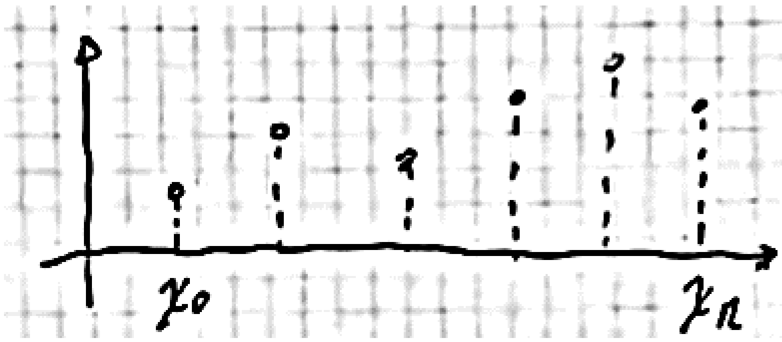
What we've looked at so far:

$$(x_0, y_0), (x_1, y_1), \dots, (x_n, y_n)$$



$$f_n(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$$

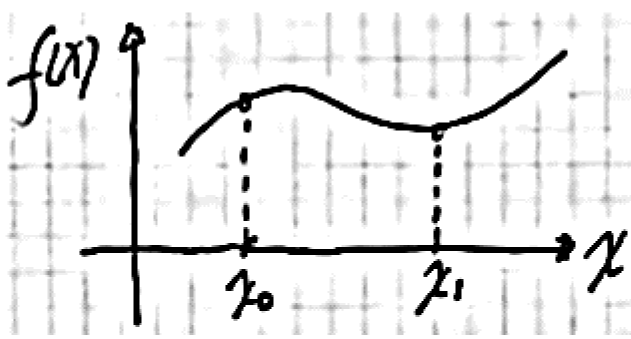
$$f_n(x) = f(x_0) + f[x_1, x_0](x - x_0) + f[x_2, x_1, x_0](x - x_0)(x - x_1) + \cdots + f[x_n, x_{n-1}, \dots, x_1, x_0](x - x_0)(x - x_1)\dots(x - x_n)$$

$$f(x_0) = y_0$$

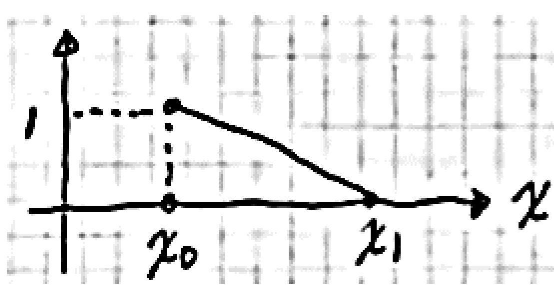
$$f[x_1, x_0] = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{y_1 - y_0}{x_1 - x_0}$$

$$f[x_2, x_1, x_0] = \frac{f[x_2, x_1] - f[x_1, x_0]}{x_2 - x_0}$$

Lagrange Interpolating Polynomial



Given $(x_0, f(x_0)), (x_1, f(x_1))$, fitting line.



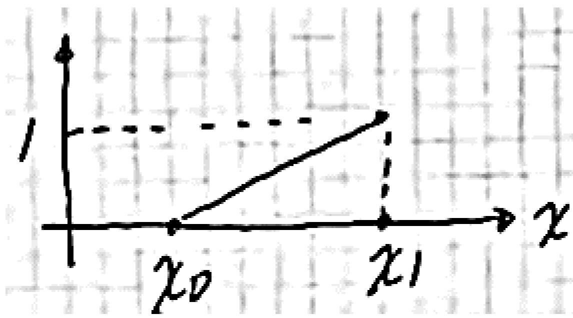
$L_0(x)$:

$L_0(x_0) = 1$

$$L_0(x_1) = 0$$

$$L_0(x) = b_0(x - x_1)$$
$$L_0(x) = b_0(x_0 - x_1) = 1$$
$$b_0 = \frac{1}{x_0 - x_1}$$

$$\boxed{L_0(x) = \frac{x - x_1}{x_0 - x_1}}$$

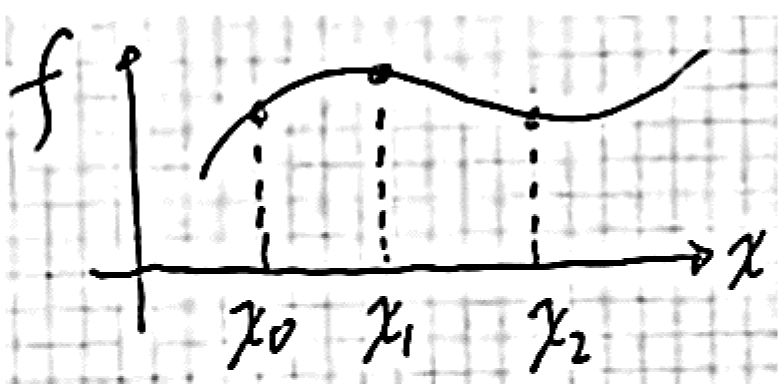


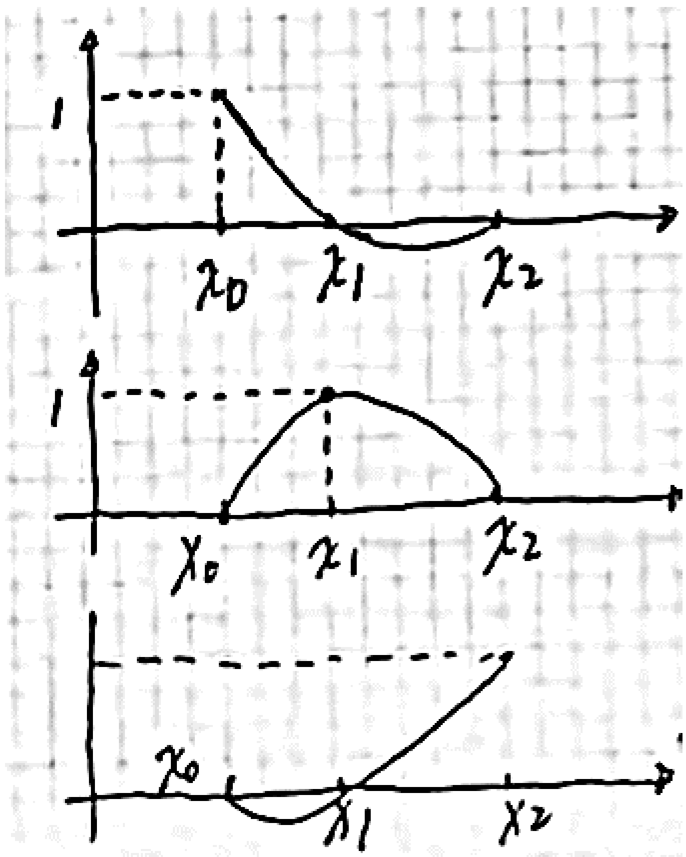
$$l_1(x):$$
$$l_1(x_0) = 0$$
$$l_1(x_1) = 1$$

$$\boxed{L_1(x) = \frac{x - x_0}{x_1 - x_0}}$$

$$\boxed{f_1(x) = f(x_0) \cdot L_0(x) + f(x_1) \cdot L_1(x)}$$

Given $(x_0, f(x_0))$, $(x_1, f(x_1))$, $(x_2, f(x_2))$ fitting polynomial of degree 2:





$$L_0(x) = \begin{cases} 1 & ; \quad x = x_0 \\ 0 & ; \quad x = x_1 \\ 0 & ; \quad x = x_2 \end{cases}$$

$$L_1(x) = \begin{cases} 0 & ; \quad x = x_0 \\ 1 & ; \quad x = x_1 \\ 0 & ; \quad x = x_2 \end{cases}$$

$$L_2(x) = \begin{cases} 0 & ; \quad x = x_0 \\ 0 & ; \quad x = x_1 \\ 1 & ; \quad x = x_2 \end{cases}$$

$$\boxed{f_2(x) = f(x_0) \cdot L_0(x) + f(x_1) \cdot L_1(x) + f(x_2) \cdot L_2(x)}$$

