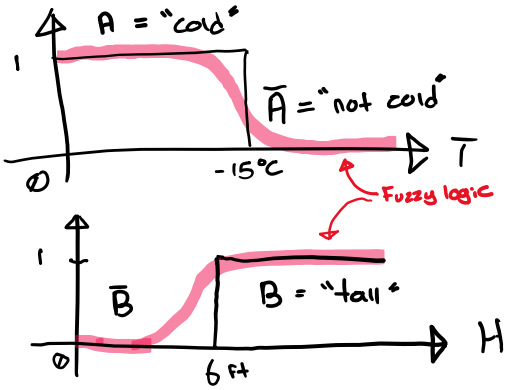
2.2 Fuzzy Sets (Cont'd)

Crisp sets: $A\begin{cases}1\\0\end{cases}$

Concepts and thoughts are abstract and imprecise $\neq$ random.

Fuzzy logic → approximate knowledge



Membership function (MF) grade.

Consider:
Height:
$H = 6.001\ ft$
"Tall" MF grade: 99.9%

$H' = 5.999\ ft$
"not Tall" MF grade: 0.01%

Fuzzy set:
$A = \{x, \mu_A(x)\}$

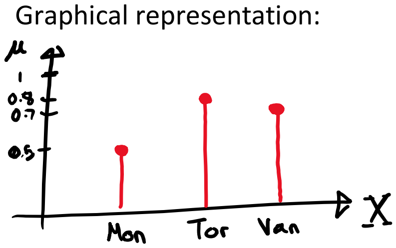
$x =$ variable $\in X$
$\mu_A = MF$
$X =$ universe of discourse

1) Fuzzy sets with a discrete non-ordered universe

$X = \{Montreal, Toronto, Vancouver\}$

$C = \text{"desired city to live in"}$
$C = \{(Mon, 0.5), (Tor, 0.8), (Van, 0.7)\}$

Graphical representation:



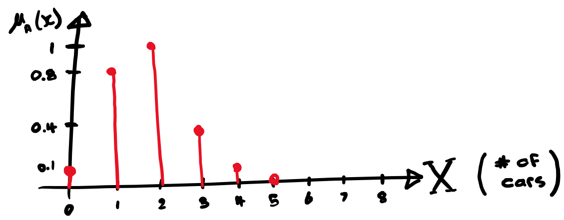
2) Fuzzy set with a discrete ordered universe

$X = \{0, 1, 2, 3, 4\}$

Fuzzy set A = "sensible number of cars in a family"

$A = \{(0, 0.1), (1, 0.8), (2, 1.0), (3, 0.4), (4, 0.1)\}$

Graphical representation:



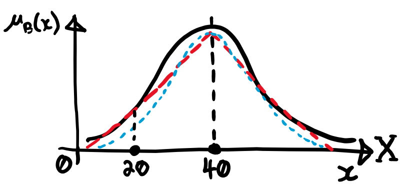
3) Fuzzy sets with a continuous space

$X =$ "ages" $(0 \sim 120)$

Fuzzy set B = "about 40 years old"

$B = \{(x, \mu_B(x)\}, x \in X$

Graphical representation:



$$\mu_B(x) = \frac{1}{1 + \left(\frac{x - 40}{10}\right)^4}$$

- Subjective (X, MF)
- Not random

4) Other fuzzy set representations

For a discrete, non-ordered universe:

$$A = \frac{\mu_A(x_1)}{x_1} + \frac{\mu_B(x_2)}{x_2} + \frac{\mu_C(x_3)}{x_3} + \cdots$$

e.g.

$$C = \frac{0.5}{Mon} + \frac{0.8}{Tor} + \frac{0.7}{Van}$$

For a discrete, ordered universe:

$$A = \frac{0.1}{0} + \frac{0.8}{1} + \frac{1.0}{2} + \frac{0.4}{3} + \frac{0.1}{4}$$

For a continuous space:

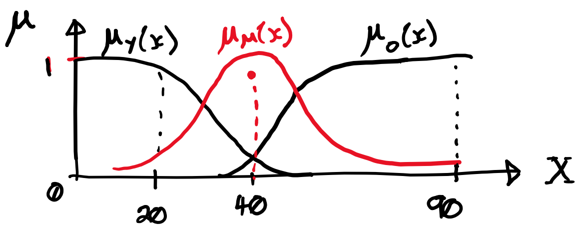
$$B = \frac{\mu_B(x)}{x}$$

$$B = \left[\frac{1}{1 + \left(\frac{x - 40}{10}\right)^4}\right] \Big/ x$$

If the universe space X is a continuous space, we can partition X into several fuzzy sets.

