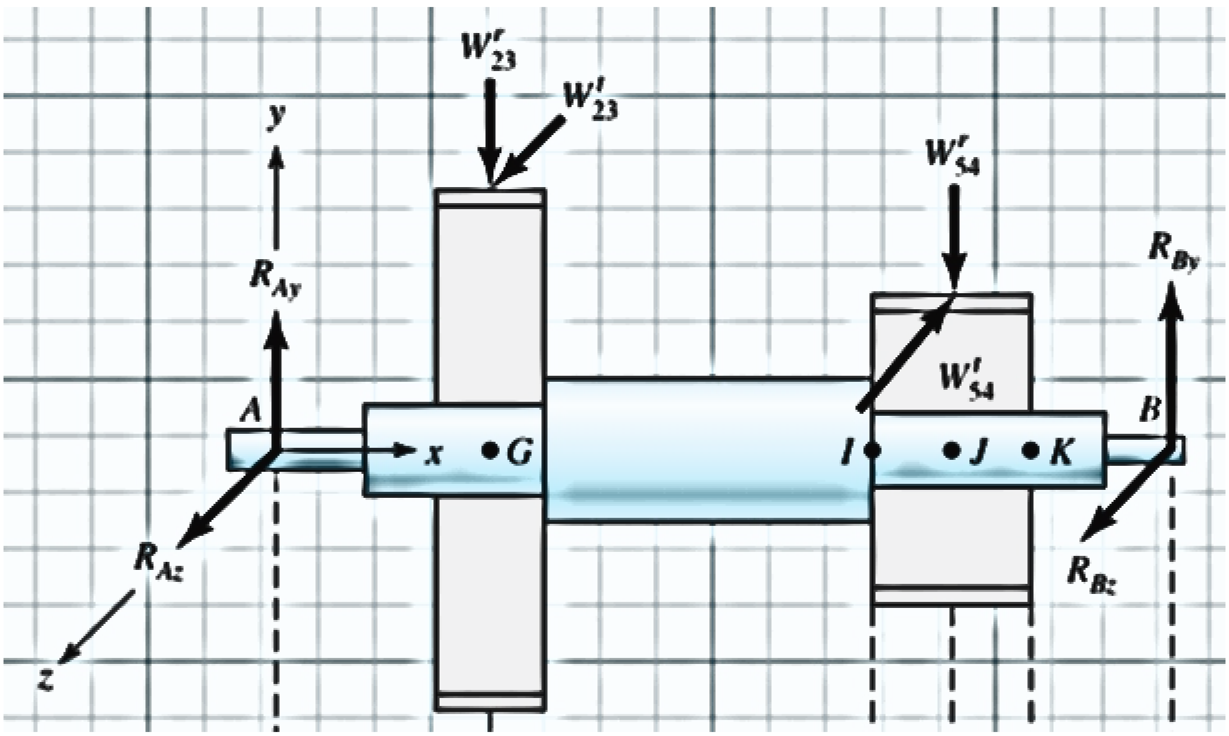
Example 7-1
Loading and design details are given. We are asked to, (1) determine n using DE-Goodman, DE-Gerber, DE-ASME, and DE-Soderberg; and (2) check against yielding failure by evaluating n_y

Example 7-2
Countershaft AB carries two spur gears at G and J; is supported by two bearings at A and B. Its layout is shown in Figure 7-10. Gear loads are

$$W_{23}^{r} = 197\ lb$$
$$W_{23}^{t} = 540\ lb$$
$$W_{54}^{r} = 885\ lb$$
$$W_{54}^{t} = 2431\ lb$$



We are to select appropriate materials and/or diameters at various cross sections, based on fatigue with infinite life. Design factor is 1.5.

The text starts with cross-section I where there are torque and bending moment, and a shoulder for stress concentration. Generous shoulder fillet (r/d = 0.1) is assumed DE-Goodman is used to determine diameters.

Example
A critical cross-section of a shat is subject to a combined bending moment of 63-lb-in and a torque of 74 lb-in. The cross-section is the seat of a rolling-element bearing, and sharp shoulder fillet is expected. Shaft material is SAE 1040 CD. Estimate the shaft's diameter at the cross-section for infinite life with $n? = 1.5$. Base calculations on ASME-Elliptic criterion. Operating conditions are typical.

Solution

(1) First iteration

$M_a = 63\ lb - in$
$T_m = 74\ lb - in$
$S_{ut} = 85\ ksi$
$S_y = 71\ ksi$
$S'_e = 42.5\ ksi$
$k_a = 2.7(85)^{-0.265} = 0.832$
$k_b = 0.9$
$k_c = 1$
$k_d = 1$
$k_e = 0.897$
$S_e = 28.55\ ksi$
$k_t = 2.7$
$k_{ts} = 2.2$

Set $q = q_{shear} = 1$, so that $K_f = K_t = 2.7$ and $K_{fs} = K_{ts} = 2.2$, and
$A = 2K_f M_a = 340.2\ lb - in$
$B = \sqrt{3} K_{fs} T_m = 282.0\ lb - in$

$$d = \sqrt[3]{\frac{16n}{\pi}\sqrt{\left(\frac{A}{S_e}\right)^2 + \left(\frac{B}{S_y}\right)^2}} = 0.458"$$

