5. Attributes of an Element:

5.1) Dimensionality:

- 1D
- 2D
- 3D

5.2) Associated with certain material and certain geometric properties such as:

- Cross-sectional area (A)
- Moments of inertia (I_x, I_y, I_{xy}, J)
- Thickness (t)
- Modulus of elasticity (E)
- Poisson's ratio (ν)

5.3) A number of <u>nodes</u>

Each node is associated with a number of DOFs (physical unknowns) such as:

- Temperature (1 DOF)
- Displacement (1 or 2 or 3 DOFs)
- Velocity (1 or 2 or 3 DOFs)
- …

5.4) DOFs of an element = (DOFs of a node) x (number of nodes)

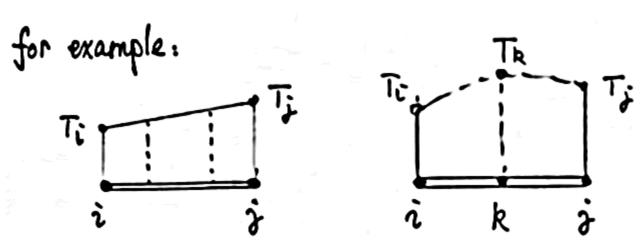
5.5) Interpolation within Element

In FEA, DOFs at the nodes are the unknowns to be solved for.

Between nodes (within element), the unknown variable is interpolated.

The interpolation function is known as the shape function.

Shape function is a key feature of FEM; its construct/form, has significant effect on the quality of the solution.



In general, the more nodes that are used, the higher the degree of interpolation, the more accurate the element; but the number of DOFs of the element is increased.

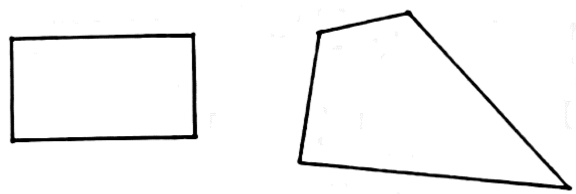
Lesson #1

Not all elements are created equal;

Some elements are better than others;

- More accurate
- Less sensitive to distortion of the element's shape

A given element does not have equal accuracy in all situations;



Consider the following diagram:

