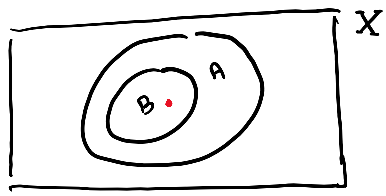
2.5 Fuzzy Operations

3) Set Inclusion

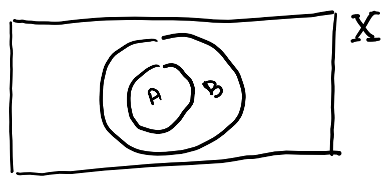
$B \subset A$



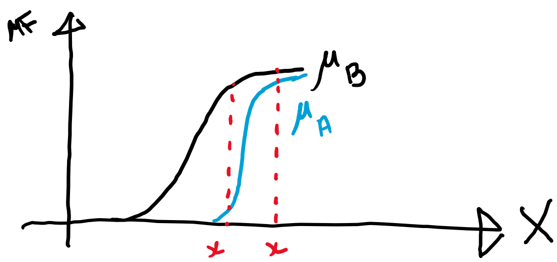
Fuzzy sets A, B

If A is a subset of fuzzy set B,

$A \subset B$

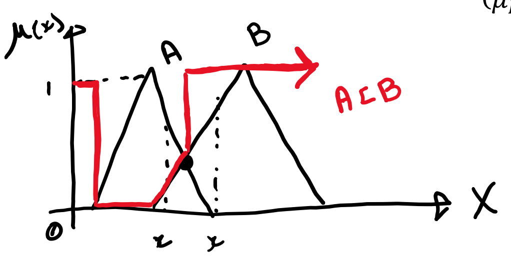


$$\mu_{A \sqsubset B}(x) = \begin{cases} 1 & ;\ \text{if } \mu_A(x) \leq \mu_B(x) \\ \mu_A(x) \mathrm{T} \mu_B(x) & ;\ \text{Otherwise} \end{cases}$$



min ~ T-norm

$$\mu_{A \sqsubset B}(x) = \begin{cases} 1 & ;\ \text{if } \mu_A(x) \leq \mu_B(x) \\ \mu_B(x) & ;\ \text{Otherwise} \end{cases}$$



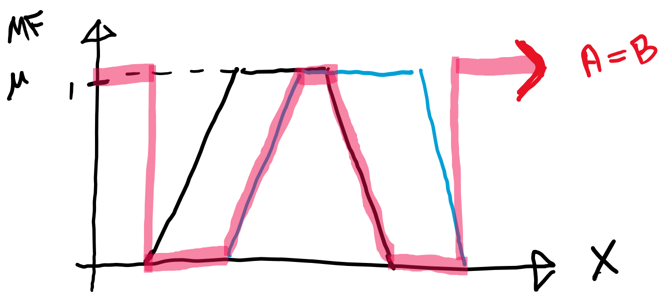
4) Set Equality ($A = B$)

$\mu_A(x) = \mu_B(x)$

$$\mu_{A=B}(x) = \begin{cases} 1 & ; \quad \text{if } \mu_A(x) = \mu_B(x) \\ \mu_A(x) \mathrm{T} \mu_B(x) & ; \quad \text{otherwise} \end{cases}$$

min ~ T-norm

$$\mu_{A=B}(x) = \begin{cases} 1 & ; \quad \text{if } \mu_A(x) = \mu_B(x) \\ \min\big(\mu_A(x), \mu_B(x)\big) & ; \quad \text{if } \mu_A(x) \neq \mu_B(x) \end{cases}$$



2.6 Implication (IF – THEN)

$$A \to B$$

IF A THEN B

$A \sim X$

$B \sim Y$

$A \to B, X \text{ x } Y$

1) Method 1 (Mamdani implication)

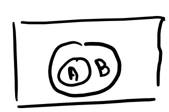
$$\mu_{A \to B}(x, y) = \min[\mu_A(x), \mu_B(y)]$$
$$x \in X, y \in Y$$