Consider:
$X =$ "age"

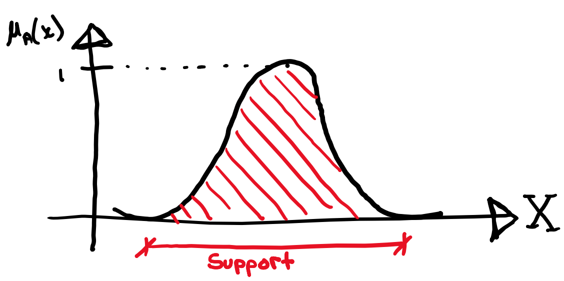
Partitions:
Y M O
"young", "middle aged", "old"
$\mu_Y(x),\ \mu_M(x),\ \mu_O(x),\ where\ x \in X$



2.3 Other Concepts of Fuzzy Sets

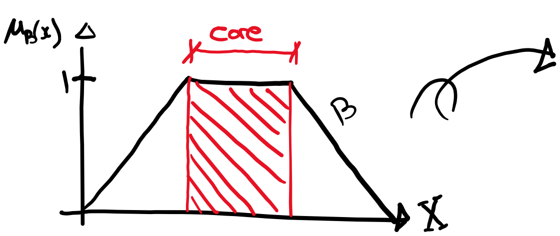
1) Support

$support(A) \rightarrow \{x | \mu_A(x) > 0\}$



2) Core

$core(B) \rightarrow \mu_B(x) = 1$

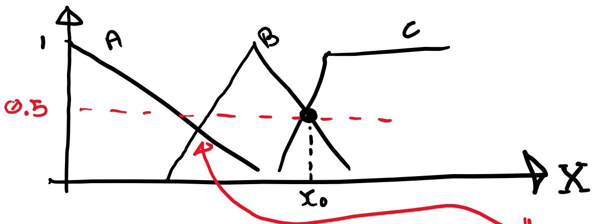


(For a triangular function, only 1 point would be in the core)

3) Normality

$normality(C) \rightarrow \max\{\mu_C(x)\} = 1$

4) Cross-over Points



this is a cross-over point, but doesn't have specified indication. If no specified indication, grade = 0.5

$\mu_B(x_0) = \mu_C(x_0) = 0.5$

x_0 = a cross-over point

5) Fuzzy singletons

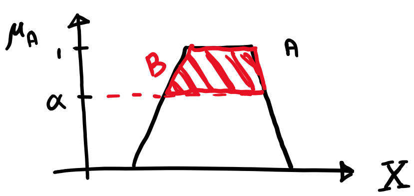
Basically, a fuzzy set in discrete form.

Diagnosis:

Class 1, Class 2, …

6) $\alpha - cut$

$B = \{x, \mu_B(x)|_{\mu_A(x) \geq \alpha}\}$

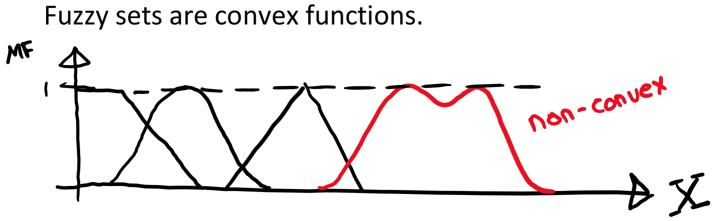


- Strong $\alpha - cut$

  $B = \{x, \mu_B(x) | \mu_A(x) > \alpha\}$

7) Convexity

Fuzzy sets are convex functions.



