2.9 Fuzzy IF-THEN Rules

Fuzzy implication

"If x is A then y is B"

Premise / condition

Conclusion / consequence

"$A \rightarrow B$"

A and B ~ linguistic values

X and Y ~ universe

$R = A \rightarrow B$

- A coupled with B

$R = A \rightarrow B = A \times B$

$$= \int_{X \times Y} \mu_A(x) \,\tilde{*}\, \mu_B(y) \Big/ (x, y)$$

$\tilde{*} = T$-norm operator

- Material implication (A entails B)

$R = A \rightarrow B = \neg A \sqcup B$

And:

$a = \mu_A(x)$

$b = \mu_B(y)$

1) A coupled with B

1. *Mamdani conjunction*

$$R_m = A \to B = A \times B = \int_{X \times Y} \mu_A(x) \wedge \mu_B(y) \Big/ (x, y)$$

$$f_m(a, b) = a \wedge b$$

2. *Larson (product) implication*

$$R_p = A \times B = \int_{X \times Y} \mu_A(x) \cdot \mu_B(y) \Big/ (x, y)$$

$$f_p = a \cdot b$$

3. *Bounded product operator*

$$R_{bp} = A \times B = \int_{X \times Y} \mu_A(x) \odot \mu_B(y) \Big/ (x, y)$$

$$= \int 0 \vee [\mu_A(x) + \mu_B(y) - 1] \Big/ (x, y)$$

$$f_{bp} = 0 \vee [a + b - 1]$$

4. *Drastic product operator*

$$R_{dp} = A \times B = \int_{X \times Y} \mu_A(x) \hat{\cdot} \mu_B(y) \Big/ (x, y)$$

$$f_{dp}(a, b) = a \hat{\cdot} b = \begin{cases} a & ; \quad b = 1 \\ b & ; \quad a = 1 \\ 0 & ; \quad \text{Otherwise} \end{cases}$$

Consider:

$a = \mu_A(x) = \text{bell}(x, 4, 3, 10)$

$b = \mu_B(y) = \text{bell}(y, 4, 3, 10)$

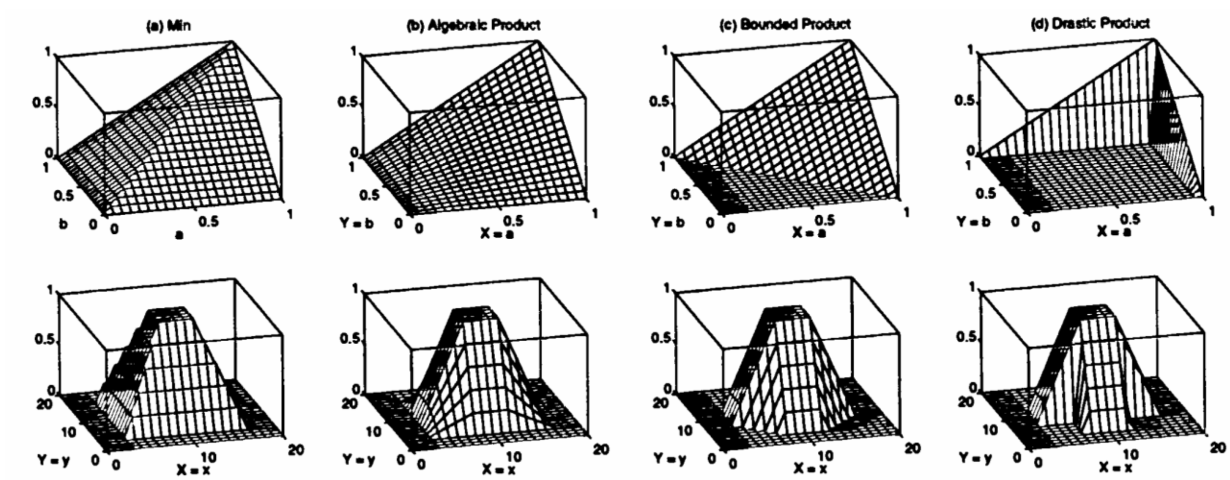


Figure 2.16. *(First row) Four T-norm operators $T_{min}(a, b)$, $T_{ap}(a, b)$, $T_{bp}(a, b)$, and $T_{dp}(a, b)$; (second row) the corresponding surfaces for a = trapezoid$(x, 3, 8, 12, 17)$ and b = trapezoid$(y, 3, 8, 12, 17)$.* (MATLAB file: `tnorm.m`)