$$L_0(x) = b_0(x - x_1)(x - x_2)$$
$$L_0(x) = b_0(x_0 - x_1)(x_0 - x_2) = 1$$

$$b_0 = \frac{1}{(x_0 - x_1)(x_0 - x_2)}$$

$$L_0(x) = \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)}$$

$$L_1(x) = \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)}$$

$$L_2(x) = \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)}$$

Example

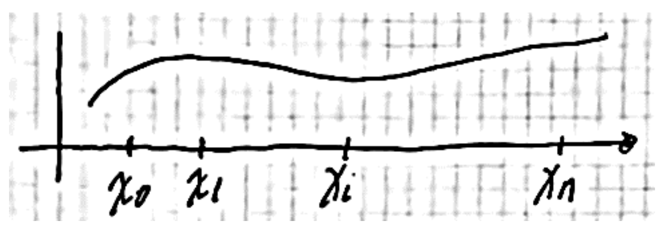
Given the following data, use the Lagrange interpolating polynomial to fit the data.

$$x_0 = 1 \;\; ; \;\; f(x_0) = 0$$
$$x_1 = 4 \;\; ; \;\; f(x_1) = 1.386294$$
$$x_2 = 6 \;\; ; \;\; f(x_2) = 1.791760$$

Solution

$$L_0(x) = \frac{(x-4)(x-6)}{(1-4)(1-6)} = \left(\frac{1}{15}\right)(x^2 - 10x + 24)$$
$$L_1(x) = \frac{(x-1)(x-6)}{(4-1)(4-6)} = -\left(\frac{1}{6}\right)(x^2 - 7x + 6)$$
$$L_2(x) = \frac{(x-1)(x-4)}{(6-1)(6-4)} = \left(\frac{1}{10}\right)(x^2 - 5x + 4)$$

$$\therefore f_2(x) = f(x_0)\cdot L_0(x) + f_1(x)\cdot L_1(x) + f_2(x)\cdot L_2(x)$$
$$= \cdots$$
$$= \cdots$$



$$L_0(x) = \begin{cases} 1 & ; \quad x = x_0 \\ 0 & ; \quad x = x_1, \dots, x_n \end{cases}$$

$$L_1(x) = \begin{cases} 1 & ; \quad x = x_1 \\ 0 & ; \quad other\ x \end{cases}$$

$$L_i(x) = \begin{cases} 1 & ; \quad x = x_i \\ 0 & ; \quad other\ x \end{cases}$$

(Where $i = 0, 1, \dots n$)

$$L_i(x) = \frac{(x-x_0)\dots(x-x_{i-1})(x-x_{i+1})\dots(x-x_n)}{(x_i-x_0)\dots(x_i-x_{i-1})(x_i-x_{i+1})\dots(x_i-x_n)}$$
$$= \prod_{j=0}^{n} \frac{x - x_j}{x_i - x_j}$$

(Where $j = 0, 1, \dots n$; but $j \neq i$)

$$\therefore f(x) = f(x_0)L_0(x) + f(x_1)L_1(x) + \cdots f(x_n)L_n(x)$$
$$= \sum_{i=0}^{n} f(x_i)L_i(x)$$

Estimate error:

$$R_n = f[x_1, x_n, x_{n-1}, \dots, x_0] \prod_{i=0}^{n} (x - x_i)$$

Inverse Interpolation

x	1	2	3	4	5	6	7
$f(x) = \frac{1}{x}$	1	0.5	0.3333	0.25	0.2	0.16667	0.1428

Find x such that $f(x) = 0.3$

1. Interchange $x \leftrightarrow f(x)$, construct the interpolation polynomial
2. Using a few point construct a polynomial then solving the equation to find x.