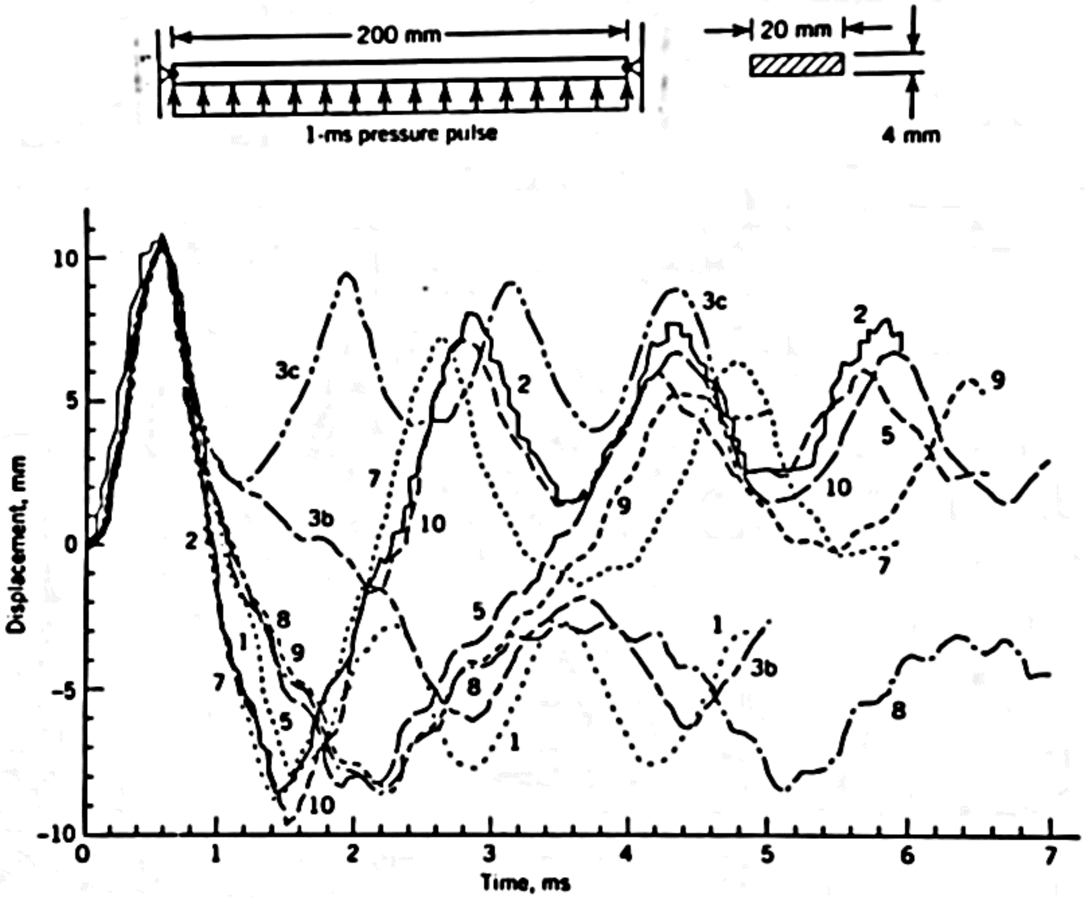


Fig. 1.5-1. Lateral midpoint displacement versus time for a beam loaded by a pressure pulse [1.6] The material is elastic–perfectly plastic. Plots were generated by various users and various codes.

Formal (General) Approach

1. Available principles (methods)
    a. Solid mechanics (structural mechanics)
        Variational methods
        Virtual work
    b. Field problems (e.g. heat transfer, fluid flow, electric potential, multi-physics and so on)
        Weighted residual methods

$$\left\{\begin{array}{l}\text{Galerkin's}\\ \text{collocation}\\ \text{least squares}\\ \text{subdomain weighted residual}\\ \ldots\end{array}\right.$$

2. Variational methods (principles)
    Variational principle is a principle used to find a function which minimizes or maximizes a physical quantity that depends upon the function to be found.

    Single variable calculus:
        Function is given,
            1st order derivative
            2nd order derivative
    Variational principles:
        Boundary conditions and loading are known (e.g. a circular plate, being clamped along outer edge, and subject to a central load);
        The unknown function is the deflection $w(r, \theta)$;
        Physical quantity: work, energy;

3. The Principle of Minimum Potential Energy
    Commonly used in solid mechanics
    Applicable to linear elastic analyses only;
    Been extended to many other “non-structural” applications.

    Statement of the principle:
    Of all the geometrically possible shapes that a body can assume, the true one, corresponding ot the satisfaction of stable equilibrium of the body, is identified by the minimum value of the total potential energy.

    2 key issues:
    - total potential energy
    - finding a function giving a minimum value of energy

    Total potential energy:

$$\pi_p = u + \Omega$$

u: strain energy due to deformation
Ω: potential energy of external forces (including body forces, surface loads, and concentrated forces/moment, etc.)
$\Omega = -(\text{work done by external forces})$

Finding a function that minimizes π_p by variational calculus.

4. The Principles of Momentum Potential energy as Applied to an Elastic Body

$$
\begin{aligned}
\pi_p = & \int_V \frac{1}{2}\{\epsilon)^T[E]\{\epsilon\}\, dV \\
& - \int_V \frac{1}{2}\left\{\underset{\sim}{u}\right)^T \{B_f\}\, dV \\
& - \int_S \frac{1}{2}\{\bar{u})^T\{\phi\}\, dS \\
& - \{u)^T\{p\}
\end{aligned}
$$

Where $\{\epsilon\}$ and $\{\sigma\}$ are strain and stress vectors, respectively.

$[E]$ is the elastic matrix, such that:

$$\{\sigma\} = [E]\{\epsilon\}$$

$\{P\}$: concentrated forces/moments vector
$\{\phi\}$: surface load vector
$\{B_f\}$: body force components vector

$\{u\}$: displaces at nodes where $\{p\}$ is applied.
$\{\bar{u}\}$: displacement evaluated on the surface of the body where $\{\phi\}$ is applied
$\left\{\underset{\sim}{u}\right\}$: displacement within the body

5. The Finite Element Form of the Principle of Minimum Potential Energy

The volume of the body is divided into NE elements, each having a volume of V_e

Similarly, S, the surface, is divided based on element formation

$$
\begin{aligned}
\therefore \pi_p = & \sum_{j=1}^{NE} \int_{V_e} \frac{1}{2}\{\epsilon)^T[E]\{\epsilon\}\, dV_e \\
& - \sum_{j=1}^{NE} \int_{V_e} \frac{1}{2}\left\{\underset{\sim}{u}\right)^T \{B_f\}\, dV_e \\
& - \sum_{j=1}^{NE} \int_{S_e} \frac{1}{2}\{\bar{u})^T\{\phi\}\, dS_e \\
& - \{u)^T\{p\}
\end{aligned}
$$

