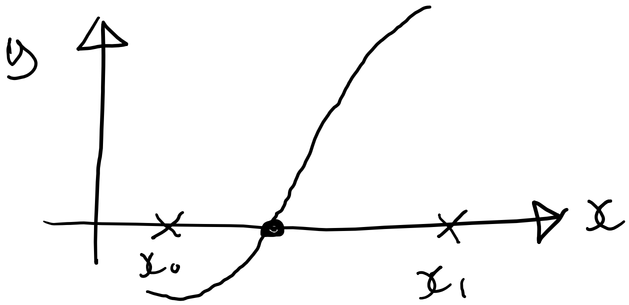
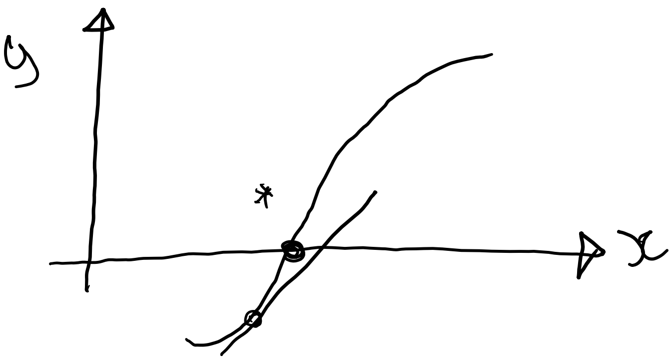
Part 3: Roots of equations

Bisection method:



Open method:

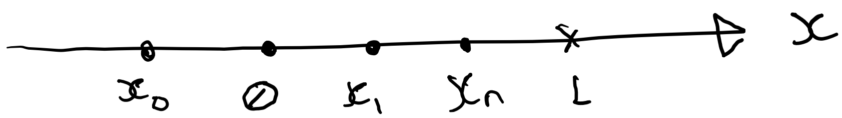


Convergence speed for iterative methods
(how do we measure the convergence speed of iterative methods?)

1. Order of convergence
2. Rate of convergence

$$\{X_n\}: x_0, x_1, x_2, \dots, x_n, \dots, \dots$$

↳ converges to L



$$|x_{n+1} - L|, \quad |x_n - L|$$

$$\lim_{n\to\infty} \frac{|x_{n+1} - L|}{|x_n - L|} = \mu \quad ; \quad 0 \le \mu \le 1$$

1st : $0 \le \mu \le 1$: the sequence $\{x_n\}$ is said to converge $Q - linearly$ to L
2nd : $\mu = 0 : Q - superlinearly\ to\ L$
3rd : $\mu = 1 : Q - sublinearly\ to\ L$

If the sequence converges $Q - sublinearly\ to\ L$, and

$$\lim_{n\to\infty} \frac{|x_{n+2} - x_{n+1}|}{|x_{n+1} - x_n|} = 1$$

Converges logarithmically to L.

Order of convergence:

$$\lim_{n\to\infty} \frac{|x_{n+1} - L|}{|x_n - L|^q} < M$$

positive constant

$q = 1$: linear convergence
$q = 2$: quadratic convergence
$q = 3$: cubic convergence
...

Example

1st sequence:

$$(x_n) = \left\{1, \frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \dots, \frac{1}{3^n}, \dots\right\}$$

$$x_n = \frac{1}{3^n} \quad ; \quad n = 0, 1, 2, \dots$$

$$x_n \to L = 0 \quad ; \quad n \to \infty$$

$$\lim_{n\to\infty} \frac{|x_{n+1} - L|}{|x_n - L|} = \frac{\left|\frac{1}{3^{n+1}} - 0\right|}{\left|\frac{1}{3^n} - 0\right|} = \frac{1}{3} < 1$$

$$\lim_{n\to\infty} \frac{|x_{n+1} - L|}{|x_n - L|^q} = \frac{1}{3} \quad ; \quad Q - linearly$$

