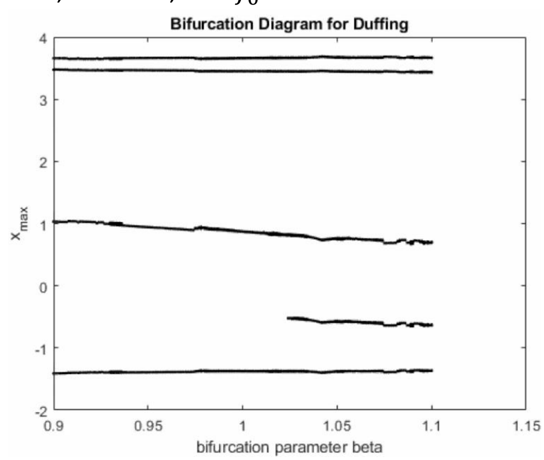
Example: Consider the following Duffing oscillator:

$$\ddot{x} - \gamma\dot{x} + \alpha x + \beta x^3 = f_0 \cos(\omega t)$$

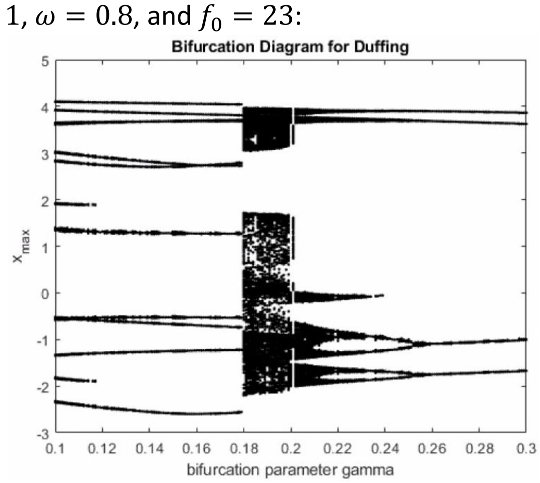
(1) $\gamma = 0.18$, $\alpha = \beta = 1$, $\omega = 0.8$, and $\boldsymbol{f_0} = \mathbf{20}$ **to** $\mathbf{23}$:



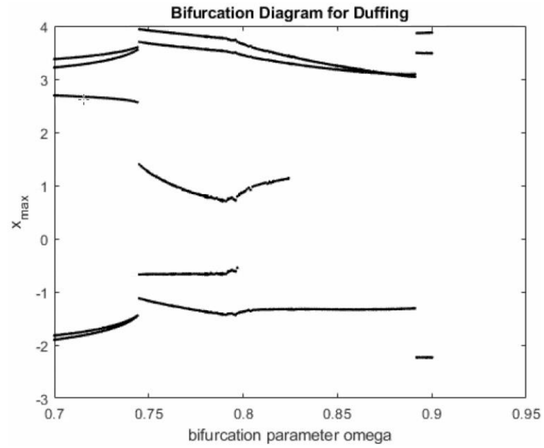
(2)TODO FIX $\gamma = 0.18$, $\alpha = \beta = 1$, $\omega = 0.8$, and $f_0 = 20$ to 23:



(3) $\boldsymbol{\gamma} = \mathbf{0.1}$ **to** $\mathbf{0.3}$, $\alpha = \beta = 1$, $\omega = 0.8$, and $f_0 = 23$:



(4) $\gamma = 0.18$, $\alpha = \beta = 1$, $\boldsymbol{\omega} = \mathbf{0.7}$ **to** $\mathbf{0.9}$, and $f_0 = 23$:



NOTE: Some potential topics for final exam: Perturbation, harmonic balance, modeling, bifurcation by computation or by theory

Bifurcation Diagram by Analytical Ways

Bifurcation and bifurcation diagram

For a dynamic system defined by $\ddot{\boldsymbol{x}} = \boldsymbol{F}(\boldsymbol{x}, \dot{\boldsymbol{x}}, t; c)$, the number of its equilibrium points and the stability of such points change as the system's parameter c is varied. This phenomenon is known as the bifurcation.