2) A entails B

1. Zadeh's arithmetic rule

$$R_a = A \rightarrow B = \neg A \sqcup B$$
$$f_a(a, b) = 1 \wedge (1 - a + b)$$

2. Zadeh's max-min rule

$$R_{mm} = A \rightarrow B = \neg A \sqcup (A \sqcap B)$$
$$a = \mu_A(x)$$
$$b = \mu_B(x)$$
$$f_{mm}(a, b) = (1 - a) \vee (a \wedge b)$$

3. Boolean fuzzy implication

$$R_B = A \rightarrow B = \neg A \sqcup B$$
$$= \int_{X \times Y} [1 - \mu_A(x)] \vee \mu_B(y) \Big/ (x, y)$$
$$f_B(a, b) = (1 - a) \vee b$$

4. Gogen's fuzzy implication

$$R_\Delta = A \rightarrow B$$
$$= \int_{X \times Y} \mu_A(x) \gtrsim \mu_B(y) \Big/ (x, y)$$
$$f_\Delta(a, b) = a \gtrsim b = \begin{cases} 1 & ; \quad a \le b \\ b/a & ; \quad a > b \end{cases}$$

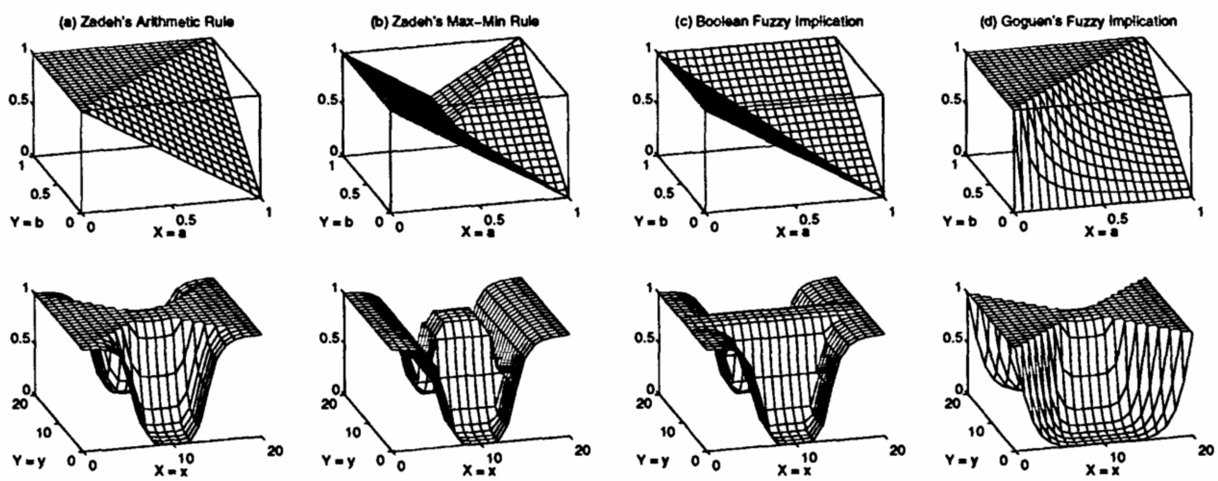


Figure 3.9. *First row: fuzzy implication functions based on the interpretation "A entails B"; second row: the corresponding fuzzy relations.* (MATLAB file: **fuzimp.m**)

2.10 Fuzzy Reasoning Rulebase

2-valued logic, modus ponens

Something like:
fact $\sim x$ is A'
premise (rule) $=$ if x is A then y is B
consequent conclusion $\sim y$ is B'
→ called approximate reasoning
→ or generalised modus ponens (GMP)

Let A, B be fuzzy sets
of X and Y, $A' \sim$ of X'

