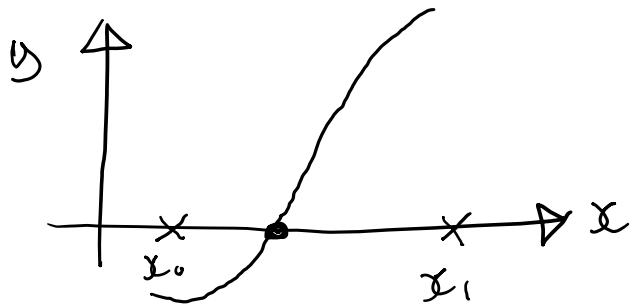
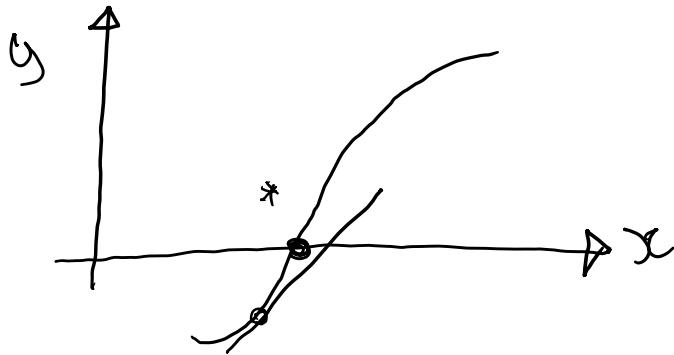


Part 3: Roots of equations

Bisection method:



Open method:



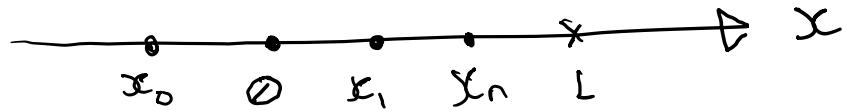
Convergence speed for iterative methods

(how do we measure the convergence speed of iterative methods?)

1. Order of convergence
2. Rate of convergence

$$\{x_n\}: x_0, x_1, x_2, \dots, x_n, \dots, \dots$$

\hookrightarrow converges to L



$$|x_{n+1} - L|, |x_n - L|$$

$$\lim_{n \rightarrow \infty} \frac{|x_{n+1} - L|}{|x_n - L|} = \mu \quad ; \quad 0 \leq \mu \leq 1$$

1st: $0 \leq \mu \leq 1$: the sequence $\{x_n\}$ is said to converge Q - linearly to L

2nd: $\mu = 0$: Q - superlinearly to L

3rd: $\mu = 1$: Q - sublinearly to L

If the sequence converges Q — sublinearly to L , and

$$\lim_{n \rightarrow \infty} \frac{|x_{n+2} - x_{n+1}|}{|x_{n+1} - x_n|} = 1$$

Converges logarithmically to L .

Order of convergence:

$$\lim_{n \rightarrow \infty} \frac{|x_{n+1} - L|}{|x_n - L|^q} < M$$

$q = 1$: linear convergence

$q = 2$: quadratic convergence

$q = 3$: cubic convergence

...

positive
constant

Example

1st sequence:

$$(x_n) = \left\{ 1, \frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \dots, \frac{1}{3^n}, \dots \right\}$$

$$x_n = \frac{1}{3^n} ; n = 0, 1, 2, \dots$$

$$x_n \rightarrow L = 0 ; n \rightarrow \infty$$

$$\lim_{n \rightarrow \infty} \frac{|x_{n+1} - L|}{|x_n - L|} = \frac{\left| \frac{1}{3^{n+1}} - 0 \right|}{\left| \frac{1}{3^n} - 0 \right|} = \frac{1}{3} < 1$$

$$\lim_{n \rightarrow \infty} \frac{|x_{n+1} - L|}{|x_n - L|^q} = \frac{1}{3} ; Q - linearly$$