Using $(2, 0.5), (3, 0.3333), (4, 0.25)$ to construct a polynomial:

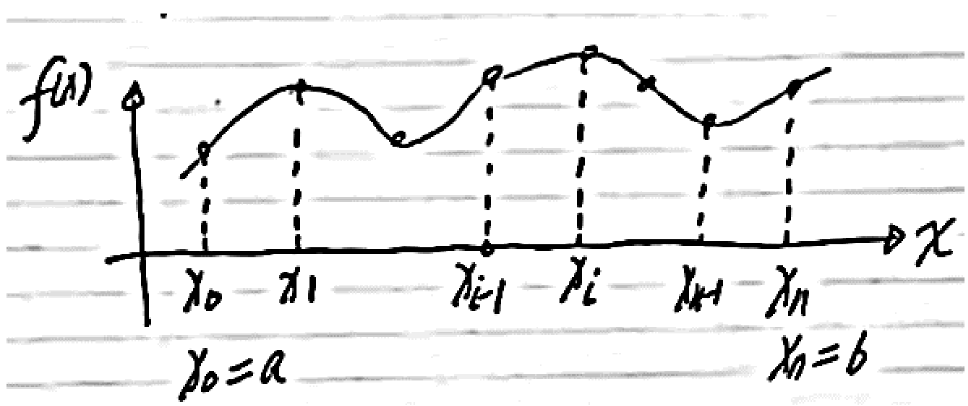
$$f_2(x) = 1.08333 - 0.375x + 0.041667x^2$$
$$0.3 = f_2(x) = 1.08333 - 0.375x + 0.041667x^2$$

$$\rightarrow x = 3.295842, 5.704158$$

The exact value of x is:

$$f(x) = \frac{1}{x} = 0.3 \rightarrow x = 3.333$$

Spline Interpolation



Given a set of $n+1$ data points (x_i, y_i) where no two x_i are the same and $a = x_0 < x_1 < \cdots x_n = b$, the spline $S(x)$ is a piecewise function satisfying:

1. $S(x) \in C^2[a,b] (S(x), S'(x), S''(x)$ exist and continuous
2. On each interval $[x_{i-1}, x_i]$, $S(x)$ is a cubic polynomial $i = 1, 2, \ldots, n$
3. $S(x_i) = f(x_i) = y_i, \quad i = 0, 1, \ldots, n$

Assume that

$$S(x) = \begin{cases} C_1(x) & ; \quad x_0 < x < x_2 \\ & \quad \vdots \\ C_2(x) & ; \quad x_1 < x < x_2 \\ & \quad \vdots \\ C_n(x) & ; \; x_{n+1} < x < x_n \end{cases}$$

And

$$C_i(x) = a_{0i} + a_{1i}x + a_{2i}x^2 + a_{3i}x^3$$
$$i = 1, 2, \dots, n$$
$$a_{3i} \neq 0$$

There are a $4n$ unknowns

The equations:

$$C_i(x)|_{x=x_{i-1}} = C_i(x_{i-1}) = f(x - i)(= y_{i-1})$$

$$C_i(x_{i-1}) = y_{i-1} \qquad (i = 1, 2, 3, \dots\ n - 1)$$
$$C_i(x_i) = y_i \qquad (i = 1, 2, 3, \dots\ n - 1)$$
$$(x_i) = C'_{i+1}(x_i) \qquad (i = 1, 2, 3, \dots, n - 1)$$
$$C''_i(x_i) = C''_{i+1}(x_i) \qquad (i = 1, 2, 3, \dots, n - 1)$$

Total of $4n - 2$ equations – boundary conditions are needed.

Case 1: The first derivatives at the endpoints are given

Consider clamped boundary conditions

$$C'_1(x_0) = f'_0$$
$$C'_n(x_n) = f_n{}'$$

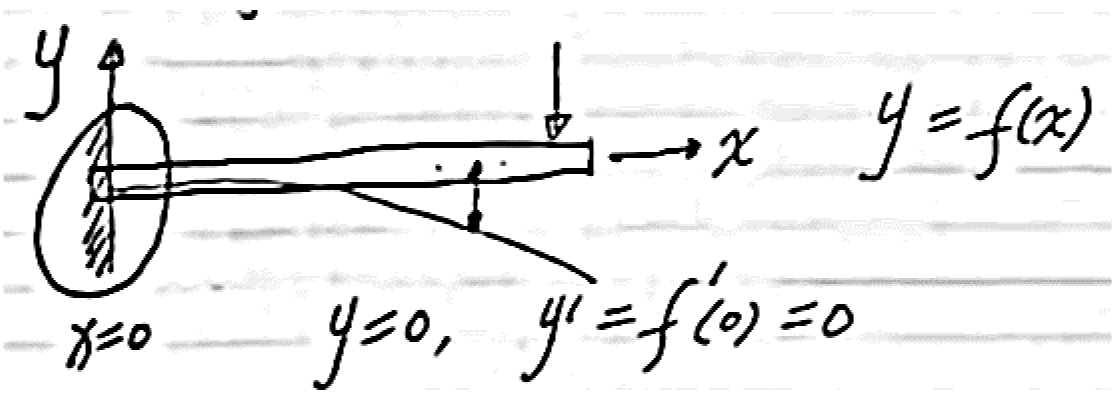
Case 2: The second derivatives at the endpoints are given.