Rule – fuzzy implication
$R = A \rightarrow B \ \ ; \ \ X \text{ x } Y$

$$\mu_B'(y) = \max\min[\mu_A'(x), \ \ \mu_R(x,y)]$$

$$= \vee_x \ [\mu_A(x) \wedge \mu_B(x,y)]$$

or

$$B' = A' \circ R = A' \circ (A \rightarrow B)$$

" $\circ$ " $=$ composition operator

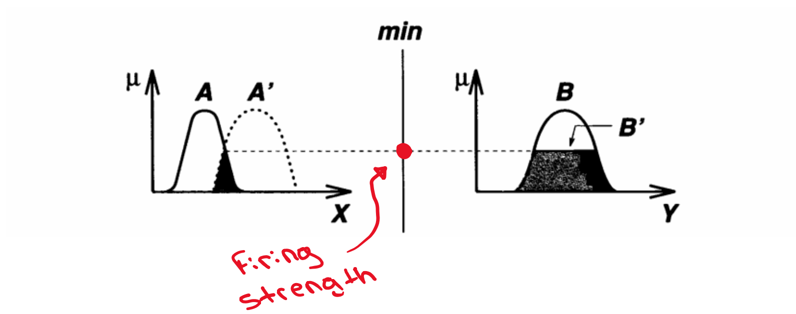
1) Single rule with single antecedent

Premise 1 (fact):	x is A'
Premise 2 (rule):	If x is A then y is B
Consequence (conclusion):	y is B'

$$\mu_B'(y) = \vee_x \ [\mu_A'(x) \wedge \mu_R(x,y)]$$
$$A \rightarrow B = A \wedge B$$
$$= \vee_x \ \big[\mu_A'(x) \wedge [\mu_A(x) \wedge \mu_B(y)]\big]$$
$$= \vee_x \ \big[\underbrace{[\mu_A'(x) \wedge \mu_A(x)]}_{\text{degree of match}} \wedge \mu_B(y)\big] = \omega \text{ (intersection)}$$

$$\mu_B'(y) = \vee_x \ [\omega \wedge \mu_B(y)]$$

$$\mu_B'(y) = \overset{\vee_x}{\max} \ \underbrace{[\mu_A'(x) \wedge \mu_A(x)]}_{\omega} \wedge \mu_B(y)$$
$$= \omega \wedge \mu_B(y)$$



2) Single rule with multiple antecedents

antecedent ∼ something existing before (or logically proceeding) another.

Premise 1 (fact):	x is A' and y is B'
Premise 2 (rule):	If x is A and y is B then z is C
Consequence (conclusion):	z is C'

$$R = A \times B \to C$$

(and)

Mamdani's implication:

$$R_m(A, B, C) = A \times B \to C$$
$$= \int \mu_A(x) \wedge \mu_B(y) \wedge \mu_C(z) \Big/ (x, y, z)$$

