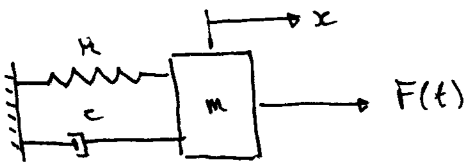
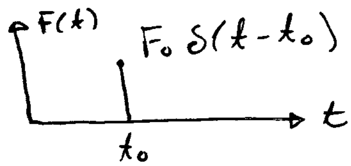


$$\begin{cases} m\ddot{x} + c\dot{x} + kx = F(t) \\ t = 0 : \ x(0) = x_0 , \quad \dot{x}(0) = v_0 \end{cases}$$

$F(t) = F_0 \cos(\omega t)$

(Constant Force)

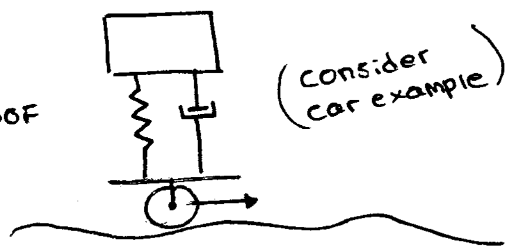
(Impulsive Force)



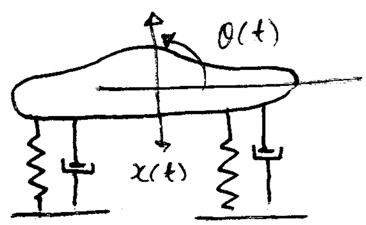
(Variable Force)

Chapter 4 : Multiple Degree of Freedom Systems

1 DOF



(consider car example)

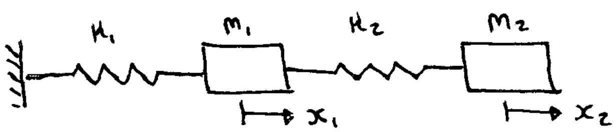


2 DOF

DOF's :

$$\underbrace{1 \rightarrow 2}_{\text{Introduce concepts } (1 \rightarrow 2)} \underbrace{\rightarrow 3 \rightarrow \cdots \rightarrow n}_{\text{Same concepts } (2 \rightarrow n)} \underbrace{\rightarrow \cdots \rightarrow \infty}_{\text{Introduce concepts } (n \rightarrow \infty)}$$

4.1 : 2 DOF Model



$$k_1 x_1 \longleftarrow \boxed{m_1} \longrightarrow k(x_2 - x_1)$$

$$(\xrightarrow{+ve}) \quad m_1 \ddot{x}_1 = -k_1 x_1 + k(x_2 - x_1)$$

$$\longrightarrow m_1 \ddot{x}_1 + (k_1 + k_2) x_1 - k_2 x_2 = 0$$

$$k_2(x_2 - x_1) \longleftarrow \boxed{m_2}$$

$$(\xrightarrow{+ve}) \quad m_2 \ddot{x}_2 = -k_2(x_2 - x_1)$$

$$\longrightarrow m_2 \ddot{x}_2 - k_2 x_1 + k_2 x_2 = 0$$

Initial conditions:

$$x_1(0) = x_{10}, \qquad \dot{x}_1(0) = \dot{x}_{10}$$

given numbers

$$x_2(0) = x_{20}, \qquad \dot{x}_2(0) = \dot{x}_{20}$$

Define:

$$\vec{x}(t) = \begin{Bmatrix} x_1(t) \\ x_2(t) \end{Bmatrix} = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

$$\dot{\vec{x}}(t) = \begin{Bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{Bmatrix} \qquad \ddot{\vec{x}}(t) = \begin{Bmatrix} \ddot{x}_1(t) \\ \ddot{x}_2(t) \end{Bmatrix}$$

$$\begin{cases} m_1 \ddot{x}_1 + (k_1 + k_2) x_1 - k_2 x_2 = 0 \\ m_2 x_2 - k_2 x_1 + k_2 x_2 = 0 \end{cases}$$

$$[M] = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \qquad [K] = \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix}$$

$$\boxed{[M]\ddot{\vec{x}} + [K]\vec{x} = 0}$$

Initial conditions:

$$\vec{x}(0) = \begin{Bmatrix} x_{10} \\ x_{20} \end{Bmatrix} \qquad \dot{\vec{x}}(0) = \begin{Bmatrix} \dot{x}_{10} \\ \dot{x}_{20} \end{Bmatrix}$$