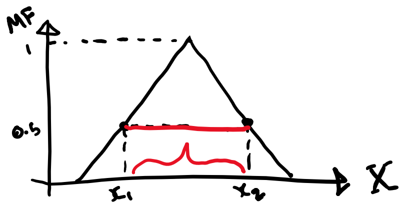
8) Fuzzy Numbers

A fuzzy number is a fuzzy set

→ normality

→ convexity (monotonically increasing, followed by monotonically decreasing, or constant)

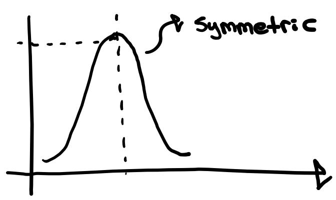
9) Bandwith



$x_2 - x_1 =$ bandwith

$\mu_A(x_1) = \mu_A(x_2) = 0.5$

10) Symmetry



compared to

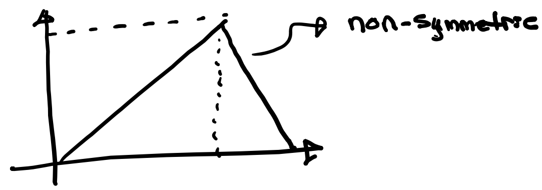


Table 2.1: Some properties of fuzzy sets

Property name	Relation
Commutativity	$A \cap B = B \cap A$ $A \cup B = B \cup A$
Associativity	$(A \cap B) \cap C = A \cap (B \cap C)$ $(A \cup B) \cup C = A \cup (B \cup C)$
Distributivity	$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
Absorption	$A \cup (A \cap B) = A$ $A \cap (A \cup B) = A$
Idempotency (Idem = same; potent = power) (Similar to unity or identity operation)	$A \cup A = A$ $A \cap A = A$
Exclusion: Law of excluded middle Law of contradiction	 $A \cup A' \subset X$ $A \cap A' \supset \phi$
DeMorgan's Laws	$(A \cap B)' = A' \cup B'$ $(A \cup B)' = A' \cap B'$
Boundary conditions	$A \cup X = X$ $A \cap X = A$ $A \cup \phi = A$ $A \cap \phi = \phi$

2.4 Set Operations

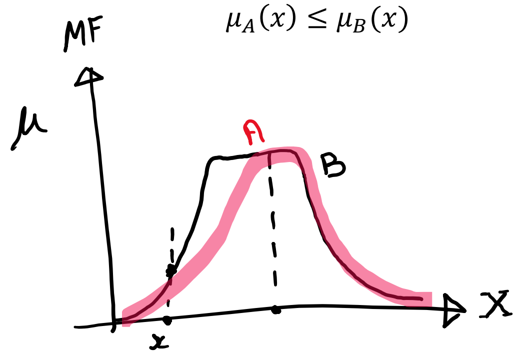
1) Subset

Consider fuzzy set A & B

If A is a subset of B:

$$A \subset B$$

$$\mu_A(x) \leq \mu_B(x)$$

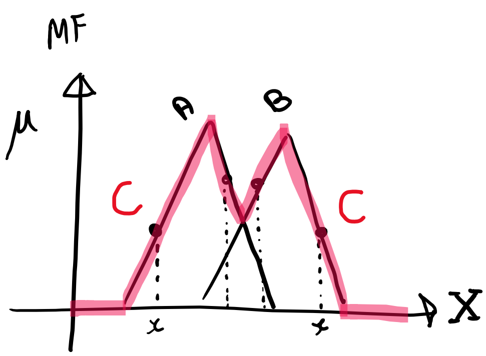


2) Union (Disjunction) - OR

Given A & B

$$C = A \cup B$$

$$\mu_c(x) = \max\{\, \mu_A(x),\ \ \mu_B(x) \,\}$$

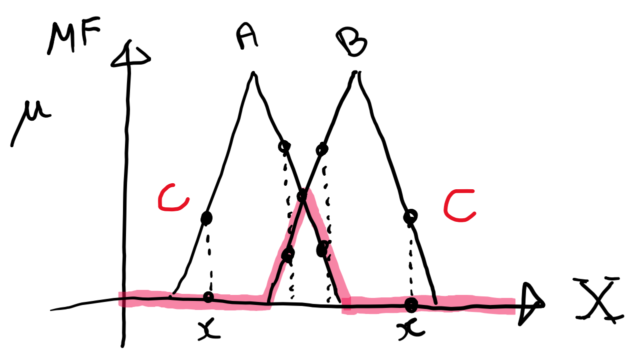


3) Intersection (Conjunction) – AND

Given A & B

$$C = A \cap B$$

$$\mu_C(x) = \min\{\, \mu_A(x),\ \ \mu_B(x) \,\}$$

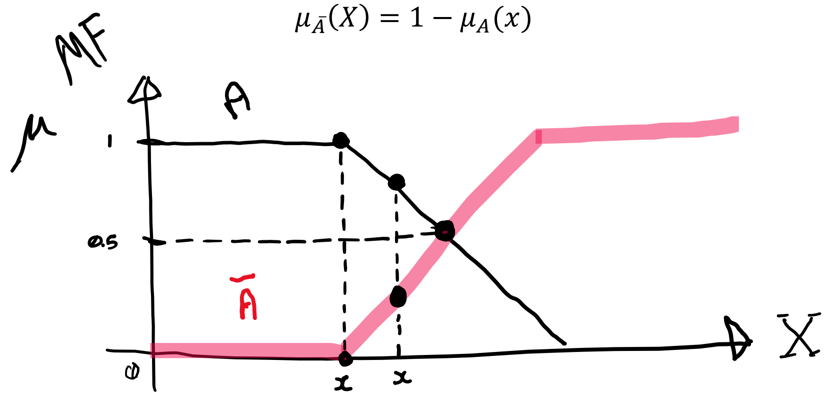


4) Complement (Negation) – NOT

Given A & B

$$not\ A \text{ or } \bar{A} \text{ (fuzzy set)}$$

$$\mu_{\bar{A}}(X) = 1 - \mu_A(x)$$



5) Cartesian Product / Co-product

$A \sim$ fuzzy set in X
$B \sim$ fuzzy set in Y } "different universes / domains"

Cartesian product A x B
is in X x Y

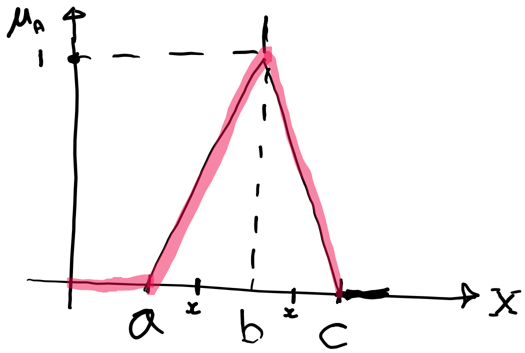
$$\mu_{A \times B}(x, y) = \min\{\ \mu_A(x),\ \ \mu_B(y)\ \}$$

Cartesian co-product $A + B$
is in $X + Y$

$$\mu_{A+B}(x, y) = \max\{\ \mu_A(x),\ \ \mu_B(y)\ \}$$

2.4 Membership Functions (MF)

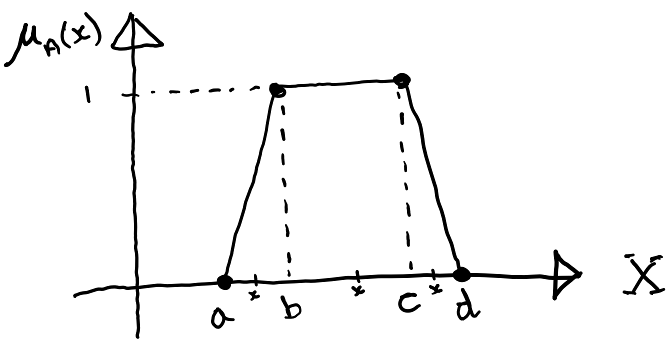
1) Triangular Membership Functions



$$\mu_A(x;a,b,c)=\begin{cases}0 & when\ x\le a\\ \left(\frac{x-a}{b-a}\right) & when\ a<x<b\\ \left(\frac{c-x}{c-b}\right) & when\ b<x<c\\ 0 & when\ x>c\end{cases}$$