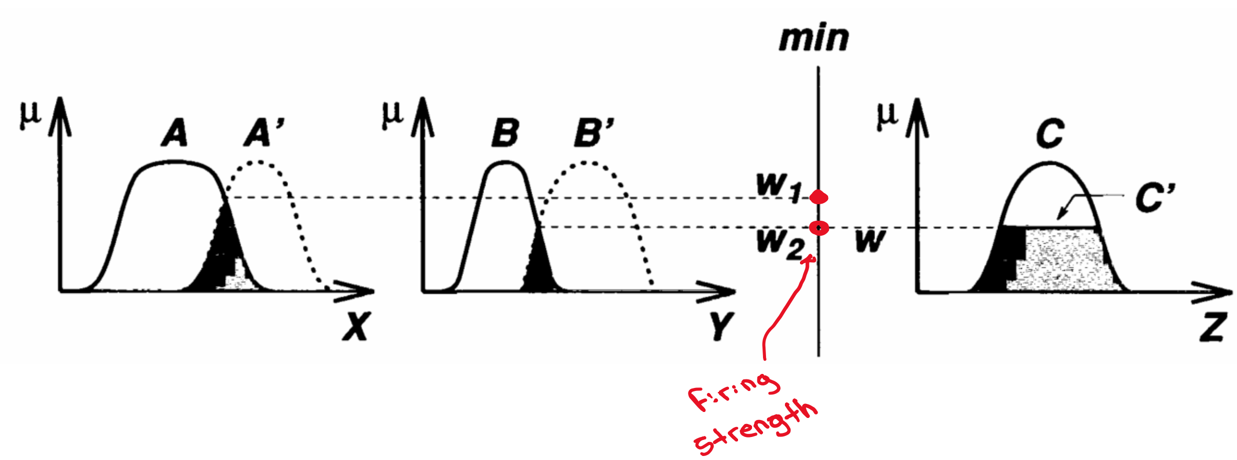
$A' \times B'; C' = ?$

$$C' = (A' \times B') \times R_m$$
$$= (A' \times B') \cdot (A \times B \to C)$$
$$= (A' \times B') \wedge (A \times B \times C)$$

$$\mu'_C(z) = max - min$$
$$= \vee_{x,y} \{[\mu'_A(x) \wedge \mu'_B(y)] \wedge [\mu_A(x) \wedge \mu_B(y) \wedge \mu_C(z)]\}$$
$$= \vee_{x,y} \{[\mu'_A(x) \wedge \mu_A(x)] \wedge [\mu'_B(y) \wedge \mu_B(y)]\} \wedge \mu_C(z)$$

$$= \underbrace{\vee_x [\mu'_A(x) \wedge \mu_A(x)]}_{\omega_1} \wedge \underbrace{\vee_y [\mu'_B(y) \wedge \mu_B(y)]}_{\omega_2} \wedge \mu_C(z)$$