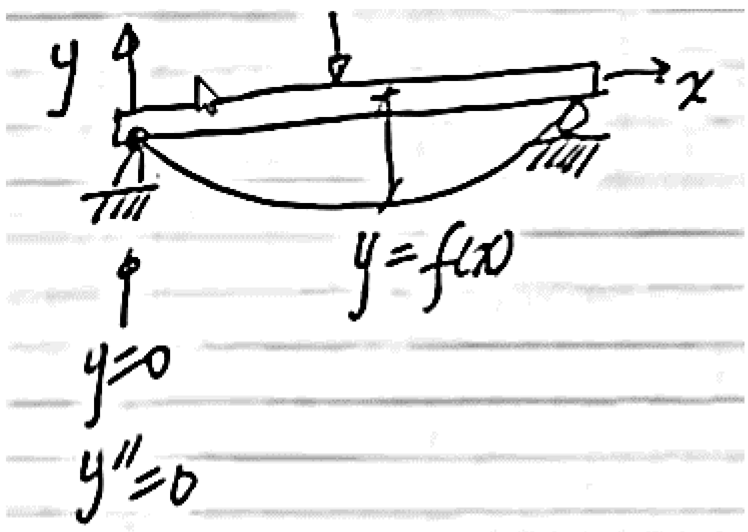
$$C''_1(x_0) = f''_0$$
$$C''_n(x_n) = f_n{}''$$

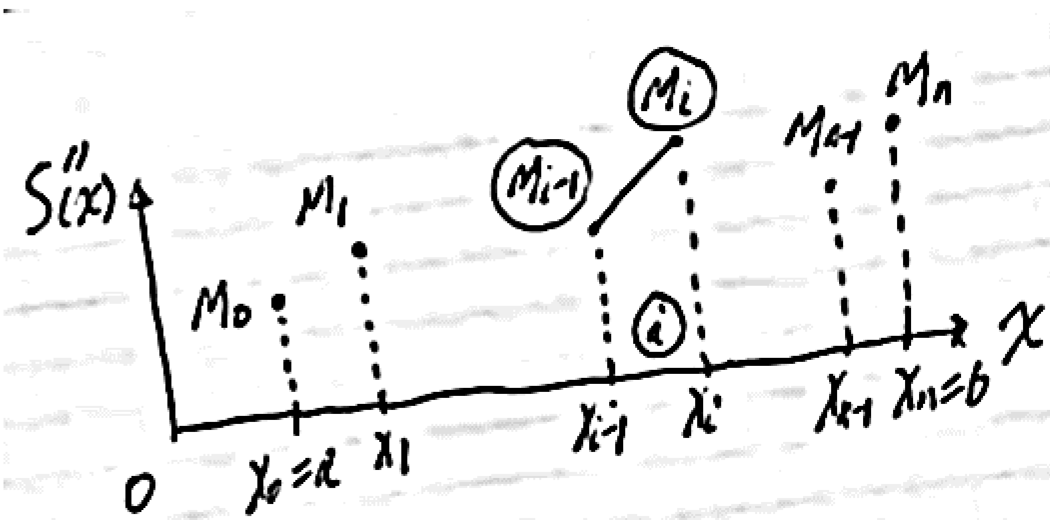
Special case $f''_0 = f_n{}''$ is called natural or simple B.C.'s