1

Within an element,

$$\left\{\underset{\sim}{u}\right\} = [N]\{u\}$$

$[N]$: shape function matrix

Then $\{\epsilon\}$ can be written as, symbolically

$$\begin{aligned}\{\epsilon\} &= [\partial]\left\{\underset{\sim}{u}\right\} \\ &= [\partial][N]\{u\} \\ &= [B]\{u\}\end{aligned}$$

$[B]$: strain-displacement matrix
$[\partial]$: a matrix of partial differentiation operators

Eqn. (1) becomes:

$$\begin{aligned}&\therefore \pi_p = \sum_{j=1}^{NE} \int_{V_e} \frac{1}{2}\{u\}^T[B]^T[E][B]\{u\}\,dV_e \\ &-\sum_{j=1}^{NE} \int_{V_e} \frac{1}{2}\{u)^T[N]^T\{B_f\}\,dV_e \\ &-\sum_{j=1}^{NE} \int_{S_e} \frac{1}{2}\{u)^T[\bar{N}]^T\{\phi\}\,dS_e \\ &-\{U)^T\{p\}\end{aligned}$$

Where $[U] = \sum\{u\}$ (symbolically)
And $[\bar{N}]$ is $[N]$ but evaluated over S_e

Minimization: $\frac{\partial \pi_p}{\partial\{U\}} = \{0\}$

Finally:

$$\begin{aligned}&\left(\sum_{j=1}^{NE} \int_{V_e} [B]^T[E][B]dV_e\right) \cdot \{U\} \\ &= \{P\} + \sum_{j=1}^{NE} \int_{V_e} [N]^T\left[B_f\right] dV_e + \sum_{j=1}^{NE} \int_{S_e} [\bar{N}]^T[\phi]\,dS_e\end{aligned} \tag{2}$$

In Eqn. (2):

$$\int_{V_e} [B]^T[E][B]dV_e = [k] \tag{3}$$

The element stiffness matrix

$$\sum_{j=1}^{NE}[k] = [K]$$

The structure stiffness matrix

$$\sum_{j=1}^{NE} \int_{V_e} [N]^T [B_f] \, dV_e + \sum_{j=1}^{NE} \int_{S_e} [\bar{N}]^T [\phi] \, dS_e = \{f_{eq}\} \qquad 4$$

The element equivalent nodal force vector

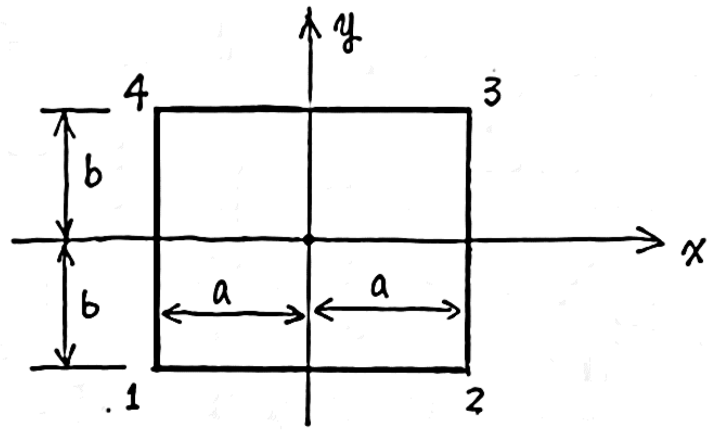
$$\sum_{j=1}^{NE} [f_{eq}] = [F_{eq}]$$

The structure equivalent nodal force vector

Eqn. (2) can be further written as:

$$[K]\{U\} = \{P\} + \{f_{eq}\} \qquad 5$$

4-Noded Quadrilateral Element (Q4)



4 nodes, 1, 2, 3, and 4

$$\begin{cases} \text{counter} - \text{clockwise} \\ 1 \text{ in the 3rd quadrant} \\ 1-2 \text{ defined local x} \\ 2-3 \text{ defines local y} \end{cases}$$

2DOFs per node:
$u -$ displacement in the x −direction
v −displacement in the y −direction

8DOFs per element:
$\therefore\ [k]_{8x8} \qquad \{f_{eq}\}_{8x1}$

Element nodal DOFs:

$$\{u\}_e = [\, u_1\ v_1\ u_2\ v_2\ u_3\ v_3\ u_4\ v_4\,]^T$$

Within the element, any point (x, y) will have displacements.

$$u(x,y) \text{ and } v(x,y)$$

$u(x,y)$ and $v(x,y)$ are related to $\{u\}_e$ via shape functions.