2) Method 2 (Larson implication)

$$\mu_{A \to B}(x, y) = \mu_A(x) \cdot \mu_B(y)$$

3) Method 3 (Bounded sum implication)

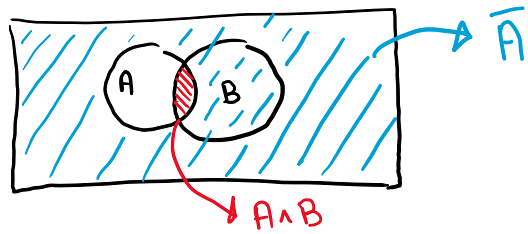
$$\mu_{A \to B}(x, y) = \min[1, \{1 - \mu_A(x) + \mu_B(y)\}]$$



Proof:

A	B	$A \to B$
T	T	T
T	F	F
F	T	T
F	F	T

$$A \rightarrow B = (A \wedge B) \vee \bar{A}$$



$$A \rightarrow B = (A \vee \bar{A}) \wedge (B \vee \bar{A})$$
$$= X \wedge (B \vee \bar{A})$$
$$= B \vee \bar{A}$$
$$= 1 - (B \vee A)$$

4) Method 4 (Zadeh implication)

$$\mu_{A \rightarrow B}(x, y) = \max[\min\{\mu_A(x), \mu_B(y)\},\ 1 - \mu_A(x)]$$

$\mu_{\bar{A}}(x)$

5) Method 5 (Dienes-Rescher implication)

$$\mu_{A \rightarrow B}(x, y) = \max[1 - \mu_A(x), \mu_B(y)]$$

$$\text{IF } \mu_A(x) = 0.6$$
$$\text{IF } \mu_B(x) = 0.5$$

Method 1 (Mamdani):

$= \ \min(0.6, 0.5)$

$= 0.5$

Method 2 (Larson):

$= \text{product}(\mu_A(x), \mu_B(x))$

$= 0.6 * 0.5$

$= 0.3$

Method 3:

$= \min[1, \{1 - 0.6 + 0.5\}]$

$= \min\ [1, 0.9]$

$= 0.9$

Method 4:

$= \max[\min\{0.6, 0.5\}, 1 - 0.6]$

$= \max[0.5, 0.4]$

$= 0.5$

Method 5:

$= \max[1 - 0.6, 0.5]$

$= \max[0.4, 0.5]$

$= 0.5$

Example 2-6 (Problem 2.16)

Example 2.16

Consider the membership functions of fuzzy sets A and B as shown in Figure 2.10, and expressed below:

$$\mu_A(x) = \max\left\{0, \frac{10x-3}{2}\right\} \quad 0.3 \le x \le 0.5$$
$$= \max\left\{0, \frac{7-10x}{2}\right\} \quad 0.5 < x \le 0.7$$
$$= 0 \quad \text{otherwise}$$

$$\mu_B(y) = \max\left\{0, \frac{10y-3}{2}\right\} \quad 0.3 \le y \le 0.5$$
$$= \max\left\{0, \frac{7-10y}{2}\right\} \quad 0.5 < y \le 0.7$$
$$= 0 \quad \text{otherwise}$$

The resulting expressions for the combined membership functions, which represent the five implication relations, are given in (a)–(e) below, and sketched in Figure 2.11.

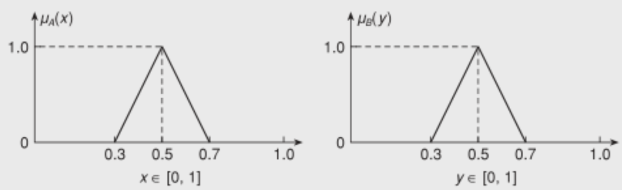


Figure 2.10: Membership functions of fuzzy sets A and B

Solution

(b) Mamdani implication (min operation)

$$\mu_{A\to B}(x,y) = \begin{cases} \min\left[\frac{10x-3}{2}, \frac{10y-3}{2}\right] & \text{if } 0.3 \le x \le 0.5 \text{ and } 0.3 \le y \le 0.5 \\ \min\left[\frac{10x-3}{2}, \frac{7-10y}{2}\right] & \text{if } 0.3 \le x \le 0.5 \text{ and } 0.5 < y \le 0.7 \\ \min\left[\frac{7-10x}{2}, \frac{10y-3}{2}\right] & \text{if } 0.5 < x \le 0.7 \text{ and } 0.3 \le y \le 0.5 \\ \min\left[\frac{7-10x}{2}, \frac{7-10y}{2}\right] & \text{if } 0.5 < x \le 0.7 \text{ and } 0.5 < y \le 0.7 \\ 0 & \text{otherwise} \end{cases}$$

