$$[M]\ddot{\vec{x}} + [K]\vec{x} = \emptyset$$

$\vec{x}$: the physical coordinates

Transformation :

$$\vec{x} = [M]^{-1/2}\vec{q}$$ not physical coordinate (general coordinate)

$$\longrightarrow \ddot{\vec{q}} + [\bar{K}]\vec{q} = \emptyset$$

Here $[\bar{K}] = [M]^{-1/2}[K][M]^{-1/2}$

Free vibration $\vec{q}(t) = \vec{v}e^{i\omega t}$

$$\longrightarrow ([\bar{K}] - \omega^2[I])\vec{v} = \emptyset$$

$$[I] = diag(1)$$

$$[\bar{K}]\vec{v} = \omega^2\vec{v}$$

$\vec{v}$ → eigenvector

ω^2 → eigenvalue

* Real eigenvalue & real eigenvector
* Eigenvalues are positive if and only if $[\bar{K}]$ is positive definite
* The set of eigenvectors can be chosen to be orthogonal

$$\vec{x} = \begin{Bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{Bmatrix}, \quad \vec{y} = \begin{Bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{Bmatrix}$$

Inner product

$$\vec{x}^T \cdot \vec{y} = \underset{1\times n}{(x_1, x_2, \dots, x_n)} \underset{n\times 1}{\begin{Bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{Bmatrix}} = x_1y_1 + x_2y_2 + \dots + x_ny_n$$

Same

results in 1×1

Norm :

$$\|\vec{x}\| = \sqrt{\vec{x}^T \cdot \vec{x}} = \sqrt{x_1^2 + x_2^2 + \dots + x_n^2}$$

Vector :

$$\frac{1}{\|x\|}\vec{x}$$ ← normal vector

$\vec{x}$ and $\vec{y}$ orthogonal $\vec{x}^T\vec{y} = \emptyset$

$\boxed{\text{Example :}}$ Normalize $\vec{x} = \begin{Bmatrix} 1 \\ 2 \\ 2 \end{Bmatrix}$

$$\vec{x}^T \cdot \vec{x} = 1^2 + 2^2 + 2^2 = 9$$

$$\therefore \|\vec{x}\| = \sqrt{\vec{x}^T \cdot \vec{x}} = \sqrt{9} = 3$$

$\therefore$ the normalized vector of $\vec{x}$ is

$$\frac{1}{\|\vec{x}\|}\vec{x} = \left(\frac{1}{3}\right)\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 1/3 \\ 2/3 \\ 2/3 \end{pmatrix}$$

$\boxed{\text{Example :}}$ $[M] = \begin{bmatrix} 9 & \emptyset \\ \emptyset & 1 \end{bmatrix}$; $[K] = \begin{bmatrix} 27 & -3 \\ -3 & 3 \end{bmatrix}$

Since $[M]^{1/2} = \begin{bmatrix} 3 & \emptyset \\ \emptyset & 1 \end{bmatrix} \longrightarrow [M]^{-1/2} = \begin{bmatrix} 1/3 & \emptyset \\ \emptyset & 1 \end{bmatrix}$

$$\therefore [\bar{K}] = [M]^{-1/2}[K][M]^{-1/2}$$

$$= \begin{bmatrix} 1/3 & \emptyset \\ \emptyset & 1 \end{bmatrix}\begin{bmatrix} 27 & -3 \\ -3 & 3 \end{bmatrix}\begin{bmatrix} 1/3 & \emptyset \\ \emptyset & 1 \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix}$$

Eigenvalues and eigenvectors

$$([\bar{K}] - \lambda[I])\vec{v} = \emptyset$$

$$\longrightarrow \begin{pmatrix} 3-\lambda & -1 \\ -1 & 3-\lambda \end{pmatrix}\vec{v} = \emptyset$$

$$\det\begin{pmatrix} 3-\lambda & -1 \\ -1 & 3-\lambda \end{pmatrix} = \emptyset$$

$$(3-\lambda)^2 - (-1)^2 = \emptyset$$

$(3-\lambda) = (-1)$ or $(3-\lambda) = 1$

$\lambda_1 = 2$ $\qquad$ $\lambda_2 = 4$

For $\lambda_1 = 2$

$$\begin{pmatrix} 3-2 & -1 \\ -1 & 3-2 \end{pmatrix}\begin{Bmatrix} v_{11} \\ v_{21} \end{Bmatrix} = \emptyset$$