Bifurcation diagram is a widely used technique for examining the pre- or post-chaotic changes in a dynamic system under parameter variations.

Bifurcation diagrams can be drawn through analytical ways or by computation.

Bifurcation diagram by the analytical ways

Focusing on autonomous dynamic systems defined by first order ODEs $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}; c)$;

<u>The existence and uniqueness of theorem</u>

Detail of the existence and uniqueness theorem can be found form Boyce et al, Theorem 2.4.2 and Theorem 2.8.1, for example.

The essence of the theorem governing the existence and uniqueness of the solutions to first order ODEs $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}; c)$ is,

If the functions $\boldsymbol{f}$ and their first order partial derivatives are continuous over a certain domain for x and t, then there exists a unique solution of the system of ODEs that satisfied the initial condition.

<u>The equilibrium points (or critical points)</u>

They are those that meet the condition of $\dot{\boldsymbol{x}} = \boldsymbol{0}$, or $\boldsymbol{f}(\boldsymbol{x_e}; c) = \boldsymbol{0}$, with $\boldsymbol{x_e}$ denoting the equilibrium points.

<u>Definition of stability</u>

There is not a universally agreed upon definition. But the most fundamental definition is attributed to Lyapunov.

Let $\boldsymbol{x_e} \in R^n$ be an equilibrium point,

(1) $\boldsymbol{x_e}$ is stable if, for any $h > 0$, there is a $\delta > 0$ such that if a solution $\boldsymbol{x}(t)$ satisfies $||\boldsymbol{x}(0) - \boldsymbol{x}_e|| < \delta$, then:

$$||\boldsymbol{x}(t) - \boldsymbol{x_e}|| < h, \qquad for\ all\ t > 0$$

(2) x_e is asymptotically stable is there is a $\delta > 0$ such that if a solution $x(t)$ satisfies $||\boldsymbol{x}(0) - \boldsymbol{x}_e|| < \delta$, then:

$$\lim_{t\to\infty} \boldsymbol{x}(t) = \boldsymbol{x_e};$$

(3) $\boldsymbol{x_e}$ is monotonically stable if it is asymptotically stable and $||\boldsymbol{x}(t) - \boldsymbol{x_e}||$ decreases monotonically with time;

(4) $\boldsymbol{x_e}$ is globally asymptotically stable if it is asymptotically stable and $\boldsymbol{x}(t) \to \boldsymbol{0}\ \ and\ t \to \infty$ for all $\boldsymbol{x}(0)$; and

(5) $\boldsymbol{x_e}$ is unstable if it is not stable as defined above in (1).

Parts (1), (2) and (5) of the definition appear often in the literature.
Detail of (1), (2) and (5) can be found from Boyce et al, Sec. 9.2 for example.

Linearization of nonlinear ODEs

For ODEs $x = f(x; c)$, its Jacobian matrix J evaluated at x_e, is:

$$\boldsymbol{J}(\boldsymbol{x_e}; c) = \boldsymbol{A}(\boldsymbol{x_e}; c) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}_{\boldsymbol{x_e}}$$

Taking the Taylor expansion of $\boldsymbol{f}(\boldsymbol{x}; c)$ at $\boldsymbol{x_e}$ and keeping terms up to the linear,

$$\dot{x} = \boldsymbol{f}(\boldsymbol{x}; c) \approx \boldsymbol{f}(\boldsymbol{x_e}; c) + \boldsymbol{J}(\boldsymbol{x} - \boldsymbol{x_e})$$

Defining $\boldsymbol{u} = \boldsymbol{x} - \boldsymbol{x_e}$, and noting that $\boldsymbol{f}(\boldsymbol{x_e}; c) = 0$ and $\dot{\boldsymbol{x}}_{\boldsymbol{e}} = \boldsymbol{0}$, then linearized ODEs at $\boldsymbol{x_e}$ is:

$$\dot{\boldsymbol{u}} = \boldsymbol{J} \cdot \boldsymbol{u}$$

In other words, $\boldsymbol{x_e}$ is the "reference", $\boldsymbol{u}$ is the growth or shrinkage from the reference.