(a) Larsen implication (product or dot operation)

$$\mu_{A\to B}(x,y) = \begin{cases} \frac{(10x-3)(10y-3)}{4} & \text{if } 0.3 \le x \le 0.5 \text{ and } 0.3 \le y \le 0.5 \\ \frac{(10x-3)(7-10y)}{4} & \text{if } 0.3 \le x \le 0.5 \text{ and } 0.5 < y \le 0.7 \\ \frac{(7-10x)(10y-3)}{4} & \text{if } 0.5 < x \le 0.7 \text{ and } 0.3 \le y \le 0.5 \\ \frac{(7-10x)(7-10y)}{4} & \text{if } 0.5 < x \le 0.7 \text{ and } 0.5 < y \le 0.7 \\ 0 & \text{otherwise} \end{cases}$$

2.7 Extension Principle and Fuzzy Relations

$f \sim$ from X to Y

↳A ↳B

$$A = \frac{\mu_A(x_1)}{x_1} + \frac{\mu_A(x_2)}{x_2} + \cdots + \frac{\mu_A(x_n)}{x_n}$$

For fuzzy sets A and B

$$B = f(A)$$

$$y = f(x)$$

$$= \frac{\mu_A(x_1)}{y_1} + \frac{\mu_A(x_2)}{y_2} + \cdots + \frac{\mu(x_n)}{y_n}$$

Example:

$$A = \frac{0.1}{-2} + \frac{0.4}{-1} + \frac{0.8}{0} + \frac{0.9}{1} + \frac{0.3}{2}$$

→ MF grade

→ x

$$y = f(x) = x^2 - 3$$

$A \sim x \in X$
$B \sim y \in Y$

$$B = \frac{0.1}{1} + \frac{0.4}{-2} + \frac{0.8}{-3} + \frac{0.9}{-2} + \frac{0.3}{1}$$

Many to one mapping $\sim$ max

$$B = \frac{0.1 \vee 0.3}{1} + \frac{0.4 \vee 0.9}{-2} + \frac{0.8}{-3}$$

$$= \frac{0.7}{1} + \frac{0.9}{-2} + \frac{0.8}{-3}$$

Given fuzzy sets (X, Y)
Where: $x \in X, y \in Y$

$$\mu(x),\ \ \mu(y),\ \ 0\sim 1 \text{ (binary relations)}$$

Binary fuzzy sets
Let X and Y be two universes of discourse.

$$R = \{(x,y), \mu_R(x,y)|_{XxY}\}$$

$\mu_R(x,y) \sim$ 2D membership function

$R =$ "y is greater than x"

$$\mu_R(x,y) = \begin{cases} \dfrac{y-x}{x+y-2} & ; if\ y > x \\ 0 & ; \text{if } y \le x \end{cases}$$

- $X = \{3, 4, 5\}$
- $Y = \{3, 4, 5, 6, 7\}$

$$R = \begin{array}{c} \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{ccccc} 3 & 4 & 5 & 6 & 7 \end{array} \\ \begin{bmatrix} 0 & 0.111 & 0.200 & 0.273 & 0.353 \\ 0 & 0 & 0.091 & 0.167 & 0.231 \\ 0 & 0 & 0 & 0.077 & 0.143 \end{bmatrix} \end{array}$$

1) Max-Min Composition

$R_1 \sim$ fuzzy relation on X x Y
$R_2 \sim$ fuzzy relation on Y x Z
R_1 and $R_2 \sim$ fuzzy set X and Z