Round off $d = 12\ mm = 0.472"$

(2) Second iteration

$M_a = 63\ lb - in$
$T_m = 74\ lb - in$
$S_{ut} = 85\ ksi$
$S_y = 71\ ksi$
$S'_e = 42.5\ ksi$
$k_a = 2.7(85)^{-0.265} = 0.832$
$k_b = 0.879(0.472)^{-0.107} = 0.953$
$k_c = 1$
$k_d = 1$
$k_e = 0.897$
$S_e = 30.23\ ksi$

$\frac{r}{d} = 0.02$, then $r = 0.009$" , and $q = 0.57$, $q_{shear} = 0.6$, so that $K_f = 1.91$ and $K_{fs} = 1.66$

Finally,
$A = 2K_f M_a = 240.7\ lb - in$
$B = \sqrt{3} K_{fs} T_m = 212.8\ lb - in$

$$n = (\frac{\pi d^3}{16})(\frac{1}{\sqrt{\left(\frac{A}{S_e}\right)^2 + \left(\frac{B}{S_y}\right)^2}} = 2.4$$

$$\sigma'_{max} = \frac{16}{\pi d^3}\sqrt{A^2 + B^2} = 15.56\ ksi$$

$$n_y = \frac{S_y}{\sigma'_{max}} = 4.6$$

Therefore, $d = 12\ mm$ or $0.472"$ is sufficient, giving a factor of safety against fatigue at 2.4 and a factor of safety against yielding at 4.6.

(7-5) Deflection Consideration
Why deflection considerations?

- Shaft deflects transversely like a beam.
- Shaft also has torsional deflection like a torsion bar.
- Excessive deflections affect the proper functioning of gears and bearings, for example. Permissible slopes and transverse deflections are listed in Table 7-2.
- To minimize deflections, keep shaft short, and avoid cantilever or overhang.

How to Determine Beam Deflection

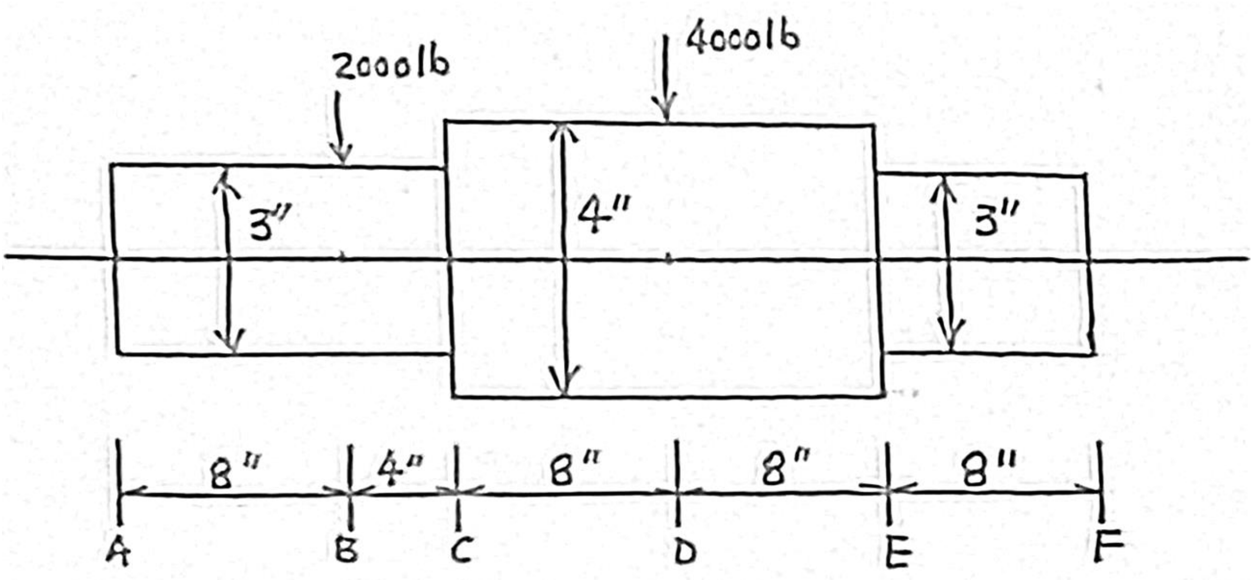
- Analytically
  Closed-form solutions (4-4)
  Superposition (4-5)
  Singularity Functions (4-6)
  Strain Energy (4-7)
  Castigliano's 2nd Theorem (4-8)
  Statically indeterminate beams (4-10)
  …..
  and so on.

Limitations: effective when EI=const. (but a stepped shaft does not have constant I)

- Numerical integration
  Simplified geometry; for example, small shoulders (diameter-wise and length-wise), fillets, keyways, notches, etc., can be omitted.
  May be tedious.

Example

A stepped shaft is shown below, which is supported by ball bearings at A and F. Determine its maximum (in magnitude) lateral deflection, and the slopes at A and F. Use $E = 30\ Mpsi$.