Classification of equilibrium point

Earlier and in a not-too-specific way, equilibrium point, or stability, is classified as:

- Centers or stable equilibrium points
- Saddle points/nodes or unstable equilibrium points.

A few terminologies are in order.

Linear stability:

- The stability of a system of linear ODEs
- The stability of a system of nonlinear ODEs which is linearized as $\dot{\boldsymbol{u}} = \boldsymbol{J} \cdot \boldsymbol{u}$

The latter is also known as the local stability.

Non-linear stability:

The stability of a system of nonlinear ODEs, typically by making use of the Lyapunov function.

Proper nodes and improper nodes:

When $\boldsymbol{J}$ has identical eigenvalues, if the corresponding eigenvectors are independent of each other, the node is proper; otherwise it is improper.

The more specific classification:

If the eigenvalues associated with $\boldsymbol{J}$ evaluated at $\boldsymbol{x_e}$ are, $\lambda_1, \dots, \lambda_n$,

$n = 2$:

λ_1 and λ_2	Classification	Linear Stability
Both real and positive	Source	Unstable
Both real and negative	Sink	Asymptotically stable
Both real, one positive, one negative	Saddle	Unstable
Identical, real and positive	Improper Node	Unstable
Identical, real and negative	Improper Node	Asymptotically stable
Complex, with positive real part	Outward spiral	Unstable
Both imaginary*	Center	Stable

* Stability is indeterminate if local stability is concerned.

Detail can be found from Boyce et al, Theorem 9.3.3 and Table 9.3.1, for example.

$n > 2$:

All eigenvalues have negative real parts, then $\boldsymbol{x_e}$ is a stable equilibrium point;

If at least one of the eigenvalues has a positive real part, then $\boldsymbol{x_e}$ is an unstable equilibrium point.

For other cases, nonlinear stability analysis is required.

<u>Classification of bifurcation</u>

Saddle point bifurcation or fold bifurcation (two equilibrium points move towards each other, collide, and become one; or the opposite)

Tranncritical bifurcation (a pair of equilibrium points exchange stability; i.e., one point goes from stable to unstable while the other does the opposite; but the change takes place at the same parameter value)

Pitchfork bifurcation (equilibrium points go from one to three, or the opposite; the former is known as supercritical pitchfork bifurcation and the latter subcritical pitchfork bifurcation)

Andronov-Hopf bifurcation or simply *Hopf bifurcation* (bifurcation from periodic solutions; e.g., the creation or destruction of a limit cycle)

<u>Saddle point Bifurcation</u>

Consider:

$$\dot{x} = a - x^2$$

where x and a are real

The equilibrium points:

$$x = \begin{cases} 0 & a \le 0 \\ \pm\sqrt{a} & a > 0 \end{cases}$$

The Jacobian at $\boldsymbol{x_e}$:

$$J = \begin{cases} [0] & x_e = 0 \\ [-2\sqrt{a}] & x_e = \sqrt{a} \\ [2\sqrt{a}] & x_e = -\sqrt{a} \end{cases}$$

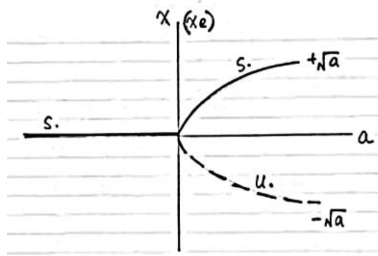
Eigenvalues:

$$\lambda = \begin{cases} [0] & x_e = 0 \\ [-2\sqrt{a}] & x_e = \sqrt{a} \\ [2\sqrt{a}] & x_e = -\sqrt{a} \end{cases}$$

The solution to $\dot{\boldsymbol{u}} = \boldsymbol{J} \cdot \boldsymbol{u}$**:**

$$\boldsymbol{u}(t) = \begin{cases} \alpha e^{0 \cdot t} & x_e = 0 \\ \alpha e^{-2\sqrt{a} \cdot t} & x_e = \sqrt{a} \\ \alpha e^{2\sqrt{a} \cdot t} & x_e = -\sqrt{a} \end{cases}$$

The bifurcation diagram is:



Transcritical Bifurcation

Consider:

$$\dot{x} = ax - bx^2$$

where x, a and b are real, $a \neq 0$, $b > 0$. The parameter is a/b.