Max-Min Composition:

$$\mu_{R_1 \circ R_2}(x,z)$$
$$= \max\min[\mu_{R_1}(x,y), \mu_{R_2}(y,z)]$$
$$= V_y\left[\mu_{R_1}(x,y) \land \mu_{R_2}(y,z)\right]$$

Where:
$\lor \sim$ max (or)
$\land \sim$ min (and)

- Properties:

$$R: X \text{ x } Y$$
$$S: Y \text{ x } Z$$
$$T: Z \text{ x } W$$

1) Associativity

$$R \circ (S \circ T) = (R \circ S) \circ T$$

2) Distributivity

$$R \circ (S \cup T) = (R \circ S) \cup (R \circ T)$$

3) Weak distributivity over intersection

$$R \circ (S \sqcap T) \sqsubseteq (R \circ S) \sqcap (R \circ T)$$

4) Monotonicity

$$S \sqsubseteq T \rightarrow R \circ S \sqsubseteq R \circ T$$

$T - \text{norm} \sim \text{min product}$
$S - \text{norm} \sim \text{max product}$

2) Max-Product Composition

$R_1 \sim X\text{x}Y$
$R_2 \sim Y\text{x}Z$
$\mu_{R_1 \circ R_2}(x, z) = \max_y \left[\mu_{R_1}(x, y) * \mu_{R_2}(y, z)\right]$

Example 2-7

Let:

$\mathcal{R}_1 =$ "x is relevant to y"
$\mathcal{R}_2 =$ "y is relevant to z"

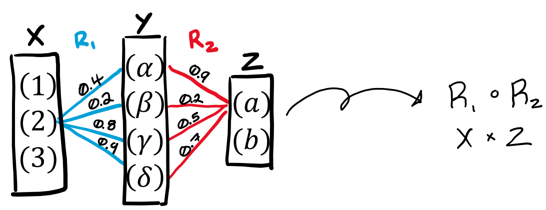
be two fuzzy relationships defined on X x Y and Y x Z, respectively, where $X = \{1, 2, 3\}$, $Y = \{\alpha, \beta, \gamma, \delta\}$, and $Z = \{a, b\}$. Assume that $\mathcal{R}_1$ and $\mathcal{R}_2$ can be expressed as the following matrices:

$$\mathcal{R}_1 = \begin{bmatrix} 0.1 & 0.3 & 0.5 & 0.7 \\ 0.4 & 0.2 & 0.8 & 0.9 \\ 0.6 & 0.8 & 0.3 & 0.2 \end{bmatrix}$$

$$\mathcal{R}_2 = \begin{bmatrix} 0.9 & 0{,}1 \\ 0.2 & 0.3 \\ 0.5 & 0.6 \\ 0.7 & 0.2 \end{bmatrix}$$

Now, we want to find $\mathcal{R}_1 \circ \mathcal{R}_2$ which can be interpreted as a derived fuzzy relation "x is relevant to z" based on $\mathcal{R}_1$ and $\mathcal{R}_2$. For simplicity, suppose that we are only interested in the degree of relevance between $2 (\in X)$ and $a (\in Z)$. If we adopt max min composition, then:

Solution



$X = \{1, 2, 3\}$
$Y = \{\alpha, \beta, \nu, \delta\}$
$Z = \{a, b\}$

1) Max-min composition operator

$$\mu_{R_1 \circ R_2}(x, z) \rightarrow \mu_{R_1 \circ R_2}(2, a)$$
$$= \max_y \min[\mu_{R_1}(x, y),\ \ \mu_{R_2}(y, z)]$$
$$= \max_y [0.4 \wedge 0.9,\ \ 0.2 \wedge 0.2,\ \ 0.8 \wedge 0.5,\ \ 0.9 \wedge 0.7\,)]$$
$$= \max_y [0.4,\ \ 0.2,\ \ 0.5,\ \ 0.7\,)]$$
$$= 0.7$$