$$= \omega_1 \wedge \omega_2 \wedge \mu_C(z)$$

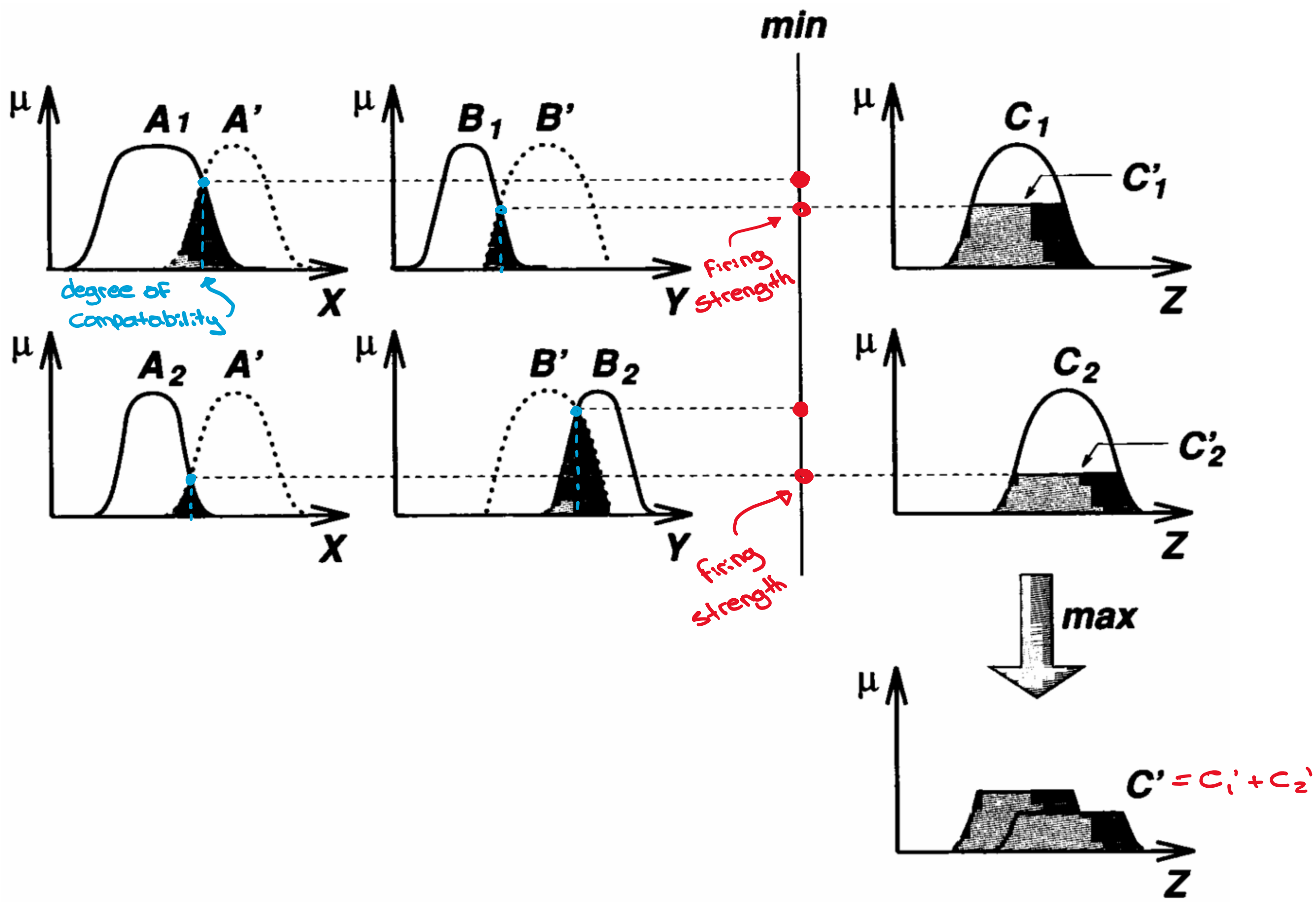


3) Multiple rules with multiple antecedents

Premise 1 (fact):	x is A' and y is B'
Premise 2 (rule 1):	If x is A_1 and y is B_1 then z is C_1
Premise 3 (rule 2):	If x is A_2 and y is B_2 then z is C_2
Consequence (conclusion):	z is C'

Rule 1: $R_1 = A_1 \times B_1 \rightarrow C_1$
Rule 2: $R_2 = A_2 \times B_2 \rightarrow C_2$
Fact: $A' \times B'$

Use max min composition operator " ∘ "
$C' = (A' \times B') \circ (R_1 \cup R_2)$



$C' = (A' \times B') \wedge (R_1 \sqcup R_2)$
$= \underbrace{[(A' \times B') \wedge R_1]}_{C_1'} \sqcup \underbrace{[(A' \times B') \wedge R_2]}_{C_2'}$

$= C_1' \sqcup C_2'$

Theorem 2.1 Decomposition Method

$R \to (A \text{ x } B \to C)$

Given fact: A' x B'

$C' = (A' \text{x } B') \cdot (A \text{ x } B \to C)$

$= \underbrace{[A' \cdot (A \to C)]}_{C_1'} \sqcap \underbrace{[B' \cdot (B \to C)]}_{C_2'}$

$= C_1' \sqcap C_2'$

Proof:

$\mu_{C'}(z) = \vee_{x,y} \{[\mu_{A'}(x) \wedge \mu_{B'}(y)] \wedge [\mu_A(x) \wedge \mu_B(y) \wedge \mu_C(z)]\}$

$= \vee_x [\mu_{A'}(x) \wedge \mu_A(x) \wedge \mu_C(z)] \wedge \vee_y [\mu_{B'}(y) \wedge \mu_B(y) \wedge \mu_C(z)]$

$= \mu_{A' \circ (A \to C)} \wedge \mu_{B' \circ (B \to C)}$

$= C_1' \wedge C_2'$

In Summary

Degree of compatibility Compare the known facts with the antecedents of fuzzy rules to find the degrees of compatibility with respect to each antecedent MF.

Firing strength Combine degrees of compatibility with respect to antecedent MFs in a rule using fuzzy AND or OR operators to form a firing strength that indicates the degree to which the antecedent part of the rule is satisfied.

Qualified (induced) consequent MFs Apply the firing strength to the consequent MF of a rule to generate a qualified consequent MF. (The qualified consequent MFs represent how the firing strength gets propagated and used in a fuzzy implication statement.)

Overall output MF aggregates all the qualified consequent MFs to obtain an overall MF.