Case 3: Periodic conditions

$$C_1(x_0) = C_n(x_n)$$
$$C'_1(x_0) = C'_n(x_n)$$
$$C''_1(x_0) = C_n{}''(x_n)$$







Use the second derivatives

$$S''(x_i) = M_i \quad i = 0, 1, 2, \ldots, n$$

To find $S(x)$ In the interval $x_{i-1} < x < x_i$:

$$C_i''(x) = M_{i-1}\frac{x_i - x}{x_i - x_{i-1}} + M_i\frac{x - x_{i-1}}{x_i - x_{i-1}} \quad i = 1, 2, \dots, n$$

Integrate the moment function twice:

$$C_i'(x) = -M_{i-1}\frac{(x_i - x)^2}{2h_i} + M_i\frac{(x - x_{i-1})^2}{2h_i} + \alpha$$

Here $h_i = x_i - x_{i-1}$

$$C_i(x) = M_{i-1}\frac{(x_i - x)^3}{6h_i} + M_i\frac{(x - x_{i-1})^3}{6h_i} + \alpha(x - x_{i-1}) + \beta$$

At $x = x_{i-1}$:

$$C_i(x) = y_{i-1} = f(x_{i-1})$$

$$\therefore C_i(x_{i-1}) = M_{i-1}\frac{(x_i - x_{i-1})^3}{6h_i} + 0 + 0 + \beta = f(x_{i-1})$$

$$\beta = f(x_{i-1}) - M_{i-1}\frac{h_i^2}{6}$$

At $x = x_i$:

$$C_i(x) = y_i = f(x_i)$$

$$\therefore C_i(x_i) = M_i\frac{(x_i - x_{i+1})^3}{6h_i} + \alpha(x_i - x_{i-1}) + \beta = f(x_i)$$

$$\alpha = (M_{i-1} - M_i)\frac{h_i}{6} + \frac{f(x_i) - f(x_{i-1})}{h_i}$$

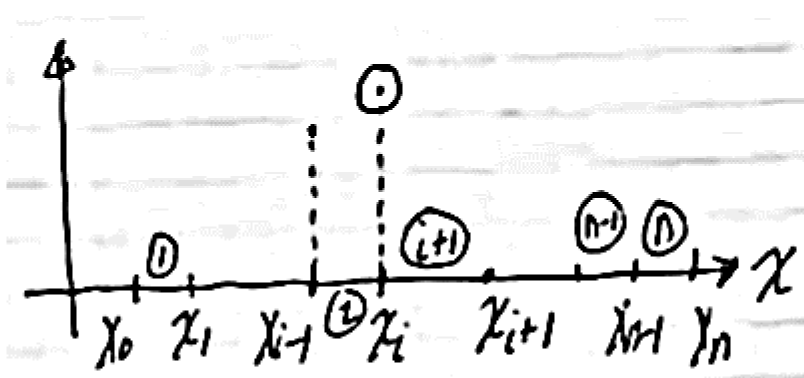
The cubic function:

$$C_i(x) = M_{i-1}\frac{(x_i - x)^3}{6h_i} + M_i\frac{(x - x_{i-1})^3}{6h_i} + \left[(M_{i-1} - M_i)\frac{h_i}{6} + \frac{f(x_i) - f(x_{i-1})}{h_i}\right](x - x_{i-1}) + f(x_{i-1}) - M_{i-1}\frac{h_i^2}{6}$$

$$C_i(x) = M_{i-1}\frac{(x_i - x)^3}{6h_i} + M_i\frac{(x - x_{i-1})^3}{6h_i} + \left(f(x_{i-1}) - M_{i-1}\frac{h_i^2}{6}\right)\frac{x_i - x}{h_i} + \left(f(x_i) - M_i\frac{h_i^2}{6}\right)\frac{x - x_{i-1}}{h_i}$$

The first derivative of $C_i(x)$:

$$C_i'(x) = -M_{i-1}\frac{(x_i - x)^2}{2h_i} + M_i\frac{(x - x_{i-1})^2}{2h_i} - \left(f(x_{i-1}) - M_{i-1}\frac{h_i^2}{6}\right)\frac{1}{h_i} + \left(f(x_i) - M_i\frac{h_i^2}{6}\right)\frac{1}{h_i}$$