2) Max-product composition operator

$$\mu_{R_1 \circ R_2}(x, z) \rightarrow \mu_{R_1 \circ R_2}(2, a)$$
$$= \max[0.4 * 0.9,\ \ 0.2 * 0.2,\ \ 0.8 * 0.5,\ \ 0.9 * 0.7]$$
$$= \max[0.36,\ \ 0.04,\ \ 0.14,\ \ 0.63]$$
$$= 0.63$$

2.7 Fuzzy IF-THEN Rules

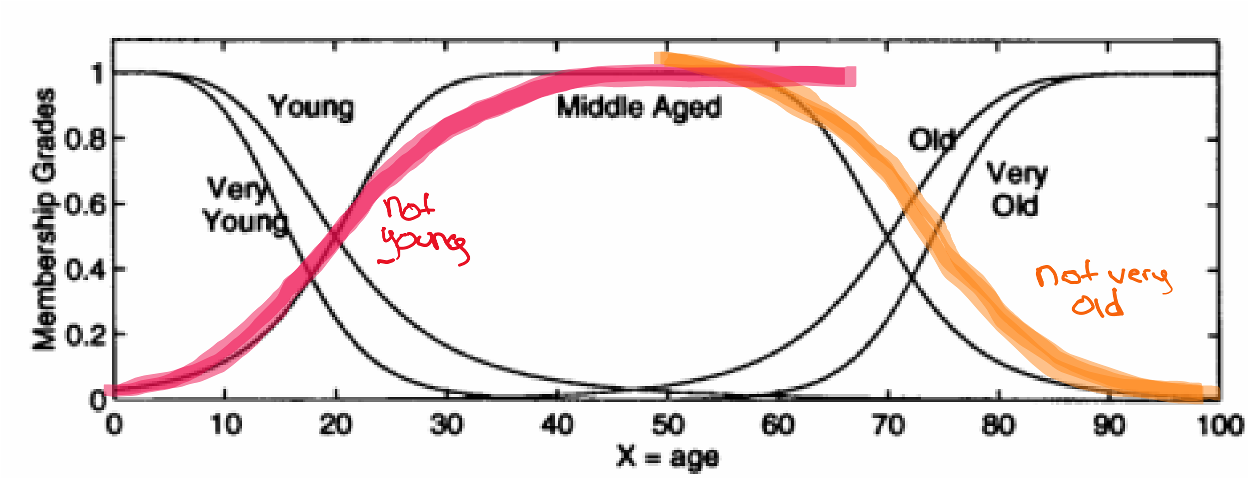
1) Linguistic Variables

$$\{\text{fuzzy set, universe, syntactic rule, semantic rule}\}$$

$$age \sim \text{linguistic variable}$$
$$set\ T\ (age)$$

$$T\ (age) = \left\{ \begin{matrix} young; \\ not\ young; \\ very\ young; \\ middle\ aged; \\ very\ old; \\ not\ very\ old; \\ more\ or\ less\ old \end{matrix} \right\}$$

$$X = [0, 100]$$



- **Primary terms**: young, middle aged, old
- **Negation**: not
- **Hedges**: very, quite, more or less
- **Connectives**: and, or, either, neither
- **Concentration** and **dilation**

Example:
$A \sim$ linguistic term
$MF: \mu_A(x)$
$A^k \sim$ modified version of the linguistic value
$A^k \sim \int \mu_A^k(x)/x$

- **Concentration**
  $CON(A) = A^2$
- **Dilation**
  $DIL(A) = \sqrt{A}$

- **Not**

  $$NOT(A) = \neg A = \frac{\int [1 - \mu_A(x)]}{x}$$

  Consider two terms A, B:

  $$A \text{ AND } B = A \cap B = \frac{\int \mu_A(x) \wedge \mu_B(x)}{x}$$

  $$A \text{ OR } B = A \cup B = \frac{\int \mu_A(x) \vee \mu_B(x)}{x}$$

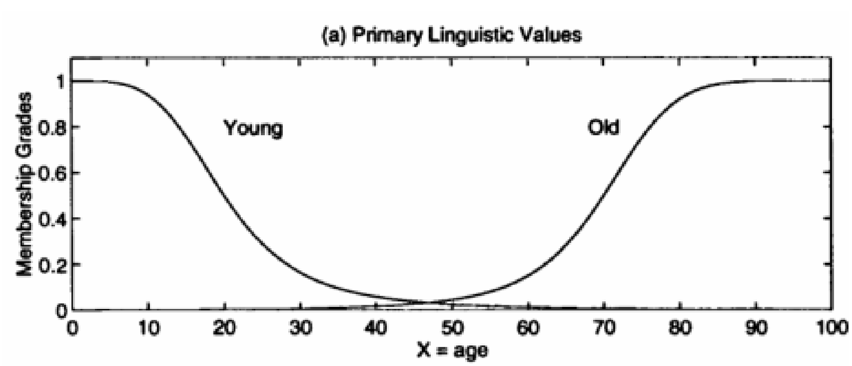
  Example:

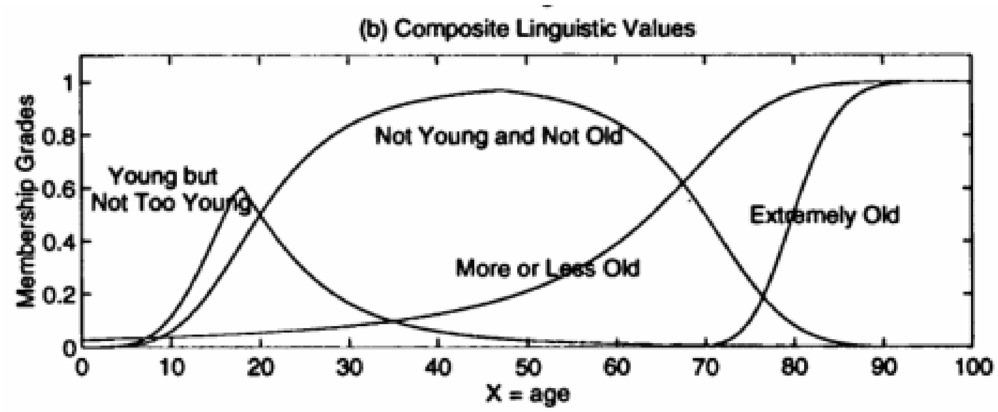
  $\mathrm{T}(age)$

  $$\mu_{young}(x) = \mathrm{bell}(x, 20, 2, 0)$$
  $$= \frac{1}{1 + \left(\frac{x}{20}\right)^4}$$

  $$\mu_{old}(x) = \mathrm{bell}(x, 30, 3, 100)$$
  $$= \frac{1}{1 + \left(\frac{x - 100}{30}\right)^6}$$

  For $x = [0, 100]$:





- **More or less**

  $DIL(old) = old^{0.5}$

  $$= \frac{\int \sqrt{\frac{1}{1+\left(\frac{x-100}{30}\right)^6}}}{x}$$

- **Not young AND not old**

  $= (\neg\ \text{young}) \sqcap (\neg\ \text{old})$

  $$= \frac{\int \left[1-\frac{1}{1+\left(\frac{x}{20}\right)^4}\right] \wedge \left[1-\frac{1}{1+\left(\frac{x-100}{30}\right)^6}\right]}{x}$$

- **Young but not very (too) young**

  $= \text{young} \sqcap (\neg\ \text{young}^2)$

  $$= \frac{\int \left[\frac{1}{1+\left(\frac{x}{20}\right)^4}\right] \wedge \left[1-\left(\frac{1}{1+\left(\frac{x}{20}\right)^4}\right)^2\right]}{x}$$

- **Extremely old**

  $= \text{con}\left(\text{con}\left(\text{con}(\text{old})\right)\right)$

  $((old^2)^2)^2 = old^8$

  $$= \frac{\int \left(\frac{1}{1+\left(\frac{x-100}{30}\right)^6}\right)^8}{x}$$

2) Orthogonality

$$T = \{t_1, t_2, \dots, t_n\}$$

Universe X

$$\mu_{t_1}(x) + \mu_{t_2}(x) + \cdots + \mu_{t_n}(x) = 1$$
$$\sim \text{orthogonal}$$