$$v_{11} - v_{21} = \emptyset$$

$$\longrightarrow \vec{v}_1 = \begin{Bmatrix} v_{11} \\ v_{21} \end{Bmatrix} = \alpha\begin{Bmatrix} 1 \\ 1 \end{Bmatrix} \qquad \alpha \neq \emptyset$$

$$\vec{v}_1^T \cdot \vec{v}_1 = (\alpha \ \alpha)\begin{pmatrix}\alpha \\ \alpha\end{pmatrix} = 2\alpha^2 = 1$$

$$\alpha = \frac{1}{\sqrt{2}} \quad \text{or} \quad \alpha = \frac{-1}{\sqrt{2}}$$

$$\therefore \quad \vec{v}_1 = \frac{1}{\sqrt{2}}\begin{pmatrix}1 \\ 1\end{pmatrix} = \begin{pmatrix}1/\sqrt{2} \\ 1/\sqrt{2}\end{pmatrix}$$

For $\lambda_2 = 4$

$$\vec{v}_2 = \frac{1}{\sqrt{2}}\begin{pmatrix}1 \\ -1\end{pmatrix} = \begin{pmatrix}1/\sqrt{2} \\ -1/\sqrt{2}\end{pmatrix}$$

Since $\vec{v}_1^T \vec{v}_2 = (1/\sqrt{2})^2 - (1/\sqrt{2})^2 = \emptyset$

↳ means they're perpendicular to each other

Mode shape :

$$\begin{cases} \vec{u}_1 = [M]^{-1/2}\, \vec{v}_1 \\ \vec{u}_2 = [M]^{-1/2}\, \vec{v}_2 \end{cases}$$

Define :

$$[P] = [\vec{v}_1 \ \vec{v}_2]$$

Since :

$$[P^T][P] = \begin{bmatrix}\vec{v}_1^T \\ \vec{v}_2^T\end{bmatrix}\begin{bmatrix}\vec{v}_1 & \vec{v}_2\end{bmatrix}$$

$$= \begin{bmatrix}\vec{v}_1^T\vec{v}_1 & v_1^T v_2 \\ \vec{v}_2^T\vec{v}_1 & v_2^T v_2\end{bmatrix} = \begin{bmatrix}1 & \emptyset \\ \emptyset & 1\end{bmatrix} = [I]$$

Then $[P^T][P] = [I]$

$[P]$: orthogonal matrix

$[P]^{-1} = [P]^T$

$$[P]^T[\bar{K}][P] = [P]^T[\bar{K}][\vec{v}_1\ \vec{v}_2]$$
$$= [P]^T[\bar{K}\vec{v}_1\ \ \bar{K}\vec{v}_2]$$

Since $[\bar{K}]\vec{v}_1 = \lambda_1\vec{v}_1$

$[\bar{K}]\vec{v}_2 = \lambda_2\vec{v}_2$

$$\rightarrow [P]^T[\bar{K}][P] = \begin{bmatrix} \vec{v}_1^{\,T} \\ \vec{v}_2^{\,T} \end{bmatrix}[\lambda_1\vec{v}_1\ \ \lambda_2\vec{v}_2]$$
$$= \begin{bmatrix} \lambda_1\vec{v}_1^{\,T}\vec{v}_1 & \lambda_2\vec{v}_1^{\,T}\vec{v}_2 \\ \lambda_1\vec{v}_2^{\,T}\vec{v}_1 & \lambda_2\vec{v}_2^{\,T}\vec{v}_2 \end{bmatrix}$$
$$= \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = [\Lambda]$$

Mode shape :

$$[M]\ddot{\vec{x}} + [K]\vec{x} = 0$$
$$\vec{x} = \vec{u}e^{i\omega t}$$
$$\rightarrow (-[M]\omega^2 + [K])\vec{u} = 0$$
$$\omega_m \ \&\ \vec{u}_m$$

$$(-[M]\omega_m^2 + [K])\vec{u}_m = 0$$
$$\rightarrow \vec{u}_m^{\,T}(-[M]\omega_m^2 + [K])\vec{u}_m = 0$$
$$\rightarrow -\vec{u}_m^{\,T}[M]\vec{u}_m \cdot \omega_m^2 + \vec{u}_m^{\,T}[K]\vec{u}_m = 0$$
$$\omega_m^2 = \frac{\vec{u}_m^{\,T}[K]\vec{u}_m}{\vec{u}_m^{\,T}[M]\vec{u}_m}$$