In MATLAB:
$trimf(x,[a\ b\ c])$

2) Trapezoidal Membership Functions



$$\mu_A(x;a,b,c,d)=\begin{cases}0 & when\ x<a\\ \left(\frac{x-a}{b-a}\right) & when\ a\le x<b\\ 1 & when\ b\le x<c\\ \left(\frac{d-x}{d-c}\right) & when\ c\le x\le d\\ 0 & when\ x>d\end{cases}$$

In MATLAB:
$trapmf(x,[a\ b\ c\ d])$

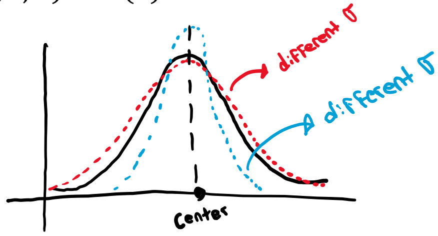
NOTE: Triangular, and trapezoidal membership functions are not continuous, which means the derivatives functions do not exist (equal to zero).

The following membership functions are continuous:

3) Gaussian Membership Functions

$\mu_A = gauss(x; c, \sigma) = G(x) = e^{-\frac{1}{2}\left(\frac{x-c}{\sigma}\right)^2}$

$c =$ center
$\sigma =$ spread



In MATLAB:
$gaussmf(x; [c, \sigma])$

$$\mu'(x) = DG(x) = e^{-\frac{1}{2}\left(\frac{x-c}{\sigma}\right)^2} \cdot \left[-\frac{1}{2} \cdot 2\left(\frac{x-c}{\sigma}\right) \cdot \frac{1}{\sigma}\right]$$

4) Generalized Bell Membership Functions

$$\mu_A = bell(x; a, b, c) = \frac{1}{1 + \left|\frac{x-c}{a}\right|^{2b}}$$

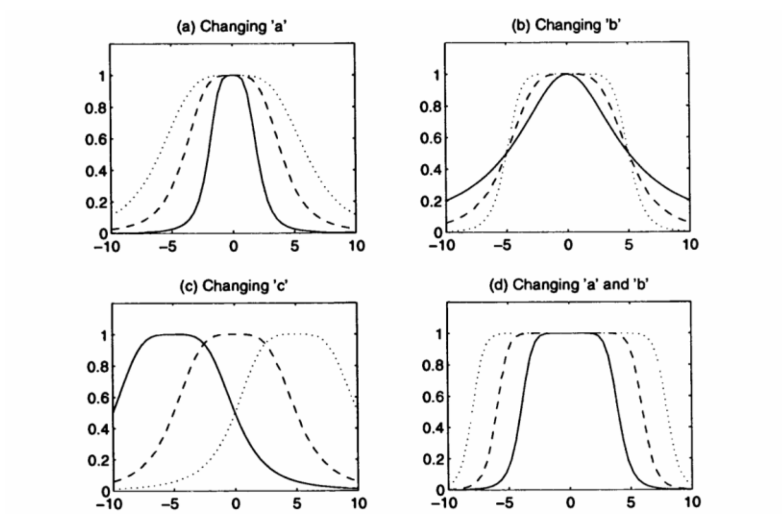


Figure 2.8. *The effects of changing parameters in bell MFs: (a) changing parameter a; (b) changing parameter b; (c) changing parameter c; (d) changing a and b simultaneously but keeping their ratio constant.* (MATLAB file: `allbells.m`)