1 DOF : $x(t) = Ae^{i\omega t} \Rightarrow A\sin(\omega t + \phi)$

2 DOF : $x_1(t) = u_1 e^{i\omega t}$

$x_2(t) = u_2 e^{i\omega t}$

$\rightarrow \quad \vec{x}(t) = \vec{u}\, e^{i\omega t}, \qquad \vec{u} = \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$

unknown

$\dot{\vec{x}}(t) = i\omega \vec{u}\, e^{i\omega t}$

$\ddot{\vec{x}}(t) = (i\omega)(i\omega)\vec{u}\, e^{i\omega t}$

$= -\omega^2 \vec{u}\, e^{i\omega t}$

$\rightarrow \quad -\omega^2 [M]\vec{u}\, e^{i\omega t} + [K]\vec{u}\, e^{i\omega t} = 0$

$(-\omega^2 [M] + [K])\vec{u}\, e^{i\omega t} = 0$

$\rightarrow \quad (-\omega^2 [M] + [K])\vec{u} = 0$

$\vec{u}$ is non-zero vector

$\rightarrow \quad \det(-\omega^2 [M] + [K]) = 0$ (condition of this equation)

the characteristic equation

1 DOF : $[M] = m \quad ; \quad [K] = K$

$\det(-\omega^2 m + K) = 0$

$\Rightarrow \quad -\omega^2 m + K = 0 \rightarrow \omega = \sqrt{K/m}$

$\det(-\omega^2 [M] + [K])$

$= \det\left(-\omega^2 \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} + \begin{bmatrix} K_1 K_2 & -K_2 \\ -K_2 & K_2 \end{bmatrix}\right)$

$= \det\begin{pmatrix} K_1 + K_2 - m_1\omega^2 & -K_2 \\ -K_2 & K_2 - m_2\omega^2 \end{pmatrix}$

$\rightarrow \quad (K_1 + K_2 - m_1\omega^2)(K_2 - m_2\omega^2) - (-K_2)(-K_2) = 0$

$\rightarrow \quad m_1 m_2 \omega^4 - (m_1 K_2 + m_2 K_1 + m_2 K_2)\omega^2 + K_1 K_2 = 0$

Two roots : ω_1^2, ω_2^2

Four roots : $\omega_1, -\omega_1, \omega_2, -\omega_2$

For $\omega = \omega_1$:

$(-\omega_1^2 [M] + [K])\vec{u} = 0$

For $\omega = \omega_2$:

$(-\omega_2^2 [M] + [K])\vec{u} = 0$

$$\rightarrow \quad \vec{x}(t) = C_1\vec{u}_1 e^{i\omega_1 t} + C_2\vec{u}_1 e^{-i\omega t} + C_3\vec{u}_2 e^{i\omega t} + C_4\vec{u}_2 e^{-i\omega t}$$
$$= A_1 \sin(\omega_1 t + \phi_1)\vec{u}_1 + A_2 \sin(\omega_2 t + \phi_2)\vec{u}_2$$

$\boxed{\text{Example:}}$ $m_1 = 9\text{ kg}$, $m_2 = 1\text{ kg}$; $k_1 = 24\text{ N/m}$, $k_2 = 3\text{ N/m}$