(For single degree of Freedom :
$$\omega^2 = \frac{K}{m} = \frac{(1/2)KA^2}{(1/2)mA^2} = \frac{(1/2)\cdot A\cdot K\cdot A}{(1/2)\cdot A\cdot m\cdot A}$$
)

Mass normalization of the mode shape :

$$\vec{u}_m^{\,T}[M]\vec{u}_m = 1$$

Then :

$$\omega_m^2 = \vec{u}_m^{\,T}[K]\vec{u}_m$$

4.3 Modal Analysis

$$\begin{cases} [M]\ddot{\vec{x}} + [K]\vec{x} = \emptyset \\ \vec{x}(0) = \vec{x}_0 \,, \quad \dot{\vec{x}}(0) = \dot{\vec{x}}_0 \end{cases}$$

$\rightarrow$ Transformation #1 :

$$\vec{x} = [M]^{-1/2}\vec{q}$$

$$\rightarrow \quad \ddot{\vec{q}} + [\bar{K}]\vec{q} = \emptyset$$

The eigenvector matrix $[P]$

$\rightarrow$ Transformation #2 :

$$\vec{q} = [P]\vec{r}$$

$$\rightarrow \quad [P]\ddot{\vec{r}} + [\bar{K}][P]\vec{r} = \emptyset$$

$$\rightarrow \quad [P]^T[P]\ddot{\vec{r}} + [P]^T[\bar{K}][P]\vec{r} = \emptyset$$

$$\rightarrow \quad \ddot{\vec{r}} + [\Lambda]\vec{r} = \emptyset$$

Here $[\Lambda] = \begin{bmatrix} \lambda_1 & \emptyset \\ \emptyset & \lambda_2 \end{bmatrix} = \begin{bmatrix} \omega_1^2 & \emptyset \\ \emptyset & \omega_2^2 \end{bmatrix}$

$\hookrightarrow$ For 2 DOF

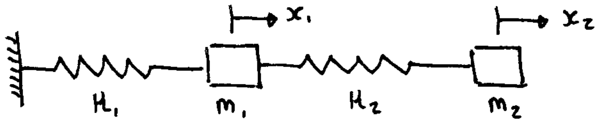
$\hookrightarrow$ (For 3 DOF, matrix would be 3×3 w/ λ_3)

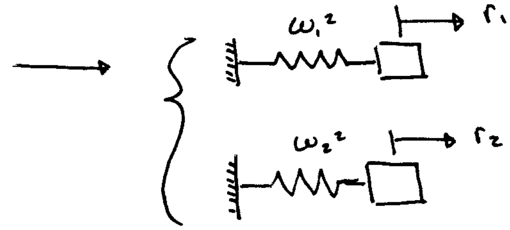
$$\vec{r} = \begin{Bmatrix} r_1(t) \\ r_2(t) \end{Bmatrix}$$

$$\rightarrow \quad \begin{Bmatrix} \ddot{r}_1 \\ \ddot{r}_2 \end{Bmatrix} + \begin{bmatrix} \omega_1^2 & \emptyset \\ \emptyset & \omega_2^2 \end{bmatrix} \begin{Bmatrix} r_1 \\ r_2 \end{Bmatrix} = \emptyset$$

$$\rightarrow \quad \begin{cases} \ddot{r}_1 + \omega_1^2 r_1 = \emptyset \\ \ddot{r}_2 + \omega_2^2 r_2 = \emptyset \end{cases}$$

$\vec{r} = \begin{Bmatrix} r_1 \\ r_2 \end{Bmatrix}$ modal coordinates





$$\vec{x} = \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = [M]^{-1/2}[P]\vec{r}$$

$$\vec{r} = \begin{Bmatrix} r_1 \\ r_2 \end{Bmatrix}$$

$$[M]\ddot{\vec{x}} + [K]\vec{x} = 0$$

$$\vec{x} = (x_1, x_2, \ldots, x_n)^T \leftarrow \text{coupled}$$

$$\boxed{\vec{x} = [M]^{-1/2}\vec{q}}$$

$$\longrightarrow \ddot{\vec{q}} + [\tilde{K}]q = 0 \qquad [\tilde{K}] = [M]^{-1/2}[K][M]^{-1/2}$$