The equilibrium points:

$$x_e = 0 \text{ and } \frac{a}{b}$$

The Jabocian at x_e:

$$\boldsymbol{J} = \begin{cases} [a] & x_e = 0 \\ [-a] & x_e = \dfrac{a}{b} \end{cases}$$

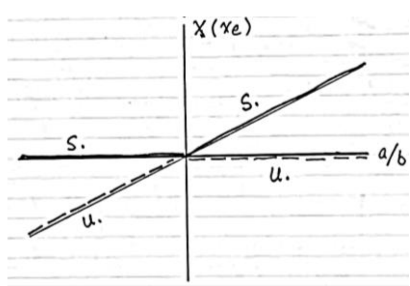
Eigenvalues:

$$\lambda = \begin{cases} a & x_e = 0 \\ -a & x_e = \dfrac{a}{b} \end{cases}$$

The solution to $\dot{\boldsymbol{u}} = \boldsymbol{J}\boldsymbol{u}$:

$$u(t) = \begin{cases} \alpha e^{at} & x_e = 0 \\ \alpha e^{-at} & x_e = \dfrac{a}{b} \end{cases}$$

The bifurcation diagram is:



Pitchfork bifurcation

Consider:

$$\dot{x} = ax - bx^3$$

Where x, a and b are real, $a \neq 0$, $b > 0$. The parameter is a/b.

The equilibrium points:

$$x_e = \begin{cases} 0 & any\ a \\ \pm\sqrt{\frac{a}{b}} & a > 0 \end{cases}$$

The Jabocian at x_e:

$$\boldsymbol{J} = \begin{cases} [a] & x_e = 0 \\ [-2a] & x_e = \pm\sqrt{\frac{a}{b}} \end{cases}$$

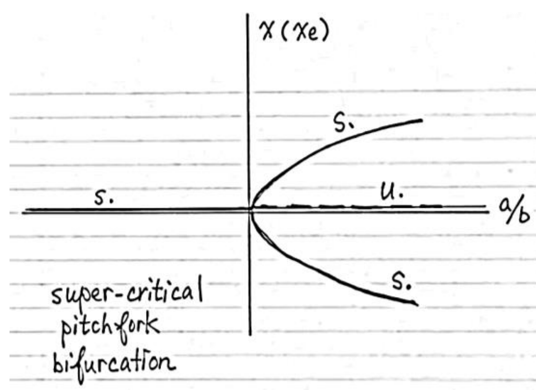
Eigenvalues:

$$\lambda = \begin{cases} a & x_e = 0 \\ -2a & x_e = \pm\sqrt{\frac{a}{b}} \end{cases}$$

The solution to $\dot{\boldsymbol{u}} = \boldsymbol{J}\boldsymbol{u}$:

$$u(t) = \begin{cases} \alpha e^{at} & x_e = 0 \\ \alpha e^{-2at} & x_e = \pm\sqrt{\frac{a}{b}} \end{cases}$$

The bifurcation is:

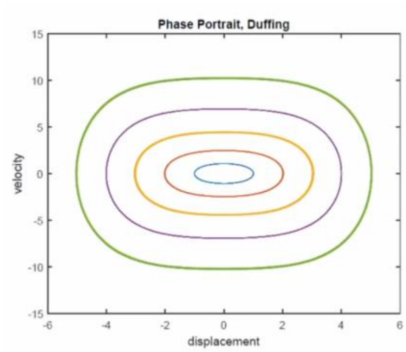


Unforced and Undamped Duffing Oscillator:

$$\ddot{x} + ax + \beta x^3 = 0$$

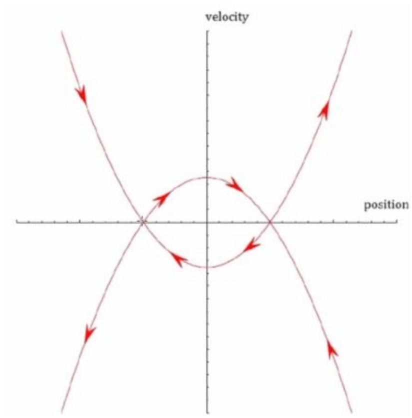
$\beta > 0$ or hardening:

- O is a center, or a stable equilibrium point.