Chapter 3: Fuzzy Inference Systems

neuro fuzzy system ~ fuzzy system (the main difference is related to parameter training)

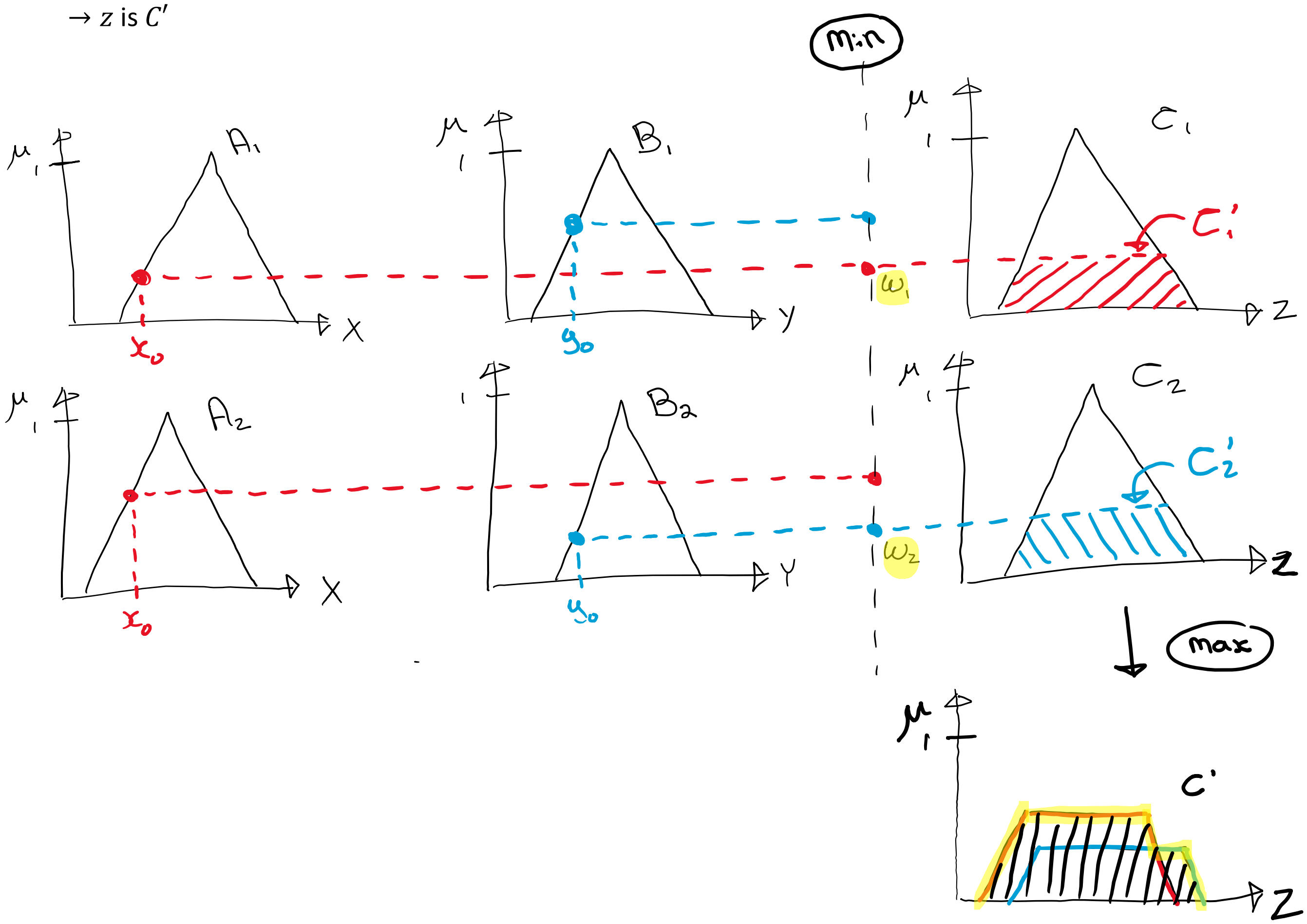
Inputs: fuzzy inputs, crisp inputs (fuzzy singletons)