1) Plot bending moment $M(x)$

2) Plot M/d^4

   $$\therefore \frac{M}{EI} = \frac{\left(\frac{64}{\pi E}\right) M}{d^4} = kM/d^4$$

3) Integrate

   $$\frac{M}{d^4} => "slope"$$

4) Integrate "$slope$"

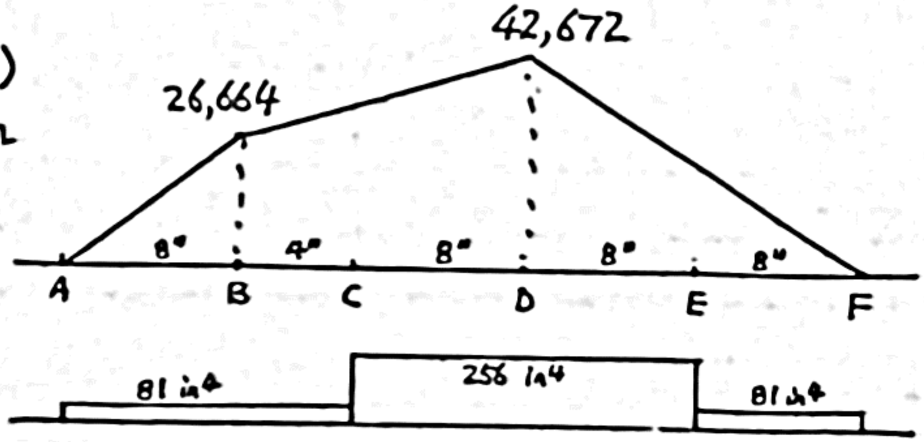
   $$\frac{M}{d^4} => "deflection"$$

5) Baseline

6) Deflection

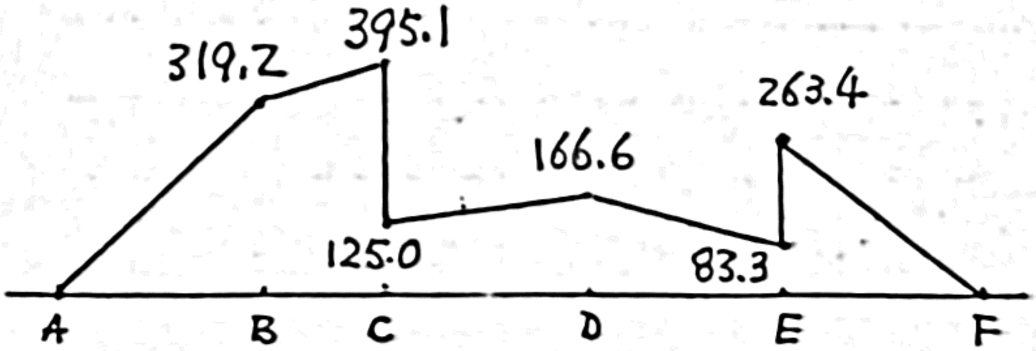
7) Slope

1) $M(x)$
lb.in

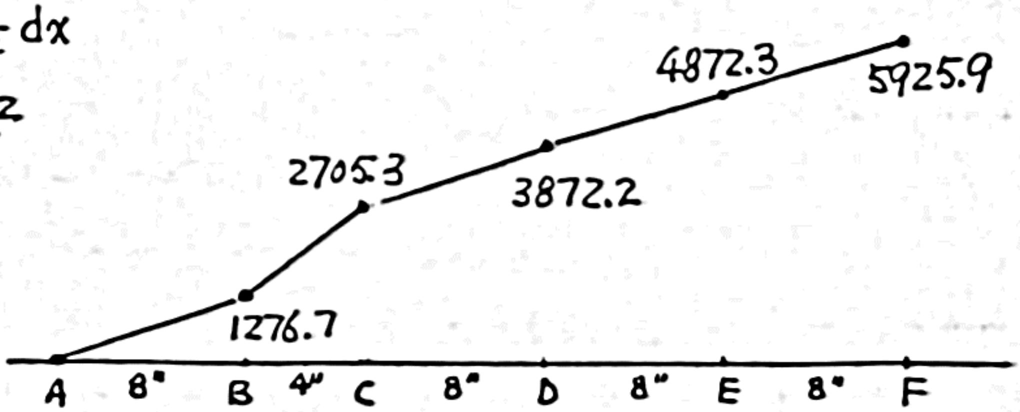


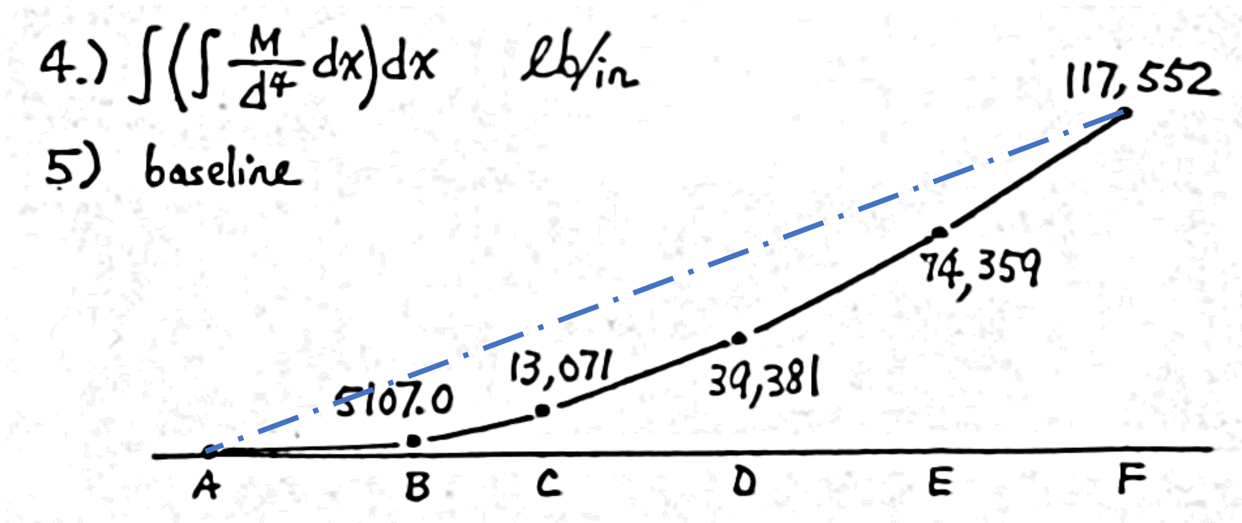
d^4

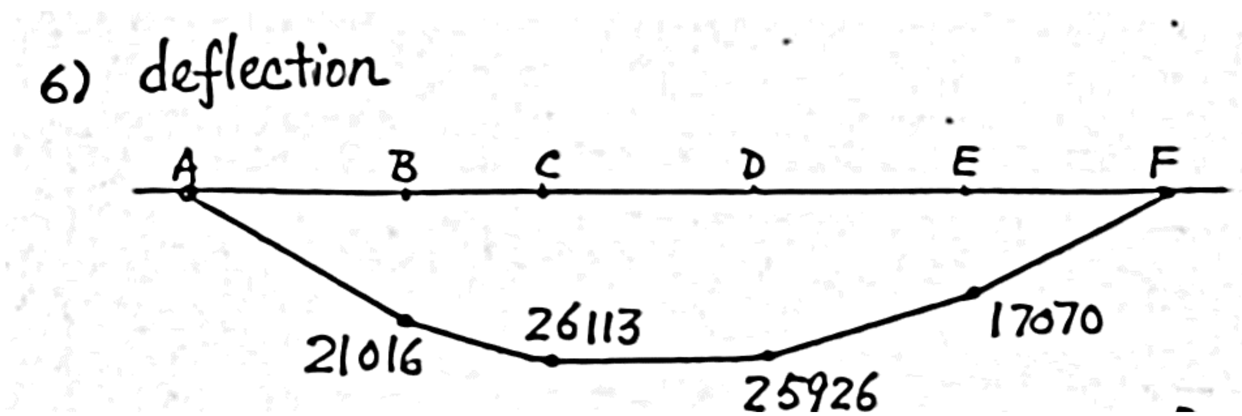
2) $\frac{M}{d^4}$
lb/in^3



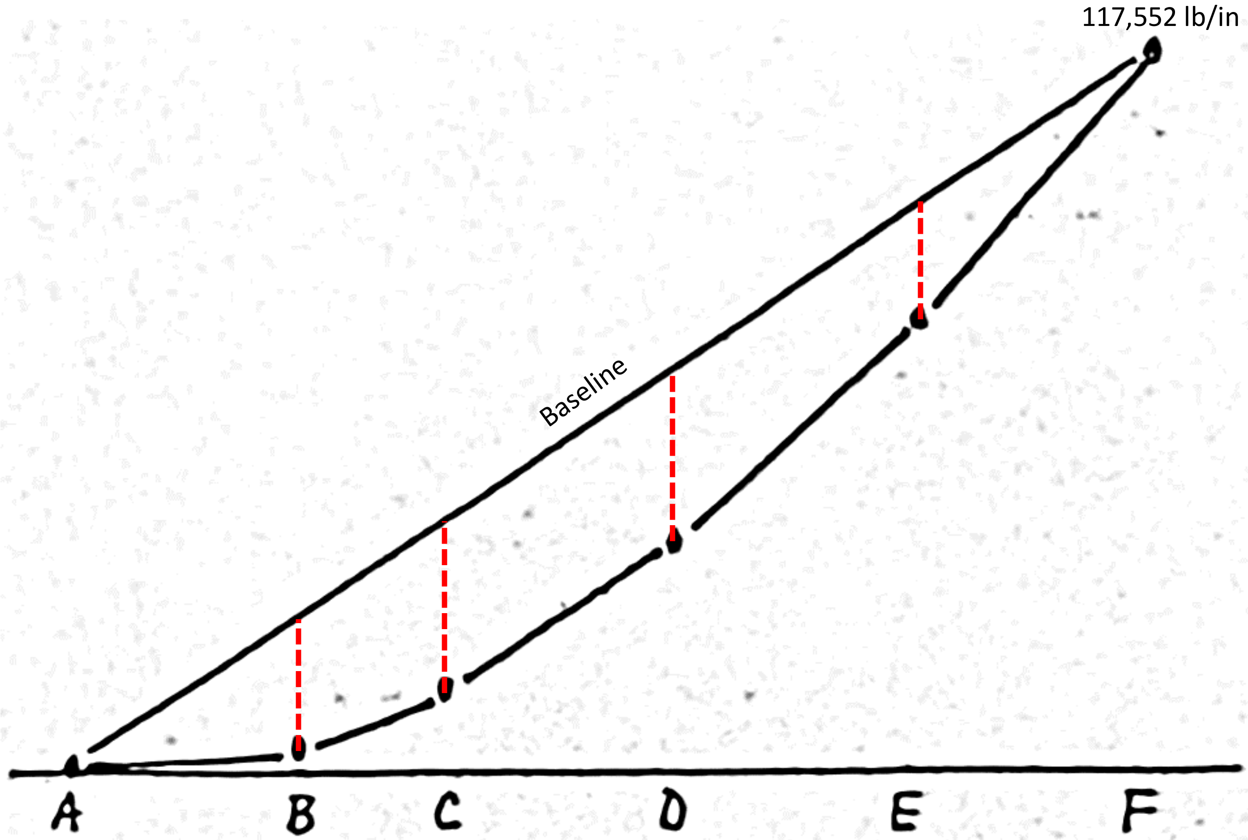
3) $\int \frac{M}{d^4} dx$
lb/in^2





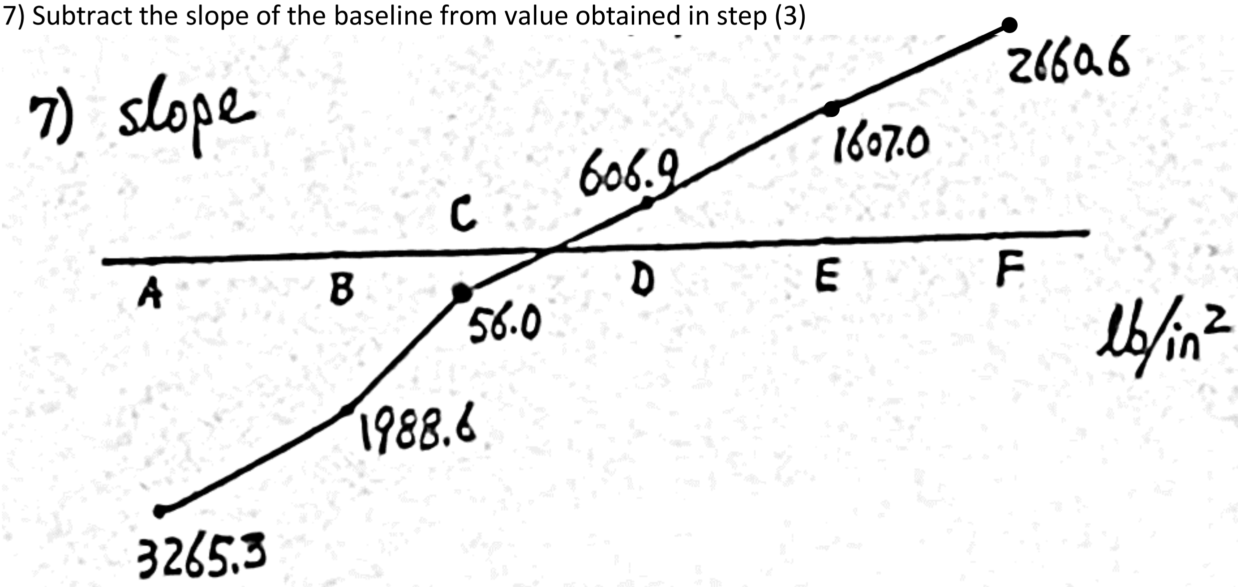


To obtain deflection curve of step (6), at any cross-section, subtract value obtained in step (4) from baseline value.
For example, at B, step (4) has 5107.0;