2nd sequence:

$$(x_n) = \left\{\frac{1}{3}, \frac{1}{9}, \frac{1}{81}, \dots, \frac{1}{3^{2^n}}, \dots\right\}$$

$$x_n = \frac{1}{3^{\,2^n}} \quad ; \quad x_{n+1} = x_n^2$$

$$x_n \to L = 0 \quad ; \quad n \to \infty$$

$$\lim_{n\to\infty} \frac{|x_{n+1} - L|}{|x_n - L|} = \lim_{n\to\infty} \left|\frac{\frac{1}{3^{2^n+1}} - 0}{\frac{1}{3^{2^n}} - 0}\right|$$

$$\lim_{n\to\infty} \frac{1}{3^{2^n}} = 0 \quad ; \quad Q - superlinearly$$

3rd sequence:

$$(x_n) = \left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots, \frac{1}{n+1}, \dots\right\}$$

$$x_n = \frac{1}{n+1} \quad ; \quad n = 0, 1, 2, \dots$$

$$x_n \to L = 0 \quad ; \quad n \to \infty$$

$$\lim_{n\to\infty} \frac{|x_{n+1} - L|}{|x_n - L|} = \lim_{n\to\infty} \left|\frac{\frac{1}{n+2}}{\frac{1}{n+1}}\right| = 1 \quad ; \quad Q - sublinearly$$

$$\lim_{n\to\infty} \left|\frac{x_{n+2} - x_{n+1}}{x_{n+1} - x_n}\right| = \lim_{n\to\infty} \left|\frac{\frac{1}{n+3} - \frac{1}{n+2}}{\frac{1}{n+2} - \frac{1}{n+1}}\right| = 1 \quad ; \quad converges\ logarithmically$$

Functional iteration and orbit

If $f: \mathcal{R} \to R$,

$$
\begin{aligned}
&f^0(x) \overset{\text{def}}{=\!=} x \\
&f'(x) \overset{\text{def}}{=\!=} f(x) \\
&f^2(x) \overset{\text{def}}{=\!=} (f^\circ f)(x) = f\big(f(x)\big) \\
&f^3(x) \overset{\text{def}}{=\!=} (f^\circ f^2)(x) = f\big(f^2(x)\big) \\
&\dots \\
&f^n(x) \overset{\text{def}}{=\!=} (f^\circ f^{n-1})(x) = f(f^{n-1}(x))
\end{aligned}
$$

$f^n(x)$: the n −th iteration of $f(x)$, $n \geq 0$

Example:

1st:

$$f(x) = x + a$$

$$
\begin{aligned}
&f^2(x) = f\big(f(x)\big) = f(x+a) = (x+a) + a \\
&= x + 2a
\end{aligned}
$$

$$
\begin{aligned}
&f^3(x) = f\big(f^2(x)\big) = f(x+2a) = (x+2a) + a \\
&= x + 3a \\
&\dots \\
&\boxed{= f^n(x) = x + na} \quad ; \quad n \geq 1
\end{aligned}
$$

2nd:

$$f(x) = \frac{x}{1+bx}$$

$$
\begin{aligned}
&f^2(x) = f(f(x) = f\left(\frac{x}{1+bx}\right) = \frac{\frac{x}{1+bx}}{1 + b\frac{x}{1+bx}} \\
&= \frac{x}{1+2bx} \\
&f^n(x) = \frac{x}{1+nbx}
\end{aligned}
$$

3rd:

$$
\begin{aligned}
&f(x) = \frac{ax+b}{x+c} (b \neq ac) \\
&f^2(x) = \frac{(a^2+b)x + ab + bc}{(a+c)x + b^2}
\end{aligned}
$$

Let $x_0 \epsilon\, \mathcal{R}$, the orbit of x_0 under function $f(x)$ is defined as the sequence of points:

$$x_0, f(x_0), f^2(x_0), \dots, f^n(x_0), \dots$$

x_0: seed of the orbit

Example $f(x) = \cos x$, $x_0 = 0.5$

The orbit

$$\begin{aligned}
&\cos(0.5) = 0.8775825619\\
&\cos(\cos(0.5)) = 0.6390124942\\
&\cos^3(0.5) = \cos(0.6390\ldots) = 0.8206851007\\
&\vdots\\
&\cos^{56}(0.5) = 0.7390851332\\
&\cos^{57}(0.5) = 0.7390851332\\
&\vdots
\end{aligned}$$

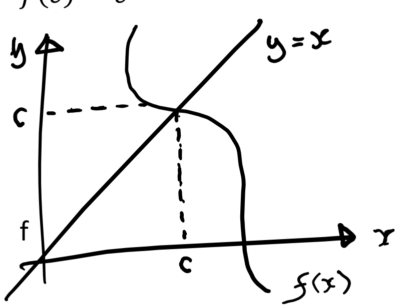
Example $f(x) = x^2 - 1, x_0 = 0.5$

$$\begin{aligned}
&x_0 = 0.5\\
&x_1 = f(x_0) = -0.75\\
&x_2 = f(x_1) = -0.4375\\
&x_3 = f(x_2) = -0.80859375\\
&\vdots\\
&x_{19} = f(x_{18}) = -1\\
&x_{20} = f(x_{19}) = 0\\
&x_{21} = f(x_{20}) = -1\\
&x_{22} = f(x_{21}) = 0\\
&\vdots\\
&does\ not\ converge
\end{aligned}$$