Find $\underbrace{\omega}_{\text{natural freq.}}$ and $\vec{u}$ — mode shape

Solution:

$$[M] = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \rightarrow \begin{bmatrix} 9 & 0 \\ 0 & 1 \end{bmatrix}$$

$$[K] = \begin{bmatrix} K_1+K_2 & -K_2 \\ -K_2 & K_2 \end{bmatrix} \rightarrow \begin{bmatrix} 27 & -3 \\ -3 & 3 \end{bmatrix}$$

$$\det\left[-\omega^2[M] + [K]\right] = 0$$

$$\rightarrow \begin{vmatrix} 27-9\omega^2 & -3 \\ -3 & 3-\omega^2 \end{vmatrix} = 0$$

$$\rightarrow (27 - 9\omega^2)(3-\omega^2) - 9 = 0$$

$$\rightarrow (81 - 27\omega^2 - 27\omega^2 + 9\omega^4 - 9) = 0$$

$$\rightarrow \omega^4 - 6\omega^2 + 8 = 0$$

$$\hookrightarrow (\omega^2 - 2)(\omega^2 - 4) = 0$$

$$\omega_1 = \sqrt{2} \; ; \; \omega_2 = 2$$

For $\omega = \omega_1 = \sqrt{2}$

$$(-\omega^2[M] + [K])\vec{u}_1 = 0$$

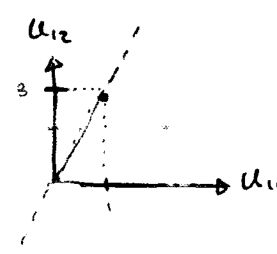
$$\rightarrow \left(-2\begin{bmatrix} 9 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 27 & -3 \\ -3 & 0 \end{bmatrix}\right)\begin{Bmatrix} u_{11} \\ u_{12} \end{Bmatrix} = 0$$

$$\rightarrow \begin{bmatrix} 9 & -3 \\ -3 & 1 \end{bmatrix}\begin{Bmatrix} u_{11} \\ u_{12} \end{Bmatrix} = 0$$

$$\rightarrow \begin{matrix} 9u_{11} - 3u_{12} = 0 \\ -3u_{11} \;\; u_{12} = 0 \end{matrix} \rightarrow \frac{u_{11}}{u_{12}} = \frac{1}{3}$$

$\rightarrow u_{11} = 1,\; u_{12} = 3$

$\rightarrow u_{11} = 1/3,\; u_{12} = 1$



Choose $u_{11} = 1/3$; $u_{12} = 1$

$$\vec{u}_1 = \begin{pmatrix} 1/3 \\ 1 \end{pmatrix}$$

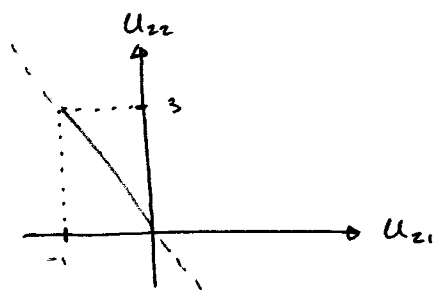
For $\omega = \omega_2 = 2$

$$\left(-\omega_2^2 [M] + [K]\right) u_2 = \emptyset$$

$$\rightarrow \left(-4 \begin{bmatrix} 9 & \emptyset \\ \emptyset & 1 \end{bmatrix} + \begin{bmatrix} 27 & -3 \\ -3 & 3 \end{bmatrix} \right) \begin{Bmatrix} u_{21} \\ u_{22} \end{Bmatrix} = \emptyset$$

$$\rightarrow \begin{bmatrix} -9 & -3 \\ -3 & -1 \end{bmatrix} \begin{Bmatrix} u_{21} \\ u_{22} \end{Bmatrix} = \emptyset$$

$$\rightarrow \begin{matrix} -9u_{21} - 3u_{22} = \emptyset \\ -3u_{21} - u_{22} = \emptyset \end{matrix} \implies \frac{u_{21}}{u_{22}} = \frac{-1}{3}$$



$u_{22} = 1$; $u_{21} = -1/3$

$$\vec{u}_2 = \begin{pmatrix} -1/3 \\ 1 \end{pmatrix}$$

$$[M]\ddot{\vec{x}} + [K]\vec{x} = 0$$

$$\vec{x} = \vec{u}e^{i\omega t}$$

$$\rightarrow \left([K] - \omega^2[M]\right)\vec{u} = 0 \quad ; \text{ where } \vec{u} \neq 0$$

$$\det\left([K] - \omega^2[M]\right) = 0$$

$\hookrightarrow$ Solving quadratic eqn yields ω_1, ω_2

natural Freq.