At $x = x_i$, we have:

$$C_i'(x_i) = 0 + M_i \frac{(x_i - x_{i-1})^2}{2h_i} + \frac{f(x_i) - f(x_{i-1})}{h_i} + M_{i-1}\frac{h_i}{6} - M_i\frac{h_i}{6}$$
$$= (M_{i-1} + 2M_i)\frac{h_i}{6} + f[x_i, x_{i-1}]$$

For interval $i + 1, x_i \le x \le x_{i+1} \quad (i = 1, 2, \dots, n-1)$

$$C_{i+1}'(x) = -M_i \frac{(x_{i+1} - x)^2}{2h_{i+1}} + M_i \frac{(x - x_i)^2}{2h_{i+1}} - \left(f(x_i) - M_{i-1}\frac{h_{i+1}^2}{6}\right)\frac{1}{h_{i+1}} + \left(f(x_{i+1}) - M_{i+1}\frac{h_{i+1}^2}{6}\right)\frac{1}{h_{i+1}}$$

At $x = x_i$:

$$C_{i+1}'(x_i) = -M_i \frac{(x_{i+1} - x_i)^2}{2h_{i+1}} + 0 + \frac{f(x_{i-1}) - f(x_i)}{h_{i+1}} + M_i \frac{h_{i+1}}{6} - M_{i+1}\frac{h_{i+1}}{6}$$

$$C_{i+1}'(x_i) = -(2M_i + M_{i+1})\frac{h_{i+1}}{6} + f(x_{i+1}, x_i)$$

Since $C_i'(x_i) = C_{i+1}{}'(x_i) \quad (i = 1, 2, \dots, n-1)$

$$(M_{i-1} + 2M_i)\frac{h_i}{6} + f[x_i, x_{i-1}]$$
$$= -(2M_i + M_{i+1})\frac{h_{i+1}}{6} + f[x_{i+1}.x_i]$$

$$M_{i-1}h_i + 2M_i(h_i + h_{i+1}) + M_{i+1}h_{i+1} = 6(f[x_{i+1}, x_i] - f[x_i, x_{i-1}])$$

$$M_{i-1} = \frac{h_i}{h_i + h_{i+1}} + 2M_i + M_{i+1}\frac{h_i}{h_i + h_{i+1}} = 6\frac{f[x_{i+1}, x_i] - f[x_i, x_{i-1}]}{h_i + h_{i+1}}$$

Define:

$$\alpha_i = \frac{h_i}{h_i + h_{i+1}} \qquad (i = 1, 2, \dots, n-1)$$

$$\beta_i = \frac{h_{i+1}}{h_i + h_{i+1}}$$

And $\alpha_i + \beta_i = 1$

Since:

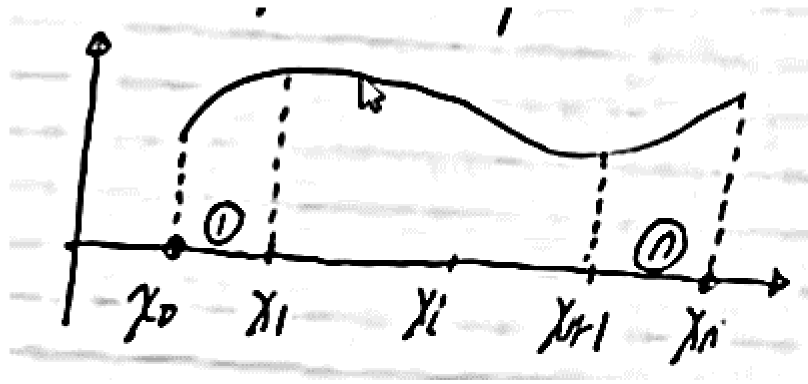
$$h_i = x_i - x_{i-1}$$
$$h_{i+1} = x_{i+1} - x_i$$
$$h_i + h_{i+1} = x_{i+1} - x_{i-1}$$

$$\boxed{\alpha_i M_{i+1} + 2M_i + \beta_i M_{i+1} = 6f[x_{i+1}, x_i, x_{i-1}] = \gamma_i \quad ; \quad i = 1, 2, \dots, n-1}$$