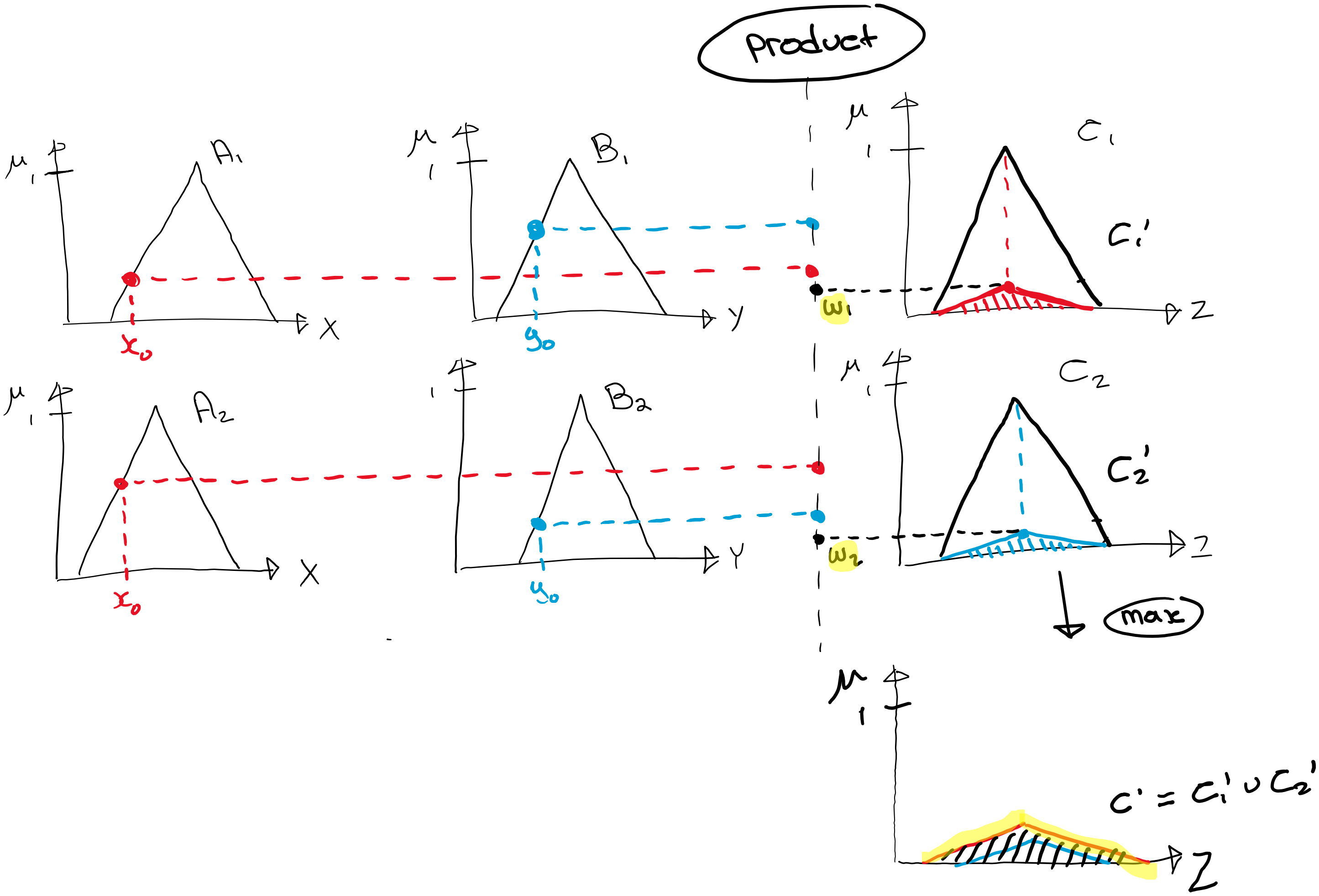
1) Mamdani Fuzzy Models

Rule: (R_1)
If $(x$ is $A_1)$ and $(y$ is $B_1)$
Then $(z$ is $C_1)$

Rule: (R_2)
If $(x$ is $A_2)$ and $(y$ is $B_2)$
Then $(z$ is $C_2)$

$x = x_0$
$y = y_0$
$z = ?$
$\rightarrow z$ is C'





Product
A1
B1
C1
C1'
x0
y0
w1
A2
B2
C2
C2'
w2
max
C' = C1' ∪ C2'

2) Defuzzification

When we want to get back a number, instead of a membership function.

- Centroid of the area (most commonly used, dividing line drawn across the centroid of the MF)
- Bisector of the area (commonly used, dividing line such that area on LHS = RHS)
- Smallest of the maximum
- Largest of the maximum
- Mean of the maximum