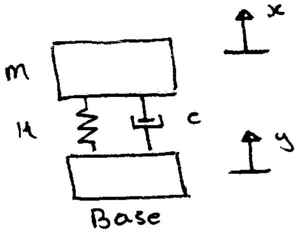
Oct. 8/19



$m\ddot{x} + c\dot{x} + kx = c\dot{y} + ky$

→ Equation of motion

$y(t) = Y\sin(\omega_b t)$

$m\ddot{x} + c\dot{x} + kx = c\omega_b Y\cos(\omega_b t) + kY\sin(\omega_b t)$

$\sqrt{k/m} = \omega_n, \quad \zeta = c/2\sqrt{mk}$

$\Rightarrow \ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = 2\zeta\omega_n\omega_b Y\cos(\omega_b t) + \omega_n^2 Y\sin(\omega_b t)$

$\Rightarrow \omega_n Y(2\zeta\omega_b\cos(\omega_b t) + \omega_n\sin(\omega_b t))$

$\Rightarrow$ Since $2\zeta\omega_b\cos(\omega_b t) + \omega_n\sin(\omega_b t)$

$= P_0\cos(\omega_b t - \theta_2)$

$= P_0\cos(\omega_b t)\cos(\theta_2) + P_0\sin(\omega_b t)\cdot\sin(\theta_2)$

$$\Rightarrow \begin{cases} P_0\cos(\theta_2) = 2\zeta\omega_b \\ P_0\sin(\theta_2) = \omega_n \end{cases}$$

$P_0 = \sqrt{2\zeta\omega_b^2 + \omega_n^2}$

$\theta_2 = \arctan\left(\dfrac{\omega_n}{2\zeta\omega_b}\right)$

$\therefore$ Equation of motion:

$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = \omega_n Y P_0 \cos(\omega_b t - \theta_2)$

The particular solution (Forced response)

$x(t) = X\cos(\omega_b t - \theta_2 - \theta_1)$

and:

$$X = \frac{f_0}{\sqrt{(\omega_n^2 - \omega_b^2)^2 + (2\zeta\omega_n\omega_b)^2}}$$

$$\theta_1 = \arctan\left(\frac{2\zeta\omega_n\omega_b}{\omega_n^2 - \omega_b^2}\right)$$

→ The magnitude of the response:

$$X = \frac{\omega_n Y P_0}{\sqrt{(\omega_n^2 - \omega_b^2)^2 + (2\zeta\omega_n\omega_b)^2}}$$

$$X = \frac{\omega_n Y\sqrt{(2\zeta\omega_b)^2 + \omega_n^2}}{\sqrt{(\omega_n^2 - \omega_b^2)^2 + (2\zeta\omega_n\omega_b)^2}}$$

$$= \omega_n Y\sqrt{\frac{\omega_n^2 + (2\zeta\omega_b)^2}{(\omega_n^2 - \omega_b^2)^2 + (2\zeta\omega_b\omega_n)^2}}$$

NOTE: ; Let $r = \omega_b/\omega_n$

Frequency ratio — excitation Frequency — natural Frequency

$\rightarrow \omega_b = r\omega_n$

$$\Rightarrow \boxed{\frac{X}{Y} = \sqrt{\frac{1 + (2\zeta r)^2}{(1 - r^2)^2 + (2\zeta r)^2}}}$$

~~transmission~~ transmissibility (or Displacement Ratio)

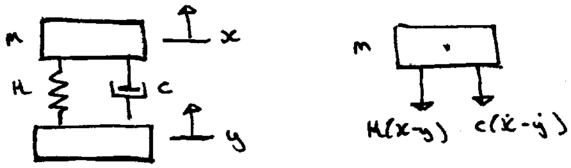
Resonance Freq. @ $r=1$ (not necessarily maximum)

To find max displacement ratio:

$$\frac{d}{dr}\left(\frac{X}{Y}\right) = 0$$

$$r = \frac{1}{2\zeta}\left[\sqrt{1+8\zeta^2} - 1\right]^{1/2}$$

Force Transmitted



$$F = k(x-y) + c(\dot{x}-\dot{y}) = -m\ddot{x}$$

$$x(t) = X\cos(\omega_b t - \theta_1 - \theta_2)$$

$$\ddot{x} = -\omega_b^2 X\cos(\omega_b t - \theta_1 - \theta_2)$$

$$\therefore\ F = m\omega_b^2 X\cos(\omega_b t - \theta_1 - \theta_2)$$

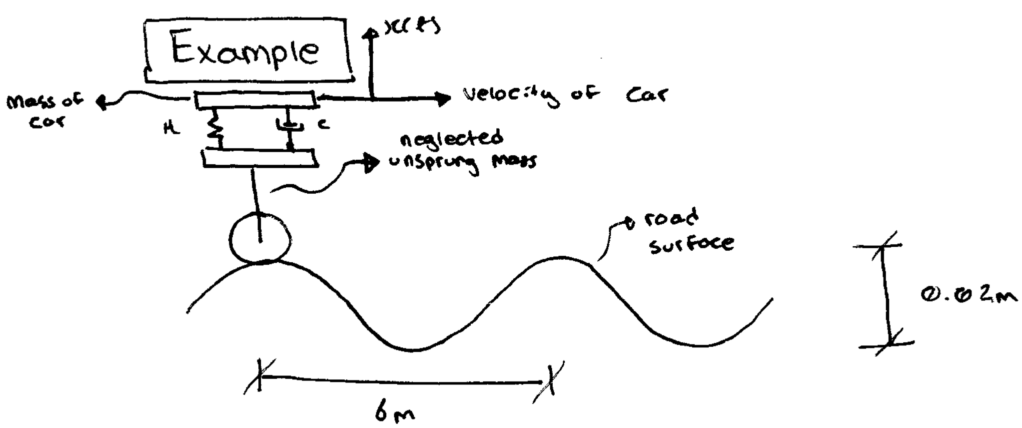
max transmitted Force $\rightarrow$

$$|F_T| = m\omega_b^2 X = m r^2 \omega_n^2 X = r^2 k X$$

$$= r^2 k Y\sqrt{\frac{1+(2\zeta r)^2}{(1-r^2)^2+(2\zeta r)^2}}$$

$$\Rightarrow \frac{|F_T|}{kY} = r^2\sqrt{\frac{1+(2\zeta r)^2}{(1-r^2)^2+(2\zeta r)^2}}$$

Example



$m = 1007$ kg

$k = 40000$ N/m

$c = 2000$ Ns/m

$V = 20$ km/h

The base excitation: $Y\sin(\omega_b t)$

$$Y = \frac{0.02}{2} = 0.01$$

Period: $$T = \frac{d}{v} = \frac{6\,\text{m}}{20\,\text{km/h}}$$

Frequency of the base excitation:

$$\omega_b = \frac{2\pi}{T} = \frac{2\pi}{6m/v} = \frac{2\pi v}{6}$$

If v is km/h:

$$\omega_b = \frac{2\pi v}{6} \times \frac{1000}{3600} \longrightarrow \text{rad/s}$$

$$\omega_b = 0.2909v \quad \text{rad/s} \quad (v \text{ is km/h})$$

When $v = 20$ km/h:

$$\omega_b = 5.818 \text{ rad/s}$$

$$r = \frac{\omega_b}{\omega_n}$$

$$r = \frac{5.818}{6.303}$$

$$r = 0.9231$$

Since $\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{4000}{1007}} = 6.303$

$$\zeta = \frac{c}{2\sqrt{mk}} = \frac{2000}{2\sqrt{1007 \times 40000}} = 0.158 < 1$$

$$\therefore \frac{X}{Y} = \sqrt{\frac{1 + (2\zeta r)^2}{(1-r^2)^2 + (2\zeta r)^2}} = 3.19$$

$$X = 3.19Y \quad (\text{Since } Y = 0.01\text{m})$$

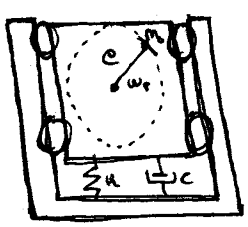
$$\longrightarrow X = 0.0319 \text{ m}$$

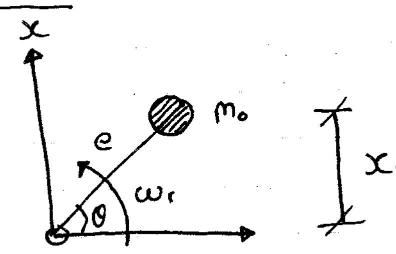
As $v \uparrow$, response $\downarrow$

Heavier objects feel less vibration (from displacement point of view)

↳ what about forces?

2.5 Rotating Unbalance





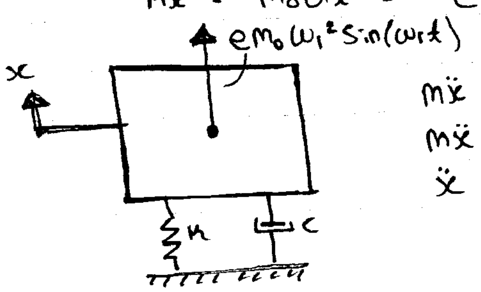
$$\theta = \omega_r t$$

$$x_r = e\sin(\omega_r t)$$

$$a_x = \ddot{x}_r = -e\omega_r^2 \sin(\omega_r t)$$

→ The force along the x-axis:

$$R_x = m_o a_x = -em_o\omega_r^2 \sin(\omega_r t)$$



$$m\ddot{x} + c\dot{x} + kx = F(t)$$

$$m\ddot{x} + c\dot{x} + kx = em_o\omega_r^2 \sin(\omega_r t)$$

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = \left(\frac{m_o}{m}\right) e\omega_r^2 \sin(\omega_r t)$$

The Forced response:

$$x(t) = X \sin(\omega_1 t - \theta)$$

Here:

$$X = \left(\frac{m_o}{m}\right) e \cdot \frac{r^2}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}}$$

(increasing mass reduces response, but can unbalance system)

$$\theta = \arctan\left(\frac{2\zeta r}{1-r^2}\right)$$

$$\left(\frac{m}{m_o}\right)\left(\frac{X}{e}\right) = \frac{r^2}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}}$$