baseline value is 26,123;
$\therefore 21{,}106 = 26{,}123 - 5107.0$ will be used in step (6)



$$slope\ of\ baseline = m = \frac{117{,}552}{36"}\left(\frac{lb}{in}\right) = 3265.3\ lb/in^2$$

7) Subtract the slope of the baseline from value obtained in step (3)



$$k = \frac{64}{\pi E} = 0.6791 * 10^{-6} \left(\frac{in^2}{lb}\right)$$

$$\therefore \delta_{max} = \left(26{,}113 \frac{lb}{in}\right)\left(0.679 * 10^{-6} \frac{in^2}{lb}\right)$$

$$= 0.0177\ in\ \downarrow$$

$$\theta_A = \left(3265.3 \frac{lb}{in^2}\right)\left(0.679 * 10^{-6} \frac{in^2}{lb}\right)$$

$$= 0.00222\ rad\ \downarrow TODO\ curve$$

$$\theta_F = 0.00181\ rad\ \uparrow TODO\ curve$$

Comments regarding the numerical integration method:
Applicable to simple supports as outlined;
Fixed supports?
More divisions for better accuracy;
Vertical plane, horizontal plane, and vector sum.

An exact numerical method for determining the bending deflection and slope of stepped shafts, C.R. Mischke, in *Advanced in reliability and stress analysis*, ASME winter annual meeting, December 1978

Mechanical Design of Machine Elements and Machines, J.A. Collins, John Wiley & Sons, 2003 (Sec. 8.5)

Example (7-3)
By the end of Example 7-2, diameters D_1 through D_7 were determined. The layout is shown below (Figure 7-10). Here we are to evaluate the slopes and deflections at key locations.

The text uses “Beam 2D Stress Analysis” (a software with FEA-core) for the evaluation.

The results are verified by the above numerical integration method implemented with MATLAB.

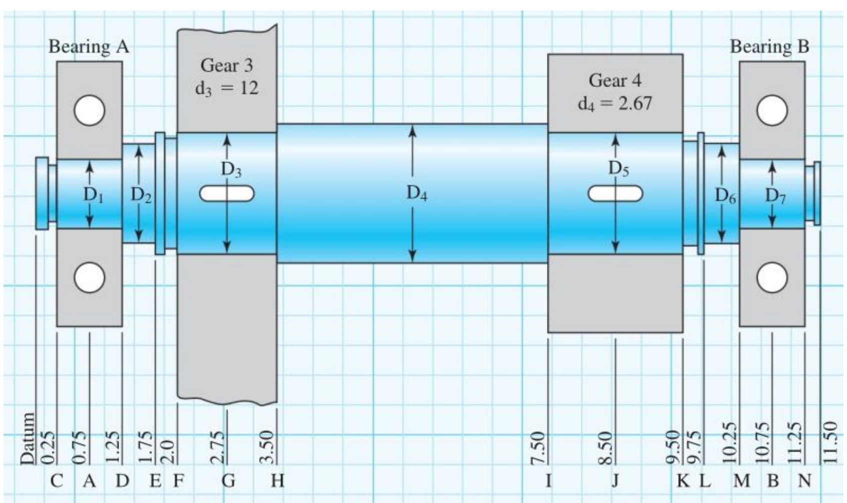


Figure 7–10
Shaft layout for Ex. 7–2. Dimensions in inches.