$$\left\{\begin{array}{l} \omega_1 \longrightarrow \vec{u}_1 \\ \omega_2 \longrightarrow \vec{u}_2 \end{array}\right\} \text{ mode shape}$$

The solution:

$$\vec{x} = \left(ae^{i\omega_1 t} + be^{-i\omega_1 t}\right)\vec{u}_1 + \left(ce^{i\omega_2 t} + de^{-i\omega_2 t}\right)\vec{u}_2$$

$$= \underline{A_1}\sin(\omega_1 t + \underline{\phi_1})\vec{u}_1 + \underline{A_2}\sin(\omega_2 t + \underline{\phi_2})\vec{u}_2$$

Initial conditions such that

$$A_2 = \phi_1 = \phi_2 = 0$$

Then:

$$\vec{x} = A_1 \sin(\omega_1 t)\,\vec{u}_1$$

Let:

$$\vec{u}_1 = \begin{Bmatrix} u_{11} \\ u_{21} \end{Bmatrix}$$

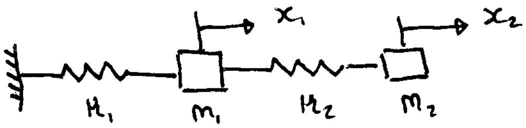
Then:

$$x_1 = A_1 \sin(\omega_1 t)\cdot u_{11}$$

$$x_2 = A_2 \sin(\omega_1 t)\,u_{21}$$

$$\rightarrow \frac{x_1}{x_2} = \frac{u_{11}}{u_{21}}$$

Example:



Initial conditions:

$x_1(0) = 1\text{ mm}$, $\quad x_2(0) = \dot{x}_1(0) = \dot{x}_2(0) = 0$

Given: $\quad m_1 = 9\text{ kg} \;/\; m_2 = 1\text{ kg}$

$\qquad\quad k_1 = 24\text{ N/m} \;/\; k_2 = 2\text{ N/m}$

Solution: $\quad \omega_1 = \sqrt{2}$ rad/s

$\qquad\qquad \omega_2 = 2$ rad/s

$$\vec{u}_1 = \begin{Bmatrix} 1/3 \\ 1 \end{Bmatrix} \;/\; \vec{u}_2 = \begin{Bmatrix} -1/3 \\ 1 \end{Bmatrix}$$

The free vibration:

$$\vec{x}(t) = A_1 \sin(\omega_1 t + \phi_1)\vec{u}_1 + A_2 \sin(\omega_2 t + \phi_2)\vec{u}_2$$

$$= A_1 \sin(\omega_1 t + \phi_1)\begin{pmatrix} 1/3 \\ 1 \end{pmatrix} + A_2 \sin(\omega_2 t + \phi_2)\begin{pmatrix} -1/3 \\ 1 \end{pmatrix}$$

$\rightarrow \; x_1(t) = (1/3)A_1 \sin(\omega_1 t + \phi_1) - (1/3) A_2 \sin(\omega_2 t + \phi_2)$

$\quad\; x_2(t) = A_1 \sin(\omega_1 t + \phi_1) + A_2 \sin(\omega_2 t + \phi_2)$

$\rightarrow \; \dot{x}_1(t) = (1/3) A_1 \omega_1 \cos(\omega_1 t + \phi_1) - (1/3) A_2 \omega_2 \cos(\omega_2 t + \phi_2)$

$\quad\; \dot{x}_2(t) = A_1 \omega_1 \cos(\omega_1 t + \phi_1) + A_2 \omega_2 \cos(\omega_2 t + \phi)$

At $t = 0$:

$x_1(0) = (1/3) A_1 \sin(\phi_1) - (1/3) A_2 \sin(\phi_2) = 1$

$x_2(0) = A_1 \sin(\phi_1) + A_2 \sin(\phi_2) = 0$

$\rightarrow \; A_1 \sin\phi_1 = 1.5 \;/\; \rightarrow \; A_2 \sin\phi_2 = -1.5$

$\dot{x}_1(0) = (1/3) A_1 \omega_1 \cos(\phi_1) - (1/3) A_2 \omega_2 \cos(\phi_2) = 0$

$\dot{x}_2(0) = A_1 \omega_1 \cos(\phi_1) + A_2 \omega_2 \cos(\omega_2) = 0$

$\rightarrow \; A_1 \cos(\phi_1) = 0 \;/\; \rightarrow \; A_2 \cos(\phi_2) = 0$

$\cos(\phi_1) = 0$ when $\phi_1 = 90°$

$\cos(\phi_2) = 0$ when $\phi_2 = 90°$

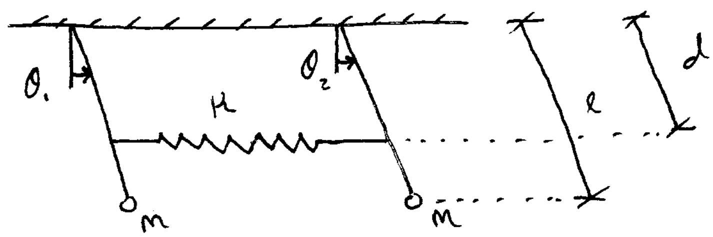
thus, $A_1 = 1.5$

$\qquad A_2 = -1.5$

$$x_1(t) = (0.5)\cos(\sqrt{2}t) + 0.5\cos(2t)$$
$$x_2(t) = (1.5)\cos(\sqrt{2}t) - 1.5\cos(2t)$$

since $\phi_{1,2} = 90°$
$\sin(\omega t + 90°) = \cos(\omega t)$

$\boxed{\text{Example:}}$ Two connected pendulums



Mass matrix: $[M] = \begin{bmatrix} ml^2 & 0 \\ 0 & ml^2 \end{bmatrix}$

Stiffness matrix: $[K] = \begin{bmatrix} mgl + Kd^2 & -Kd^2 \\ -Kd^2 & mgl + Kd^2 \end{bmatrix}$

Natural Freq: $\left| -\omega^2[M] + [K] \right| = 0$

(*) $$\begin{vmatrix} mgl + Kd^2 - \omega^2 ml^2 & -Kd^2 \\ -Kd^2 & mgl + Kd^2 - \omega^2 ml^2 \end{vmatrix} = 0$$

$\rightarrow (mgl + Kd^2 - \omega^2 ml^2)^2 - (Kd^2)^2 = 0$

$mgl + Kd^2 - \omega^2 \cdot ml^2 = \pm Kd^2$

$\omega^2 ml^2 = mgl + Kd^2 \pm Kd^2$

$\rightarrow \omega^2 ml^2 = mgl \longrightarrow \omega_1^2 = g/l$

$\rightarrow \omega^2 ml^2 = mgl + 2Kd^2 \longrightarrow \omega_2^2 = g/l + \left(2Kd^2/mgl^2\right)$

For $\omega = \omega_1 = \sqrt{g/l}$

$$\begin{pmatrix} mgl + Kd^2 - (g/l)ml^2 & -Kd^2 \\ -Kd^2 & mgl + Kd^2 - \underbrace{(g/l)}_{\omega_1^2} ml^2 \end{pmatrix} \vec{u}_1 = 0$$

$$\begin{pmatrix} Kd^2 & -Kd^2 \\ -Kd^2 & Kd^2 \end{pmatrix}\begin{pmatrix} u_{11} \\ u_{21} \end{pmatrix} = 0$$