Fixed point

c is a fixed point of function $f(x)$:

$f(c) = c$



Example:

1[st]: $f(x) = x^3 - 0.9x^2 + 1.2x - 0.3$
$x = 1$ is a fixed point

$$f(1) = 1 - 0.9 + 1.2 - 0.3 = 1$$

2[nd]: $f(x) = x + 1$
no fixed point

A periodic point:
$f^n(x_0) = x_0$ for some n

Example: $f(x) = x^2 - 4x + 5$
$x_0 = 1$, $f(1) = 2$ not a fixed point
$f(2) = 1$

$\rightarrow f^2(1) = 1$, $n = 2$, $x_0 = 1$ is a fixed point of period 2.

Theorem: $x_0, f(x_0), f^2(x_0), \ldots, f^n(x_0), \ldots$

If $\lim_{n\to\infty} f^n(x_0) = a$

Then a is a fixed point of $f(x)$
$f(a) = a$

For example, $f(x) = \cos x$, $x_0 = 0.5$

$$f^n(x) \rightarrow 0.7390851332 = a$$

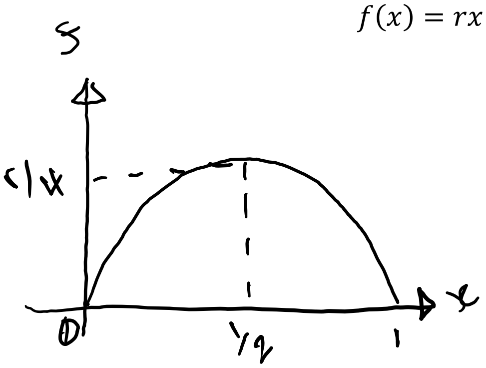
Therefore, from the theorem,

$$\cos a = a$$

(In other words, a is a fixed point of $\cos x$)

Logistic map:

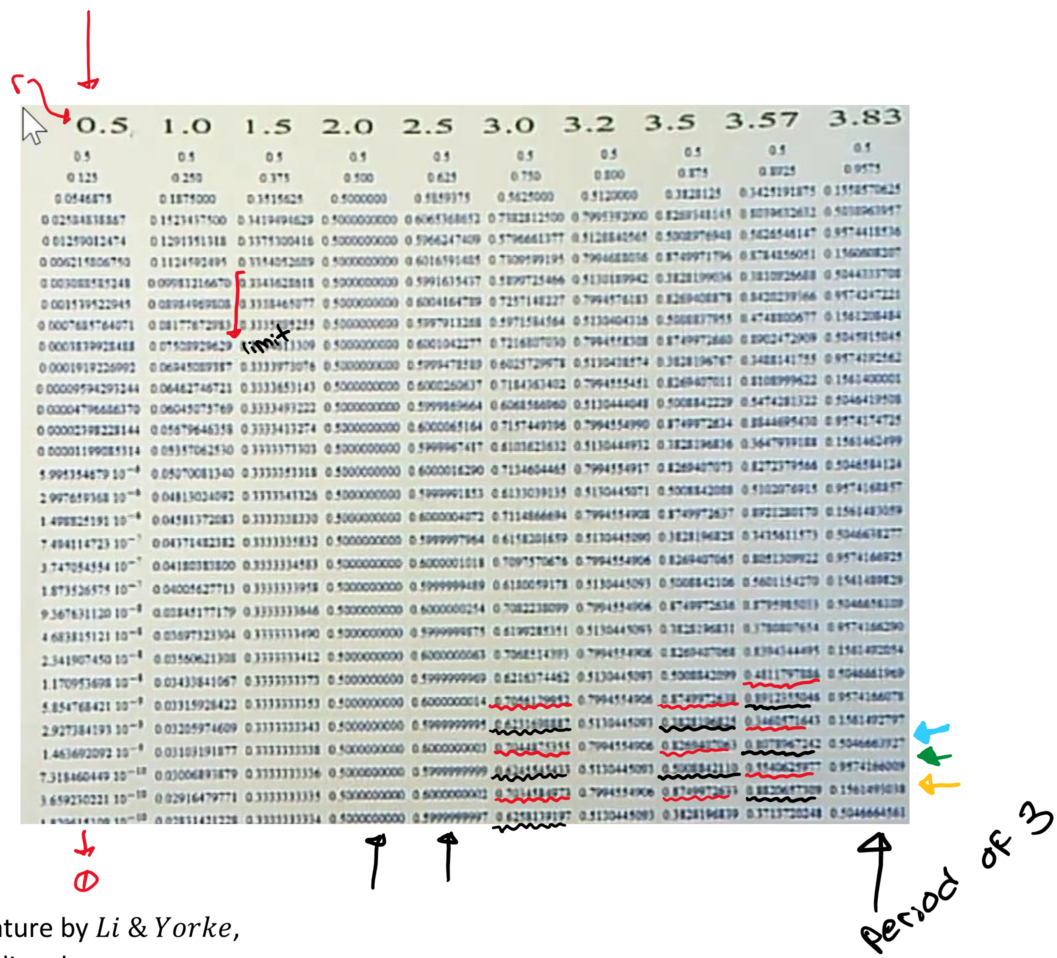
$$f(x) = rx(1-x), \quad 0 \leq x \leq 1$$
$$0 \leq r \leq 4$$