Diameter	$D_1 = D_7$	$D_2 = D_6$	$D_3 = D_5$	D_4
Example 7-2	1.0	1.4	1.625	2.0

Point of Interest	Example 7-3	Numerical Integration
Slope, left bearing (A)	0.000501 rad	0.000507 rad
Slope, right bearing (B)	0.001095 rad	0.001090 rad
Slope , left gear (G)	0.000414 rad	0.000416 rad
Slope, right gear (J)	0.000426 rad	0.000423 rad
Deflection, left gear (G)	0.0009155 in	0.0009201 in
Deflection, right gear (J)	0.0017567 in	0.0017691 in

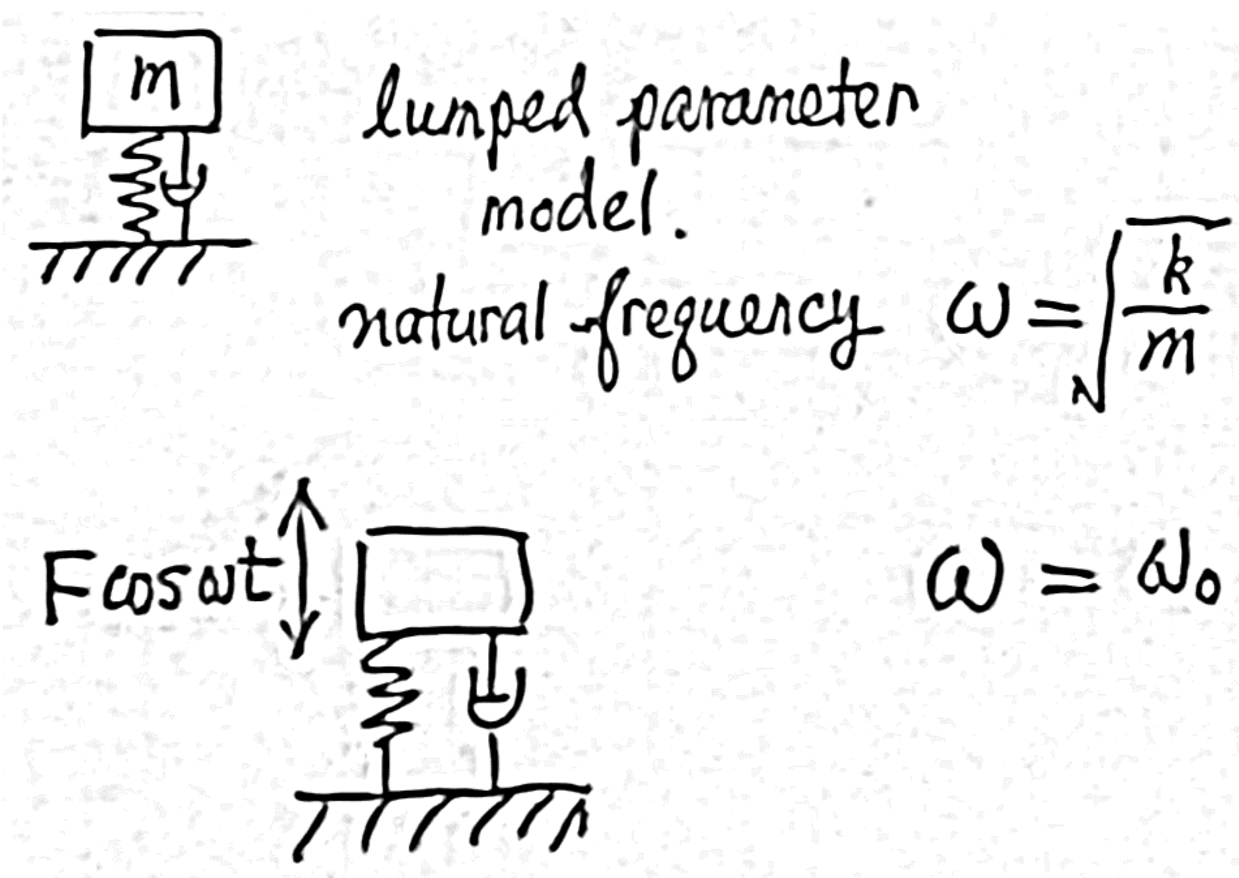
How to Determine Torsional (Angular) Deflection

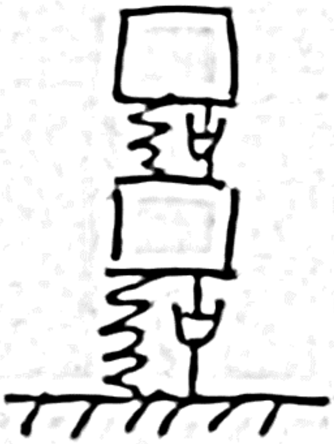
- Important for shafts carrying components that are required to function in sync with each other; for example, cam shafts;
- For a stepped shaft with individual cylinder length l_i, torque T_i and material G_i the angular deflection is,

$$\theta = \sum \theta_i = \sum \frac{T_i l_i}{G_i J_i} \qquad (7\text{–}19)$$

7-6 Critical Speeds for Shafts

It is about applying knowledge of vibrations and deflections of shafts in the design of shafts.