$\rightarrow Kd^2 \cdot u_{11} - Kd^2 \cdot u_{21} = 0$; $u_{11} = u_{21}$

$\therefore \vec{u}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

For $\omega = \omega_2 = \sqrt{g/l + 2Kd^2/ml^2}$

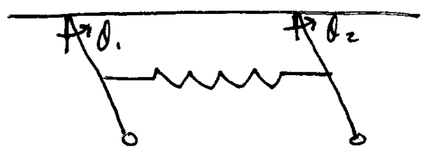
$$\begin{pmatrix} mgl + Kd^2 - (g/l + 2Kd^2/ml^2)ml^2 & -Kd^2 \\ -Kd^2 & mgl + Kd^2 - (g/l + 2Kd^2/ml^2)ml^2 \end{pmatrix} \vec{u} = \emptyset$$

$$\begin{pmatrix} -Kd^2 & -Kd^2 \\ -Kd^2 & -Kd^2 \end{pmatrix} \begin{pmatrix} u_{12} \\ u_{22} \end{pmatrix} = \emptyset$$

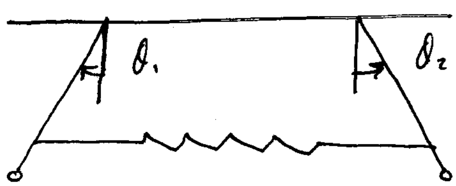
$$-Kd^2 u_{12} - Kd^2 u_{22} = \emptyset \longrightarrow u_{12} = u_{22}$$

$$\vec{u}_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$\longrightarrow \omega_1 = \sqrt{g/l}$



$\longrightarrow \omega_2 = \sqrt{\dfrac{g}{l} + \dfrac{2Kd^2}{ml^2}}$



Take $l = 1m$; $d = 0.3m$; $K = 4\ N/m$; $m = 1\ kg$

$\omega_1 = \sqrt{g/l}$; $\omega_2 = 3.245\ rad/s$

Choose $\theta_1(0) = 1$; $\theta_2(0) = 0$

$\dot{\theta}_1(0) = 0$; $\dot{\theta}_2(0) = 0$

4.2 Eigenvalues and Natural Frequencies

Symmetric Matrix :

$$[M]^T = [M]$$

A Symmetric matrix M is <u>positive definite</u> :

row vector $\rightarrow$ $\vec{x}^T [M] \vec{x} > 0$ $\rightarrow$ column vector

For all non-zero vector $\vec{x}$

A Symmetric positive definite matrix M can be Factored :

$$[M] = [L][L]^T$$

Here $[L]$ is upper triangular.

Cholesky matrix

If $[L]$ is diagonal, $[L]$ is the matrix $[M]$ square root.

<u>2 DOF</u> :

$$[M] = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \quad ; \quad [M]^{1/2} = \begin{bmatrix} \sqrt{m_1} & 0 \\ 0 & \sqrt{m_2} \end{bmatrix}$$

$$[M]^{-1/2} = \begin{bmatrix} 1/\sqrt{m_1} & 0 \\ 0 & 1/\sqrt{m_2} \end{bmatrix}$$

Equation of motion :

$$[M]\ddot{\vec{x}} + [K]\vec{x} = 0$$

Define

$$\vec{x} = [M]^{-1/2}\vec{q}(t)$$

$$\underbrace{[M]^{-1/2}[M][M]^{-1/2}}\ddot{\vec{q}} + \underbrace{[M]^{-1/2}[K][M]^{-1/2}}\vec{q} = 0$$

$$\ddot{\vec{q}} + [\tilde{K}]\vec{q} = 0 \qquad (A)$$

Here, $[K] = [M]^{-1/2}[K][M]^{-1/2}$

$[\bar{K}]^T = [\bar{K}]$

mass normalized stiffness

1 DOF :

$$m\ddot{x} + kx = 0$$

$$\ddot{x} + (k/m)\,x = 0 \qquad \text{(B)}$$

(compare (A) → (B) thus, ω_n is contained within.)

For the free vibration :

$$\ddot{\vec{q}} + [\bar{K}]\vec{q} = 0$$

Take $\vec{q} = \vec{v}e^{i\omega t}$

$$(-\omega^2\vec{v} + [\bar{K}]\vec{v})e^{i\omega t} = 0$$

$$\longrightarrow \boxed{[\bar{K}]\vec{v} = \omega^2\vec{v}}\ , \quad \vec{v} \neq 0$$

(would mean no motion at all)