$$([\tilde{K}] - \lambda[I])\vec{v} = 0$$

$$\lambda_m = \omega_m^2, \quad \vec{v}_m, \quad m = 1, 2, \ldots, n$$

$$[P] = [\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m]$$

$$[P]^T[P] = [I]$$

$$\boxed{\vec{q} = [P]\vec{r}}$$

$$\longrightarrow \ddot{\vec{r}} + [\Lambda]\vec{r} = 0 \leftarrow \text{decoupled}$$

$$[\Lambda] = \operatorname{diag}(\omega_m^2) = [P]^T[\tilde{K}][P]$$

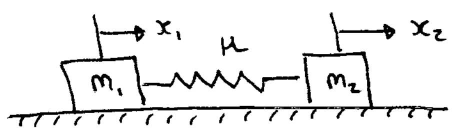
$$\longrightarrow \ddot{r}_1 + \omega_1^2 r_1 = 0$$

$$\ddot{r}_2 + \omega_2^2 r_2 = 0$$

$$\downarrow \quad \vdots \quad \downarrow$$

$$\ddot{r}_n + \omega_n^2 r_n = 0$$

Example



$m_1 = 1\text{ kg}$

$m_2 = 4\text{ kg}$

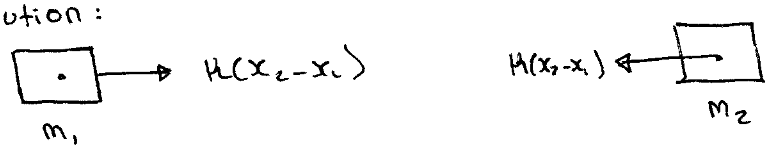
$k = 400\text{ N/m}$

$$\vec{x}(0) = \begin{bmatrix} 0.01 \\ 0 \end{bmatrix} \text{ m}$$

$$\dot{\vec{x}}(0) = 0$$

Find the response of the system.

Solution:



$\underline{m_1:}$ $\quad m_1\ddot{x}_1 = k(x_2 - x_1)$

$\underline{m_2:}$ $\quad m_2\ddot{x}_2 = -k(x_2 - x_1)$

$$\longrightarrow \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} k & -k \\ -k & k \end{bmatrix}\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = 0$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} 400 & -400 \\ -400 & 400 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = 0$$

$$[M]^{1/2} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \; ; \quad [M]^{-1/2} = \begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix} \begin{bmatrix} 400 & -400 \\ -400 & 400 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix}$$

$$= \begin{bmatrix} 400 & -200 \\ -200 & 100 \end{bmatrix}$$

$$\det([\bar{K}] - \lambda[I]) = \begin{vmatrix} 400-\lambda & -200 \\ -200 & 100-\lambda \end{vmatrix} = 0$$

$$\lambda_1 = 0 \qquad \lambda_2 = 500$$

$$\omega_1 = \sqrt{\lambda_1} = 0 \qquad \omega_2 = \sqrt{500} = 22.36 \text{ rad/s} \quad (\text{natural freq.})$$

$$\lambda_1 = 0 : \begin{bmatrix} 400-\lambda_1 & -200 \\ -200 & 100-\lambda_1 \end{bmatrix} \begin{Bmatrix} v_{11} \\ v_{12} \end{Bmatrix} = 0$$

$$\vec{v}_1 = \begin{Bmatrix} v_{11} \\ v_{21} \end{Bmatrix} = \begin{Bmatrix} 1 \\ 2 \end{Bmatrix}$$

$$\|\vec{v}\| = \sqrt{\vec{v}_1^T \vec{v}_1} = \sqrt{1^2 + 2^2} = \sqrt{5}$$

Normalized: $\vec{v}_1 = (1/\sqrt{5}) \begin{Bmatrix} 1 \\ 2 \end{Bmatrix} = \begin{Bmatrix} 0.4472 \\ 0.8944 \end{Bmatrix}$

$$\lambda_2 = 500 : \begin{bmatrix} 400-\lambda_2 & -200 \\ -200 & 100-\lambda_2 \end{bmatrix} \begin{Bmatrix} v_{12} \\ v_{22} \end{Bmatrix} = 0$$

$$\vec{v}_2 = \begin{Bmatrix} -2 \\ 1 \end{Bmatrix}$$

Normalized: $\vec{v}_2 = (1/\sqrt{5}) \begin{Bmatrix} -2 \\ 1 \end{Bmatrix} = \begin{Bmatrix} -0.8944 \\ 0.4472 \end{Bmatrix}$