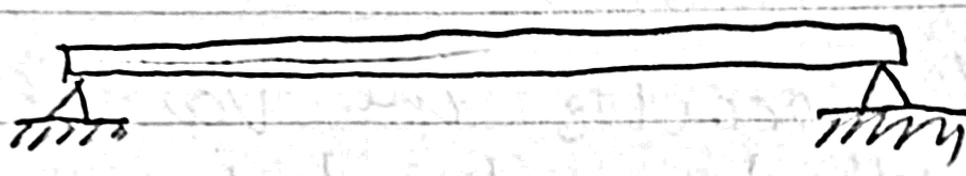




2 ω_1, ω_2

Continuum Model

∞ - DOF



$\omega_1, \omega_2, \ldots, \omega_n$

The organization of this section:

(Eq. 7-22): Exact solution of critical speed for a simply supported shaft with uniform cross section and material

$$\omega_1 = \left(\frac{\pi}{l}\right)^2 \sqrt{\frac{EI}{m}} = \left(\frac{\pi}{l}\right)^2 \sqrt{\frac{gEI}{A\gamma}} \qquad (7\text{–}22)$$

(eq. 7-23): Rayleigh's method to estimate critical speed.

$$\omega_1 = \sqrt{\frac{g \sum w_i y_i}{\sum w_i y_i^2}} \qquad (7\text{–}23)$$

(Eq. 7-24): Through (Eq. 7-32) Derivation of Dunkerley's method to estimate critical speed.

(see pp. 376-377)

Example (7-5)

Notes: (a) Rayleigh's and Dunkerley's methods only give rise to estimates; (b) They yield the upper and lower bound solutions, respectively. That is, $\omega_{1(Dunkerley)} < \omega_1 < \omega_{1(Rayleigh)}$; (c) The methods and their derivations fall under vibrations/dynamics of continuum by energy method.

Critical Speeds

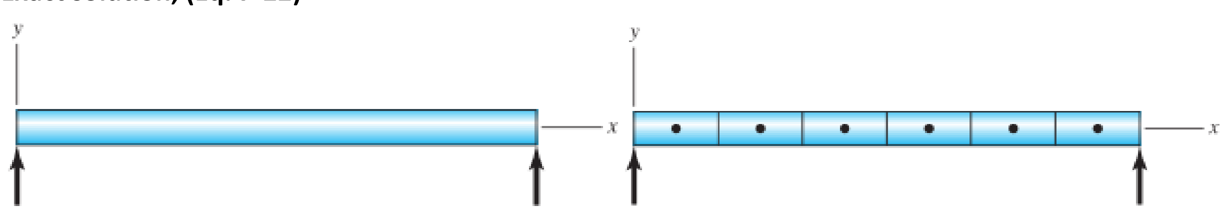
- Critical speeds refer to speeds at which the shaft becomes unstable, such that deflections (due to bending or torsion) increase without bound.
- A critical speed corresponds to the fundamental natural frequency of the shaft in a particular vibration mode.
- Three shaft vibration modes are to be concerned: lateral, vibration, shaft whirling and torsional vibration.
- Critical speeds for lateral vibration and shaft whirling are identical.
- Numerically speaking, two critical speeds can be determined, one for lateral vibration or shaft whirling, and another for torsional vibration.
- Focus will be the critical speed for lateral vibration or shaft whirling. Regarding critical speed for torsional vibration, one can reference "rotor dynamics" and the transfer matrix method.
- If the critical speed is ω_1, it is required that the operating speed ω be:

  If the shaft is rigid (shafts in heavy machinery):
  $$\frac{\omega}{\omega_1} \geq 3$$

  If the shaft is flexible (shafts that are long with small-diameters):
  $$\frac{\omega}{\omega_1} \leq 1/3$$

  The text recommends:
  $$\frac{\omega}{\omega_1} \leq 1/2$$

Exact solution, (Eq. 7-22)