2nd sequence:

$$(x_n) = \left\{ \frac{1}{3}, \frac{1}{9}, \frac{1}{81}, \dots, \frac{1}{3^{2n}}, \dots \right\}$$

$$x_n = \frac{1}{3^{2n}} ; x_{n+1} = x_n^2$$

$$x_n \rightarrow L = 0 ; n \rightarrow \infty$$

$$\lim_{n \rightarrow \infty} \frac{|x_{n+1} - L|}{|x_n - L|} = \lim_{n \rightarrow \infty} \left| \frac{\frac{1}{3^{2n+1}} - 0}{\frac{1}{3^{2n}} - 0} \right|$$

$$\lim_{n \rightarrow \infty} \frac{1}{3^{2n}} = 0 ; Q - superlinearly$$

3rd sequence:

$$(x_n) = \left\{ 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots, \frac{1}{n+1}, \dots \right\}$$

$$x_n = \frac{1}{n+1} ; n = 0, 1, 2, \dots$$

$$x_n \rightarrow L = 0 ; n \rightarrow \infty$$

$$\lim_{n \rightarrow \infty} \frac{|x_{n+1} - L|}{|x_n - L|} = \lim_{n \rightarrow \infty} \left| \frac{\frac{1}{n+2}}{\frac{1}{n+1}} \right| = 1 ; Q - sublinearly$$

$$\lim_{n \rightarrow \infty} \left| \frac{x_{n+2} - x_{n+1}}{x_{n+1} - x_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{\frac{1}{n+3} - \frac{1}{n+2}}{\frac{1}{n+2} - \frac{1}{n+1}} \right| = 1 ; converges\ logarithmically$$

Functional iteration and orbit

If $f: \mathcal{R} \rightarrow R$,

$$\begin{aligned} f^0(x) &\stackrel{\text{def}}{=} x \\ f'(x) &\stackrel{\text{def}}{=} f(x) \\ f^2(x) &\stackrel{\text{def}}{=} (f \circ f)(x) = f(f(x)) \\ f^3(x) &\stackrel{\text{def}}{=} (f \circ f^2)(x) = f(f^2(x)) \\ &\dots \\ f^n(x) &\stackrel{\text{def}}{=} (f \circ f^{n-1})(x) = f(f^{n-1}(x)) \end{aligned}$$

$f^n(x)$: the n -th iteration of $f(x)$, $n \geq 0$

Example:

1st:

$$\begin{aligned} f(x) &= x + a \\ f^2(x) &= f(f(x)) = f(x + a) = (x + a) + a \\ &= x + 2a \\ f^3(x) &= f(f^2(x)) = f(x + 2a) = (x + 2a) + a \\ &= x + 3a \\ &\dots \\ &= f^n(x) = x + na \quad ; \quad n \geq 1 \end{aligned}$$

2nd:

$$\begin{aligned} f(x) &= \frac{x}{1 + bx} \\ f^2(x) &= f(f(x)) = f\left(\frac{x}{1 + bx}\right) = \frac{\frac{x}{1 + bx}}{1 + b \frac{x}{1 + bx}} \\ &= \frac{x}{1 + 2bx} \\ f^n(x) &= \frac{x}{1 + nbx} \end{aligned}$$

3rd:

$$\begin{aligned} f(x) &= \frac{ax + b}{x + c} \quad (b \neq ac) \\ f^2(x) &= \frac{(a^2 + b)x + ab + bc}{(a + c)x + b^2} \end{aligned}$$

Let $x_0 \in \mathcal{R}$, the orbit of x_0 under function $f(x)$ is defined as the sequence of points:

$$x_0, f(x_0), f^2(x_0), \dots, f^n(x_0), \dots$$

x_0 : seed of the orbit

Example $f(x) = \cos x, x_0 = 0.5$

The orbit

$$\begin{aligned}\cos(0.5) &= 0.8775825619 \\ \cos(\cos(0.5)) &= 0.6390124942 \\ \cos^3(0.5) &= \cos(0.6390 \dots) = 0.8206851007 \\ &\vdots \\ \cos^{56}(0.5) &= 0.7390851332 \\ \cos^{57}(0.5) &= 0.7390851332 \\ &\vdots\end{aligned}$$