$$[P] = [\vec{v}_1 \; \vec{v}_2]$$

$$= \begin{bmatrix} 0.4472 & -0.8944 \\ 0.8944 & 0.4472 \end{bmatrix}$$

Verify: $[P]^T[P] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$; $[P]^T[\bar{K}][P] = \begin{bmatrix} 0 & 0 \\ 0 & 500 \end{bmatrix}$

$$\underbrace{\vec{x} = [M]^{-1/2}\vec{q}} = [M]^{-1/2}[P]\vec{r}$$

$$[S] = [M]^{-1/2}[P] = \begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix}\begin{bmatrix} 0.4472 & -0.8944 \\ 0.8944 & 0.4472 \end{bmatrix}$$

$$= \begin{bmatrix} 0.4472 & -0.8944 \\ 0.4472 & 0.2236 \end{bmatrix}$$

$$(\ddot{\vec{r}} + [\Lambda]\vec{r} = 0)$$

$$r(0) = [S]^{-1}\vec{x}(0) = \begin{bmatrix} 0.4472 & 1.7889 \\ -0.8944 & 0.8944 \end{bmatrix}\begin{Bmatrix} 0.01 \\ 0 \end{Bmatrix}$$

$$= \begin{Bmatrix} 0.004472 \\ -0.008944 \end{Bmatrix}$$

$$r_1(0) = 0.004472 \quad \& \quad r_2(0) = -0.008944$$

$$\dot{\vec{r}}(0) = [S]^{-1}\dot{\vec{x}}(0) = 0$$

$$\dot{r}_1(0) = 0, \quad \dot{r}_2(0) = 0$$

$$\ddot{r}_1 + \omega_1^2 r_1 = 0 \qquad (\omega_1 = 0)$$

$$\ddot{r}_1 = 0$$

$$\Rightarrow r_1(t) = a + bt$$

$$\Rightarrow r_1(t) = 0.004472$$

$$\ddot{r}_2 + \omega_2^2 r_2 = 0 \qquad (\omega_2 = 500)$$

$$\Rightarrow r_2(t) = -0.0089\cos(22.36t)$$

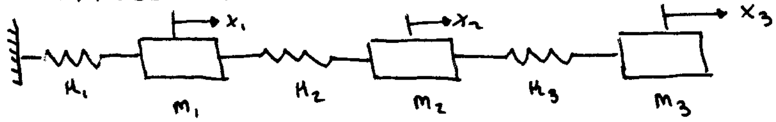
$$\therefore \vec{x} = [S]\vec{r}$$

$$= \begin{bmatrix} 0.4472 & -0.8944 \\ 0.4472 & 0.2236 \end{bmatrix}\begin{Bmatrix} 0.004472 \\ -0.0089\cos(22.36t) \end{Bmatrix}$$

$$\Rightarrow \vec{x} = \begin{Bmatrix} 2.012 + 7.60\cos(22.36t) \\ 2.012 - 1.990\cos(22.36t) \end{Bmatrix} \times 10^{-3}$$

Influence coefficients

Stiffness influence coefficients



k_{ij} : the force at point i due to a unit displacement at point j when all the other points other than j are fixed.

k_{11}:

If point j has displacement x_j ($j = 1, 2, 3$), then the force at point i :

$$F_i = k_{i1} x_1 + k_{i2} x_2 + k_{i3} x_3 = \sum_{j=1}^{3} k_{ij} x_j$$

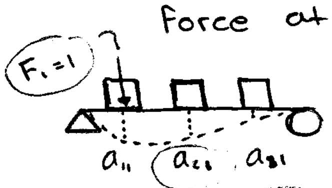
$i = 1, 2, 3$

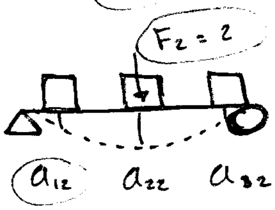
$$\rightarrow \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix} = \begin{bmatrix} k_{11} & \boxed{k_{12}} & k_{13} \\ \boxed{k_{21}} & k_{22} & \underline{\underline{k_{23}}} \\ k_{31} & \underline{\underline{k_{32}}} & k_{33} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}$$

→ symmetric matrix

Flexibility Influence Coefficients

a_{ij} : the displacement at point i due to a unit force at point j





Flexibility Influence Matrix

$$[A] = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

The relationship: $[K][A] = [I]$