$$N_1(x,y),\ \ N_2(x,y),\ \ N_3(x,y),\ \ N_4(x,y)$$

Such that,

$$u(x,y) = \sum_{i=1}^{4} N_i(x,y)u_i$$

$$v(x,y) = \sum_{i=1}^{4} N_i(x,y)v_i$$

Putting into matrix form:

$$\underbrace{\begin{Bmatrix}u\\v\end{Bmatrix}}_{\{\underset{\sim}{u}\}} = \underbrace{\begin{bmatrix}N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0\\ 0 & N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4\end{bmatrix}_{2\text{x}8}}_{[N]\ \text{shape function matrix of the Q4}} \{u\}_e$$

Where:

$$N_1(x,y) = \frac{1}{4ab}(a-x)(b-y)$$
$$N_2(x,y) = \frac{1}{4ab}(a+x)(b-y)$$
$$N_3(x,y) = \frac{1}{4ab}(a+x)(b+y)$$
$$N_4(x,y) = \frac{1}{4ab}(a-x)(b+y)$$

Next, $[B] = [\partial][N]$

Strain displacement matrix

From theory of elasticity:

$$\varepsilon_x = \frac{\partial u}{\partial x} \quad ; \quad \varepsilon_y = \frac{\partial v}{\partial y}$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$

$$\therefore \{\varepsilon\} = \begin{Bmatrix}\varepsilon_x\\ \varepsilon_y\\ \gamma_{xy}\end{Bmatrix} = \begin{Bmatrix}\dfrac{\partial u}{\partial x}\\ \dfrac{\partial v}{\partial y}\\ \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x}\end{Bmatrix}$$

$$= \underbrace{\begin{bmatrix}\dfrac{\partial}{\partial x} & 0\\ 0 & \dfrac{\partial}{\partial y}\\ \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial X}\end{bmatrix}}_{[\partial]} \begin{Bmatrix}u\\v\end{Bmatrix}$$

$$= \underbrace{[\partial][N]}_{[B]}\{u\}_e$$

$\therefore [B]$ for $Q4$ is:

$$[B] = \begin{bmatrix} \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial X} \end{bmatrix}_{3x2} \cdot \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 \end{bmatrix}_{2x8}$$

$$[B] = \begin{bmatrix} \frac{\partial N_1}{\partial x} & 0 & \frac{\partial N_2}{\partial x} & 0 & \frac{\partial N_3}{\partial x} & 0 & \frac{\partial N_4}{\partial x} & 0 \\ 0 & \frac{\partial N_1}{\partial y} & 0 & \frac{\partial N_2}{\partial y} & 0 & \frac{\partial N_3}{\partial y} & 0 & \frac{\partial N_4}{\partial y} \\ \frac{\partial N_1}{\partial y} & \frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial y} & \frac{\partial N_2}{\partial x} & \frac{\partial N_3}{\partial y} & \frac{\partial N_3}{\partial x} & \frac{\partial N_4}{\partial y} & \frac{\partial N_4}{\partial x} \end{bmatrix}_{3x8}$$

$$[k] = \int_{V_e} [B]^T [E][B] dV_e$$

If constant thickness (plane stress $t =$ const., plane strain— analyzing a thin slice of constant thickness t)

Then,

$$[k] = \int_{-b}^{b} \int_{-a}^{a} [B]^T [E][B] dx \cdot dy$$

$[B]$: 1st order polynomials in x or in y

$\therefore$ integrands are 2nd order polynomials
$\therefore$ analytical (closed-form) solutions are obtainable

Plane stress:

$$[E] = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \quad \text{(linear, elastic, isotropic)}$$

and: $\varepsilon_z = -\frac{\nu}{E}(\sigma_x + \sigma_y)$

Plane strain:

$$[E] = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & \frac{1-2\nu}{2} \end{bmatrix}$$

and: $\sigma_z = \nu(\sigma_x + \sigma_y)$

Properties of Shape Functions:

1) $\sum_i N_i = 1$ for any given point within the element, including the nodes and edges/surfaces where applicable
2) $N_i = \begin{cases} 1 & \text{at node i} \\ 0 & \text{at all other nodes} \end{cases}$

Put them in a more mathematical way:

1) Is known as the partitions of unity property.
2) Is known as the δ −function property.

Other properties include,

Consistency: to include the complete order of monomial

(Second order: x^2, y^2, xy)

(Third order: $x^3,\ y^3,\ xy^2,\ x^2y$)

Linear dependence: $N_i's$ should be linearly independent