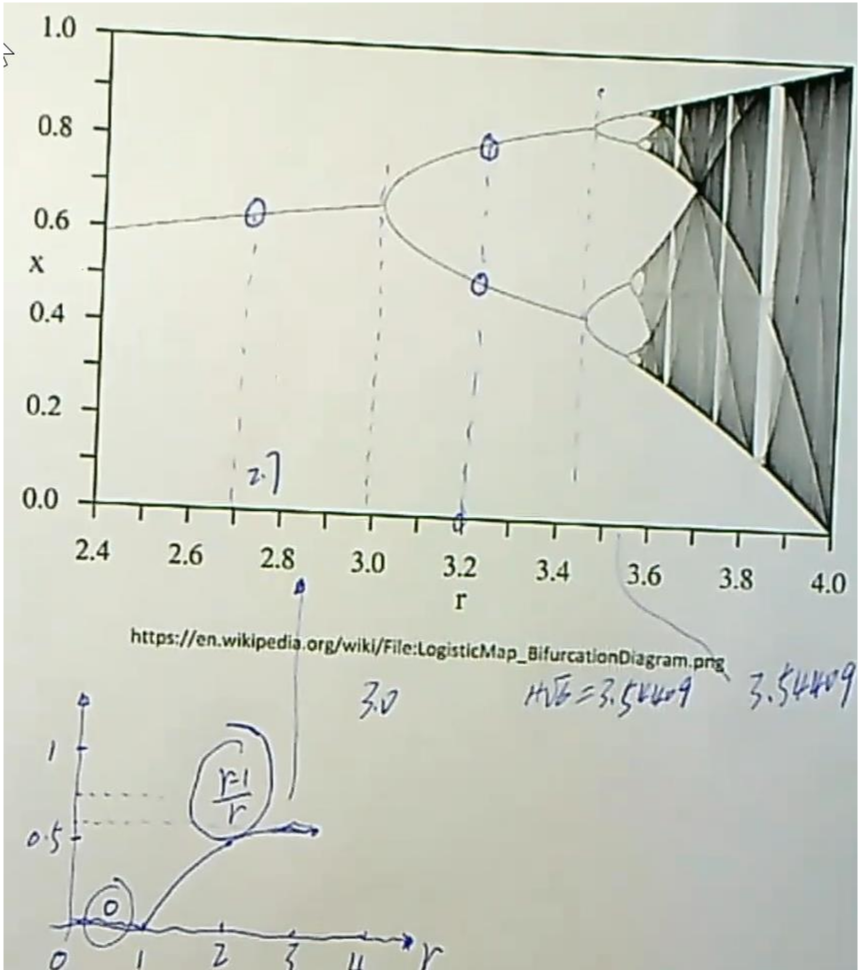
$$x_0, x_1, x_2, \dots, x_n, \dots$$

$$x_{n+1} = rx_n(1-x_n), \quad n = 0, 1, 2, \dots$$

Choose seed $x_0 = \frac{1}{2} = 0.5$



Famous literature by *Li* & *Yorke*,
Period of implies chaos


1.0
0.8
0.6
x
0.4
0.2
0.0
2.4
2.6
2.8
3.0
3.2
3.4
3.6
3.8
4.0
r
https://en.wikipedia.org/wiki/File:LogisticMap_BifurcationDiagram.png
2.7
3.0
3.54409