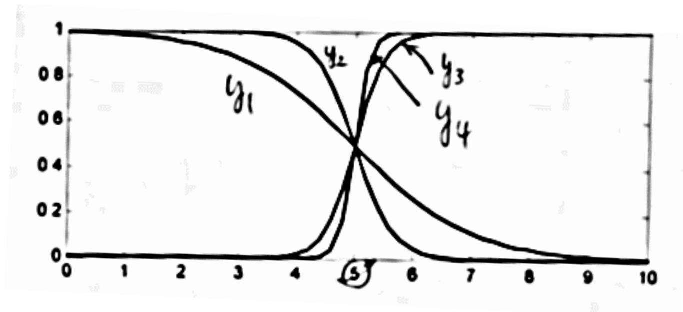
In MATLAB:
$gbellmf(x; [a, b, c])$

5) Sigmoid Membership Functions

$$\mu(x) = sig(x; a, c) = \frac{1}{1 + exp[-a(x-c)]}$$

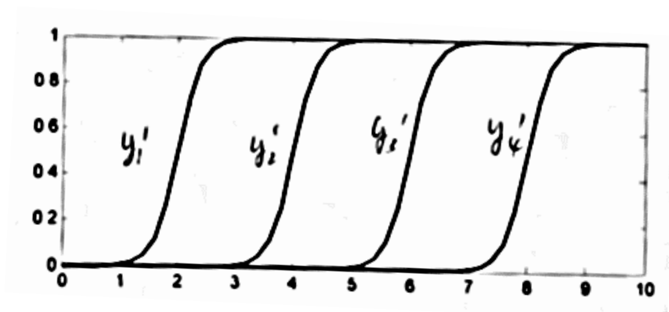
<u>In MATLAB:</u>

$sigmf(x; [a, c])$



If $a > 0$, $sigmf$ opens to the right

If $a < 0$, $sigmf$ opens to the left



$\mu'(x) = DS(x) = -1\left[1 + \mathrm{e}^{-a(x-c)}\right]^{-2} e^{-a(x-c)} \cdot (-a)$

$0 \le x < \infty$

2.5 Fuzzy Operations

MFs [0, 1]

$A \sim [0, 1]$
$B \sim [0, 1]$
$C \sim A \cup B$
$D \sim A \cap B$

$$[0, 1] \text{ x } [0, 1] \rightarrow [0, 1]$$

1) Triangular Norm (T-Norm) – Generalized Intersection

$a = \mu_A(x)$
$b = \mu_B(x)$

$T = (a, b), \quad aTb$

Properties in table below.

2) T-Conorm (S-Norm)

Properties in table below.

Table 2.2: Some properties of a triangular norm

Item description	T-norm (triangular norm)	S-norm (T-conorm)
Function	T: $[0, 1] \times [0, 1] \rightarrow [0, 1]$	Same
Nondecreasing in each argument	If $b \geq a$, $d \geq c$ then $bTd \geq aTc$	Same
Commutative	$aTb = bTa$	Same
Associative	$(aTb)Tc = aT(bTc)$	Same
Boundary conditions	$aT1 = a$ $aT0 = 0$ with $a, b, c, d \in [0, 1]$	$aS0 = a$ $aS1 = 1$
Examples	Conventional: $\min(a, b)$ Product: ab Bounded max (bold intersection): $\max[0, a + b - 1]$ General: $1 - \min[1, ((1-a)^p + (1-b)^p)^{1/p}]$ $p \geq 1$ $\max[0, (\lambda + 1)(a + b - 1) - \lambda ab]$ $\lambda \geq -1$	Conventional: $\max(a, b)$ Set addition: $a + b - ab$ Bounded min (bold union): $\min[1, a + b]$ General: $\min(1, (a^p + b)^p)^{1/p}$ $p \geq 1$ $\min[1, a + b + \lambda ab]$ $\lambda \geq -1$
DeMorgan's Laws	$aSb = 1 - (1 - a)\ T(1 - b)$ $aTb = 1 - (1 - a)\ S(1 - b)$	

Example 2.3 (Similar to Example 2.13)

Use DeMorgan's law to determine the S-norm corresponding to max(x, y), and T-norm corresponding to min(x, y).

Solution 2.3

$$
\begin{aligned}
&xSy = 1 - (1 - x)T(1 - y) \\
&T \rightarrow \min \\
&= 1 - \min[(1 - x), (1 - y)] \\
&= \begin{cases} 1 - (1 - y) = y; & x < y \\ 1 - (1 - x) = x; & x \geq y \end{cases} \\
&xSy = \max(x, y)
\end{aligned}
$$

Example 2.4 (Similar to Example 2.14)

Prove that the min operator is the largest T-norm and the max operator is the smallest S-norm.

Solution 2.4

Nondecreasing, boundary conditions

$$
\begin{aligned}
&xTy \leq 1Ty = y \\
&xTy \leq xT1 = x \\
&xTy \leq \min(x, y)
\end{aligned}
$$