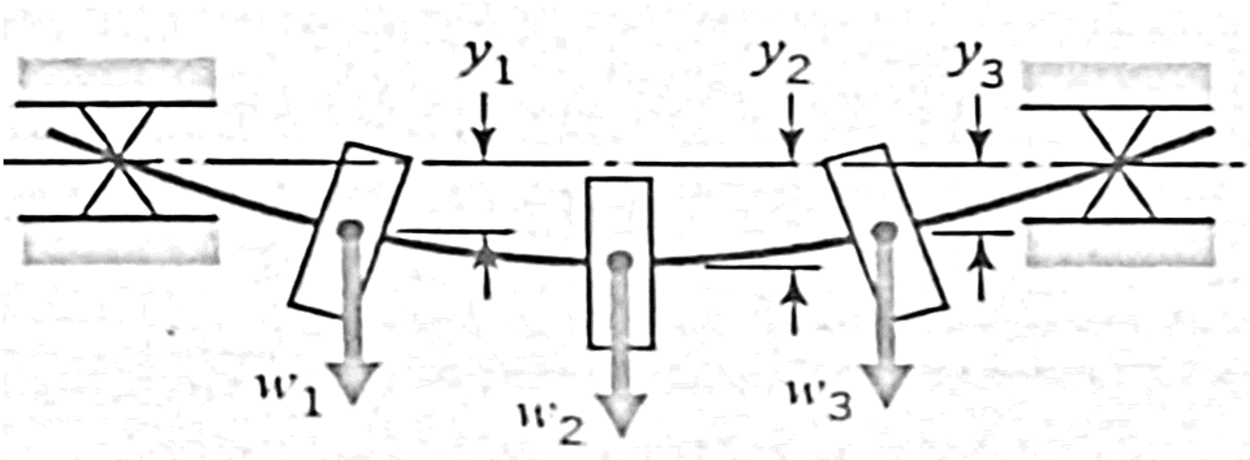


(a) (b)

Simply supported, uniform cross-section and material.

$$\omega_1 = \left(\frac{\pi}{l}\right)^2 \sqrt{\frac{EI}{m}} = \left(\frac{\pi}{l}\right)^2 \sqrt{\frac{gEI}{A\gamma}} \quad (7\text{–}22)$$

Rayleigh's Method for Critical Speed, (Eq. 7-23)
Shaft is considered massless and flexible; Components such as gears, pulleys, flywheels, and so on, are treated as lumped masses; The weight of the shaft, if significant, will be lumped as a mass or masses.



$$\omega_1 = \sqrt{\frac{g \sum w_i y_i}{\sum w_i y_i^2}} \qquad (7\text{–}23)$$

Textbook equation should be updated to include the following absolute symbols:

$$\omega_1 = \sqrt{g \frac{\sum w_i |y_i|}{\sum w_i y_i^2}}$$

Where:
w_i = weight of mass i
w_i should be treated as a force with a magnitude equation the weight of mass i;

Forces w_i ($i = 1, \dots$) should be applied in such a way that the deflection curve resembles the fundamental mode shape of lateral vibration.

y_i = lateral deflection at location i (where w_i is applied) and caused by all forces.

Dunkerley's Method for Critical Speed, (Eq. 7-32)
The model for Dunkerley's method is the same as that for Rayleigh's.

$$\frac{1}{\omega_1^2} \approx \sum_{i=1}^{n} \frac{1}{\omega_{ii}^2} \qquad (7\text{–}32)$$

and $\omega_{ii} = \sqrt{\frac{g}{|y_{ii}|}}$

Where y_{ii} is the lateral deflection at location i and caused by w_i only. ω_{ii} represents the critical speed with only w_i on the shaft.

Deflections y_i y_{ij} and δ_{ij}

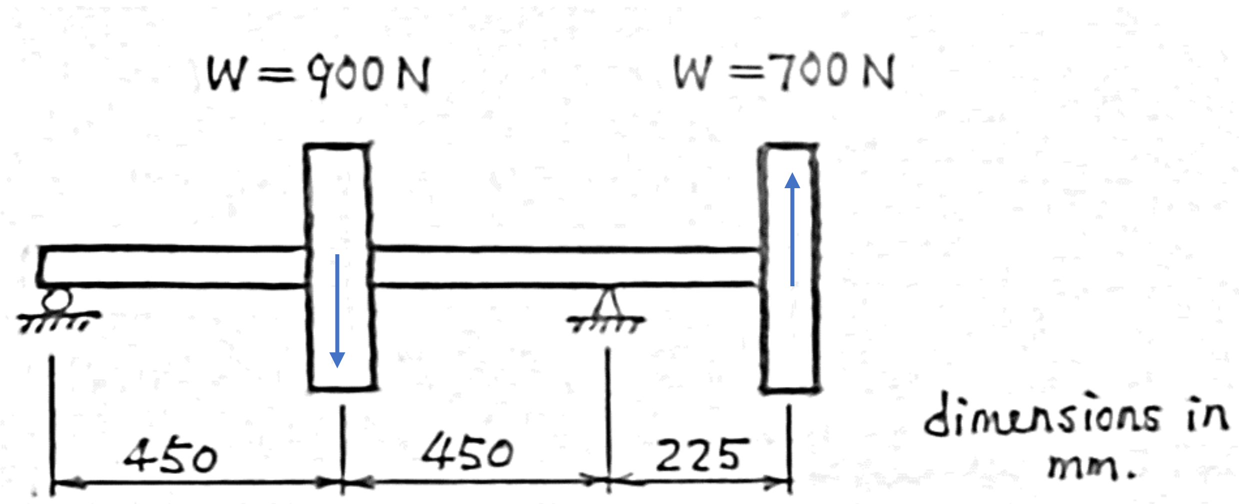
y_{ij} is the deflection at location i and due to a load applied at location j. When the load at location j is a unit load, then y_{ij} is denoted by δ_{ij}. δ_{ij} is also known as the influence coefficient.

y_i is the deflection at location i and caused by all applied loads. Therefore,

$$y_i = \sum_j y_{ij}$$

unit load, then Type equation here.unit load, then y_{ij} is denoted at δ_{ij}. δij**Problem Solving**
Closed-form solutions (Sec. 4-4 and Table A-9).
Superposition (Section 4-5);
Indexes i and $j$each runs 1 through the number of masses/forces;
y_{ij} are signed numbers;

Example:
Evaluate the range of the shaft's critical speed corresponding to its lateral vibration, in terms of EI where $EI = constant$.



$i = 1,2$
$j = 1,2$