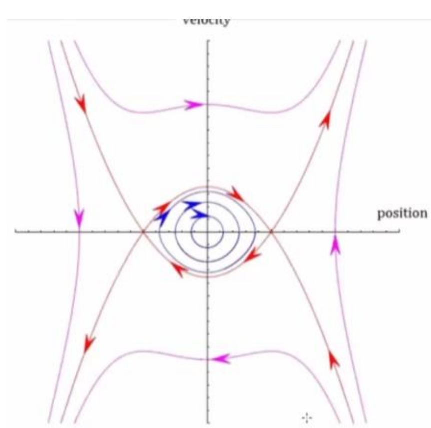
$\beta < 0$ or softening:

- Saddle points (or nodes) and separatrices:
  Saddle points (or nodes) are unstable equilibrium points.
  Separatrix refers to the boundary separating different modes of vibrations



- The two situations:

1) Continuous, or closed curves inside the separatrices; or
2) Curves "running off" to infinity outside the separatrices



Softened Duffing oscillator:

$$\dot{x} + ax + \beta x^3 = 0, \quad \text{where } \alpha \geq 0, \beta < 0$$

Define:

$$x = \begin{Bmatrix} x \\ \dot{x} \end{Bmatrix} = \begin{Bmatrix} x \\ y \end{Bmatrix}$$

The first order ODEs

$$x = \begin{Bmatrix} \dot{x} \\ \dot{y} \end{Bmatrix} = \begin{Bmatrix} y \\ -\alpha x - \beta x^3 \end{Bmatrix}$$

The equilibrium points:

$$\boldsymbol{x_e} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}, \qquad \boldsymbol{x_e} = \begin{Bmatrix} \sqrt{\dfrac{\alpha}{\beta}} \\ 0 \end{Bmatrix}, \qquad \boldsymbol{x_e} = \begin{Bmatrix} -\sqrt{\dfrac{\alpha}{\beta}} \\ 0 \end{Bmatrix}$$

Jacobians:

$$\boldsymbol{J} = \begin{bmatrix} 0 & 1 \\ -\alpha - 3\beta x^2 & 0 \end{bmatrix}$$

$$\therefore \boldsymbol{J} = \begin{bmatrix} 0 & 1 \\ -\alpha & 0 \end{bmatrix}, \qquad \boldsymbol{J} = \begin{bmatrix} 0 & 1 \\ 2\alpha & 0 \end{bmatrix}, \qquad \boldsymbol{J} = \begin{bmatrix} 0 & 1 \\ 2\alpha & 0 \end{bmatrix}$$

Eigenvalues:

$$\lambda = \begin{Bmatrix} -\sqrt{\alpha} j \\ \sqrt{\alpha} j \end{Bmatrix}, \qquad \lambda = \begin{Bmatrix} -\sqrt{2\alpha} j \\ \sqrt{2\alpha} j \end{Bmatrix}, \qquad \lambda = \begin{Bmatrix} -\sqrt{\alpha} 2j \\ \sqrt{2\alpha} j \end{Bmatrix}$$

Classification:

$\boldsymbol{x_e} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$ Center

$\boldsymbol{x_e} = \begin{Bmatrix} \sqrt{\dfrac{\alpha}{\beta}} \\ 0 \end{Bmatrix}, \boldsymbol{x_e} = \begin{Bmatrix} -\sqrt{\dfrac{\alpha}{\beta}} \\ 0 \end{Bmatrix}$ Saddle Points