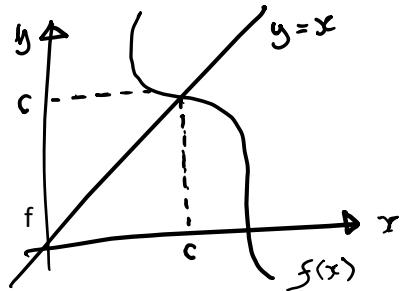
Example $f(x) = x^2 - 1, x_0 = 0.5$

$$\begin{aligned}x_0 &= 0.5 \\ x_1 &= f(x_0) = -0.75 \\ x_2 &= f(x_1) = -0.4375 \\ x_3 &= f(x_2) = -0.80859375 \\ &\vdots \\ x_{19} &= f(x_{18}) = -1 \\ x_{20} &= f(x_{19}) = 0 \\ x_{21} &= f(x_{20}) = -1 \\ x_{22} &= f(x_{21}) = 0 \\ &\vdots \\ &\text{does not converge}\end{aligned}$$

Fixed point

c is a fixed point of function $f(x)$:

$$f(c) = c$$



Example:

$$1^{\text{st}}: f(x) = x^3 - 0.9x^2 + 1.2x - 0.3$$

$x = 1$ is a fixed point

$$f(1) = 1 - 0.9 + 1.2 - 0.3 = 1$$

$$2^{\text{nd}}: f(x) = x + 1$$

no fixed point

A periodic point:

$$f^n(x_0) = x_0 \text{ for some } n$$

Example: $f(x) = x^2 - 4x + 5$

$x_0 = 1, f(1) = 2$ not a fixed point

$$f(2) = 1$$

$\rightarrow f^2(1) = 1, n = 2, x_0 = 1$ is a fixed point of period 2.

Theorem: $x_0, f(x_0), f^2(x_0), \dots, f^n(x_0), \dots$

$$\text{If } \lim_{n \rightarrow \infty} f^n(x_0) = a$$

Then a is a fixed point of $f(x)$

$$f(a) = a$$

For example, $f(x) = \cos x, x_0 = 0.5$

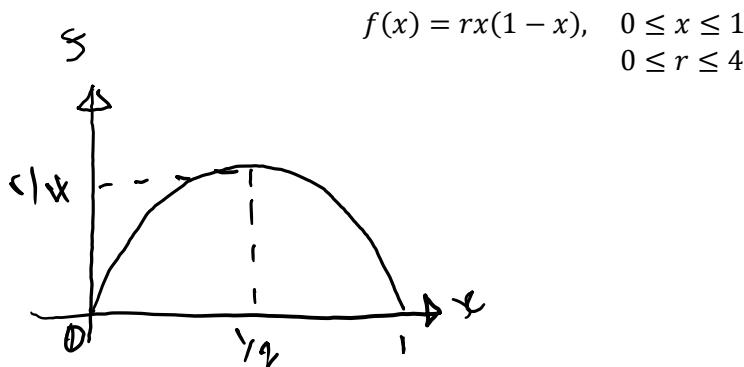
$$f^n(x) \rightarrow 0.7390851332 = a$$

Therefore, from the theorem,

$$\cos a = a$$

(In other words, a is a fixed point of $\cos x$)

Logistic map:



$x_0, x_1, x_2, \dots, x_n, \dots$

$$x_{n+1} = rx_n(1 - x_n), \quad n = 0, 1, 2, \dots$$

Choose seed $x_0 = \frac{1}{2} = 0.5$

0.5	1.0	1.5	2.0	2.5	3.0	3.2	3.5	3.57	3.83
0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5
0.125	0.250	0.375	0.500	0.625	0.750	0.800	0.875	0.925	0.9575
0.0546875	0.1875000	0.3115625	0.5000000	0.6859375	0.8625000	0.9120000	0.928125	0.942191875	0.958570625
0.02534838867	0.1123457500	0.2419494629	0.5000000000	0.6065368652	0.782312500	0.895393000	0.9280548145	0.9379632632	0.9539639347
0.01259012474	0.129151118	0.3375300416	0.5000000000	0.5966347409	0.796661377	0.912884565	0.9303976943	0.9424646147	0.9574418376
0.006215806750	0.1124792495	0.3154012689	0.5000000000	0.6016591405	0.7309999195	0.894623058	0.9149971796	0.9284256051	0.9500000000
0.003083585248	0.099813126670	0.3341628818	0.5000000000	0.5991615437	0.7807254866	0.9130189942	0.9282199054	0.9310226688	0.9444337708
0.0015139522941	0.08984969808	0.33334646077	0.5000000000	0.6034164789	0.7257148227	0.899476183	0.9269408878	0.9420239166	0.9574172221
0.00076857764071	0.08177672981	0.3315684255	0.5000000000	0.59797913268	0.5971584564	0.7133404718	0.8500837755	0.9488006777	0.9541208484
0.000381879928488	0.07502992642	0.330911309	0.5000000000	0.6021042277	0.7216807050	0.8944178108	0.9149972660	0.9302472009	0.9547915945
0.0001919236992	0.0684508787	0.3333973076	0.5000000000	0.599478189	0.6025209798	0.5130438574	0.6282096767	0.3488141755	0.917412562
0.00009594297244	0.06462746721	0.333365134	0.5000000000	0.6030260637	0.7184346402	0.7994554541	0.8269407011	0.8108995622	0.154514000001
0.0000479666370	0.06045073769	0.3333493222	0.5000000000	0.5997986964	0.6083588960	0.5130444048	0.5008842229	0.5474281322	0.5046419508
0.00002198223144	0.05679646359	0.3333413274	0.5000000000	0.600006164	0.7177449394	0.7994554990	0.8749972634	0.8544695430	0.9574174725
0.00001199085314	0.05337062630	0.3333377303	0.5000000000	0.599967417	0.6103623632	0.5130444952	0.7828196836	0.3647939188	0.1561462499
5.995354879 10^{-8}	0.05070081340	0.3333353318	0.5000000000	0.6000016290	0.7134604445	0.7994554917	0.8269407073	0.8272379566	0.5046584124
2.99765936 10^{-8}	0.04813034092	0.33333417326	0.5000000000	0.5999991851	0.6133039335	0.5130445071	0.5008842088	0.5102076915	0.9574168357
1.498822191 10^{-8}	0.04581372083	0.3333338330	0.5000000000	0.6000004072	0.7114866684	0.7994554908	0.8749972637	0.8921280170	0.1561463059
7.484114723 10^{-8}	0.04371482282	0.3333331832	0.5000000000	0.5999997964	0.6158301659	0.5130444080	0.6282196231	0.3425611173	0.5046618227
3.747054554 10^{-7}	0.04180383800	0.3333334583	0.5000000000	0.6000001018	0.7097570674	0.7994554908	0.8269407065	0.8052030922	0.9574168925
1.873526575 10^{-7}	0.04005627711	0.33333337958	0.5000000000	0.5999999489	0.6180059178	0.5130444953	0.5008842106	0.5601154270	0.1541489229
9.367631120 10^{-8}	0.03844177179	0.33333373846	0.5000000000	0.600000254	0.7082238099	0.7954554966	0.8749972638	0.8799895003	0.5046618229
4.683815121 10^{-8}	0.03697321304	0.3333334940	0.5000000000	0.5999999873	0.6195235351	0.5130445083	0.6282196371	0.3798807654	0.9774168290
2.341907450 10^{-8}	0.03456621308	0.3333333412	0.5000000000	0.6000000863	0.7068114393	0.7994554906	0.8269407068	0.8394544495	0.1561462054
1.170953698 10^{-8}	0.03433841067	0.3333333373	0.5000000000	0.5999999963	0.6214374482	0.5130445087	0.5008842099	0.4811797388	0.4046681969
5.854768421 10^{-9}	0.03115926422	0.3333333353	0.5000000000	0.6000000014	0.7044176653	0.7994554906	0.7439774631	0.8121215046	0.9574166078
2.927384193 10^{-9}	0.03205974609	0.333333334	0.5000000000	0.5999999994	0.6233605822	0.5130445087	0.5223126424	0.3468571643	0.1561462797
1.463692092 10^{-8}	0.03103191877	0.3333333338	0.5000000000	0.6000000023	0.7044873155	0.7994554906	0.8269407068	0.8079567424	0.5046660727
7.318460449 10^{-10}	0.03006893879	0.3333333336	0.5000000000	0.5999999993	0.6224440413	0.5130445087	0.5008842108	0.5154662597	0.9574166009
3.659230221 10^{-10}	0.02916479771	0.3333333335	0.5000000000	0.5999999992	0.7014584873	0.7994554906	0.8749972673	0.88236677309	0.1561461618
1.688844646 10^{-10}	0.02811421228	0.3333333334	0.5000000000	0.5999999991	0.6218132197	0.5130445089	0.5428194879	0.5713720048	0.5046664561

Famous literature by Li & Yorke,
Period of implies chaos

Period of 3