↑ ↑ ↑

The boundary conditions

Case 1: The clamped



Given $C_1'(x_0) = f_0'$; $C_n'(x_n) = f_n{}'$

$$C_1'(x_0) = -M_0 \frac{(x_1 - x_0)^2}{2h_1} + M_1 \frac{(x_0 - x_o)^2}{2h_1} - \left(f(x_0) - M_0 \frac{h_1^2}{6}\right)\frac{1}{h_1} + \left(f(x_1) - M_1 \frac{h_1^2}{6}\right)\frac{1}{h_1}$$
$$= f_0{}'$$

$$\rightarrow 2M_0 + M_1 = 6\frac{f[x_1, x_0] - f_0'}{h_1} \overset{\Delta}{=} \gamma_0$$

defined as

$$C_n'(x_n) = -M_{n-1} \frac{(x_n - x_n)^2}{2h_n} + M_n \frac{(x_n - x_{n+1})^2}{2h_n} - \left(f(x_{n+1}) - M_{n+1} \frac{h_n^2}{6}\right)\frac{1}{h_n} + \left(f(x_n) - M_n \frac{h_n^2}{6}\right)\frac{1}{h_n}$$
$$= f_n{}'$$

$$\rightarrow M_{n-1} + 2M_n = 6\, \frac{f_n' - f[x_n, x_{n-1}]}{h_n} \overset{\Delta}{=} \gamma_n$$

All the equations:

$$\begin{cases} 2M_0 + M_1 = \gamma_0 \\ \alpha_1 M_0 + 2M_1 + \beta_1 M_2 = \gamma_1 \\ \alpha_2 M_1 + 2M_2 + \beta_2 M_3 = \gamma_2 \\ \vdots \\ \alpha_{n-1} M_{n-2} + 2M_{n-1} + \beta_{n-1} M_n = \gamma_{n-1} \\ M_{n-1} + 2M_n = \gamma_n \end{cases}$$

For the first row $\beta_0 = 1$, and for the last row $\alpha_n = 1$ (β_0 is added to make the equation look consistent)

$$\begin{bmatrix} 2 & \beta_0 & & & & 0 \\ \alpha_1 & 2 & \beta_1 & & & \\ & \ddots & \ddots & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & & & \alpha_{n+1} & 2 & \beta_{n-1} \\ 0 & & & & \alpha_n & 2 \end{bmatrix} \begin{Bmatrix} M_0 \\ M_1 \\ \vdots \\ M_{n-1} \\ M_n \end{Bmatrix} = \begin{Bmatrix} \gamma_0 \\ \gamma_1 \\ \vdots \\ \gamma_{n-1} \\ \gamma_n \end{Bmatrix}$$

$$\{\alpha_n = \beta_n = 1)$$

Case 2, the natural boundary conditions:

Given:

$$M_0 = f_0''$$
$$M_n = f_n''$$

Let:

$$\beta_0 = \alpha_n = 0$$
$$\gamma_0 = 2M_0 = 2f_0''$$
$$\gamma_n = 2M_n = 2f_n''$$

Error and convergence:

Assume that $f(x) \in C^4[a,b]$, $S(x)$ is the cubic spline interpolating function that satisfies clamped or natural boundary conditions.

Let $h = \max h_i \ (1 < i < n)$
Where $h_i = x_i - x_{i-1}$

Then,

$$\left[\begin{matrix}\max \\ x \in [a,b]\end{matrix}\right] \left| f_{(x)}^{(k)} - S_{(x)}^{(k)} \right| \leq C_k \left[\begin{matrix}\max \\ x \in [a,b]\end{matrix}\right] \left| f_{(x)}^{(k)} \right| \cdot h^{4-k}$$

For $k = 0, 1, 2$ with:

$$C_0 = \frac{5}{384} \quad ; \quad C_1 = \frac{1}{24} \quad ; \quad C_2 = \frac{3}{8}$$

The interpolation is much better for the function itself, and it becomes worse for the derivatives.

As with all other functions, the accuracy of a derivative function is worse than the original function itself. Consider the coefficients as well, which get much larger as the order of the derivatives increases.

Consider $k = 0$, the function converges very quickly, at h^4

Consider $k = 1$, the derivative function converges more slowly, converging at h^3

Consider $k = 2$, the derivative functions converges even more slowly, converging at h^2