(Refer to camera example)

Solution : The forced response of the undamped mass-spring is:

$$x(t) = \frac{v_0}{\omega_n}\sin(\omega_n t) + \left(x_0 - \frac{f_0}{\omega_n^2-\omega^2}\right)\cos(\omega_n t) + \frac{f_0}{\omega_n^2-\omega^2}\cos(\omega t)$$

For zero initial conditions

$v_0 = 0, \quad x_0 = 0$

$$\boxed{x(t) = \frac{f_0}{\omega_n^2-\omega^2}\left(\cos(\omega t) - \cos(\omega_n t)\right)}$$

$$|x(t)| = \left|\frac{f_0}{\omega_n^2-\omega^2}\right| \cdot \left|\cos(\omega t) - \cos(\omega_n t)\right|$$

$(|a+b| \le |a| + |b|)$

$$|x(t)| \le \left|\frac{f_0}{\omega_n^2-\omega^2}\right| \left(|\cos(\omega t)| + |\cos(\omega_n t)|\right)$$

$$\le \frac{2f_0}{|\omega_n^2-\omega^2|}$$

$\therefore$ maximum displacement is $\dfrac{2f_0}{|\omega_n^2-\omega^2|}$

$$\boxed{\therefore \frac{2f_0}{|\omega_n^2-\omega^2|} \le 0.01}$$

<u>Case 1:</u> $\omega_n < \omega = 10\ Hz = 2\pi(10)\ rad/s = 62.832\ rad/s$

$$\frac{2f_0}{\omega^2-\omega_n^2} \le 0.01$$

Since $f_0 = \dfrac{F_0}{m} = \dfrac{15\ N}{3\ kg} = 5\ N/kg$

$\rightarrow \omega^2 - \omega_n^2 \ge 2f_0/0.01$

$\rightarrow \omega_n^2 \le \omega^2 - \dfrac{2f_0}{0.01}$

$$= (62.832)^2 - \frac{2(5)}{0.01}$$

$\omega_n^2 \le 2947.86$

$\omega_n \le 54.294$

Since $k = \dfrac{3EI}{l^3}$

(Cross-section of beam changed to 0.01 × 0.01) (*)

$$\omega_n^2 = \frac{k}{m}$$

$E = 71$ GPa,

$I = (1/12)(10\times 10^{-8})\, m^4$

$\rightarrow \omega_n^2 = k/m = \frac{3EI}{3\ell^3}$

$$\boxed{\omega_n^2 = \frac{59.1667}{\ell^3}}$$

$\Rightarrow \frac{59.1667}{\ell^3} \leq 2947.86$

$\Rightarrow \boxed{\ell \geq 0.272\ m}$

Case 2 $\quad \omega_n > \omega = 62.832$

$\frac{2f_0}{\omega_n^2-\omega^2} \leq 0.01$

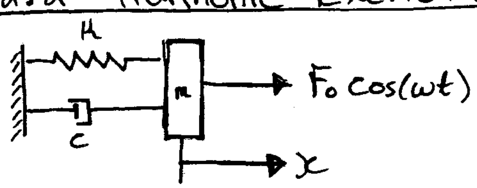
$\Rightarrow \omega_n^2 \geq \omega^2 + \frac{2f_0}{0.01} = 4947.86$

$\Rightarrow \frac{59.1667}{\ell^3} \geq 4947.86$

$\Rightarrow \boxed{\ell \leq 0.229\ m}$

Choose $\ell = 0.22m$ (requirement of $\ell > 0.2m$)

2.2 Harmonic Excitation of Damped Systems



Equation of motion:

$m\ddot{x} = F_0\cos(\omega t) - kx - c\dot{x}$

$\rightarrow m\ddot{x} + c\dot{x} + kx = F_0\cos(\omega t)$

$(k/m) = \omega_n^2 \Rightarrow k = m\omega_n^2$

$\rightarrow c/c_{cr} = \zeta \quad ; \quad c_{cr} = 2\sqrt{mk} = 2m\omega_n$

$c = 2\zeta m\omega_n$

$\Rightarrow m\ddot{x} + 2\zeta m\omega_n\dot{x} + m\omega_n^2 x = F_0\cos(\omega t)$

$\Rightarrow \boxed{\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = f_0\cos(\omega t)}$

$f_0 = F_0/m$

The general solution of the homogeneous eq. is the free vibration of the damped system.

$$x_H(t) = Ae^{-\zeta\omega_n t}\sin(\omega_d + \phi)$$

$$\omega_d = \omega_n\sqrt{1-\zeta^2}$$

The particular solution

$$x_p(t) = A_s\cos(\omega t) + B_s\sin(\omega t)$$

$$\dot{x}_p(t) = -\omega A_s\sin(\omega t) + \omega B_s\cos(\omega t)$$

$$\ddot{x}_p(t) = -\omega^2 A_s\cos(\omega t) - \omega^2 B_s\sin(\omega t)$$

$$\left(-\omega^2(A_s\cos(\omega t) + B_s\sin(\omega t)) = -\omega^2 x_p\right)$$

Sub into the eq. of motion:

$$-\omega^2(A_s\cos\omega t + B_s\sin\omega t) + 2\zeta\omega_n(-\omega A_s\sin(\omega t) + \omega B_s\cos(\omega t)) + \cdots$$

$$\cdots\ \omega_n^2(A_s\cos\omega t + B_s\sin\omega t) = f_0\cos\omega t$$

$$(-\omega^2 A_s + 2\zeta\omega_n\omega B_s + A_s\omega_n^2)\cos(\omega t) + (-\omega^2 B_s - 2\zeta\omega_n\omega A_s + B_s\omega_n^2)\sin(\omega t)$$

$$= f_0\cos(\omega t)$$

$$\Rightarrow \left.\begin{aligned}(\omega_n^2-\omega^2)A_s + 2\zeta\omega_n\omega B_s &= f_0\\ -2\zeta\omega_n\omega A_s + (\omega_n^2-\omega^2)B_s &= 0\end{aligned}\right\}$$

$$\Rightarrow A_s = \frac{\omega_n^2-\omega^2}{(\omega_n^2-\omega^2)^2 + (2\zeta\omega_n\omega)^2}f_0$$

$$B_s = \frac{2\zeta\omega_n\omega}{(\omega_n^2-\omega^2)^2 + (2\zeta\omega_n\omega)^2}f_0$$

$$\therefore x_p(t) = A_s\cos(\omega t) + B_s\sin(\omega t)$$

$$= X\cos(\omega t - \theta)$$

$$= X\cos(\omega t)\cos\theta + X\sin(\omega t)\sin\theta$$

$$A_s = X\cos\theta\ ;\quad B_s = X\sin\theta$$

$$\Rightarrow X = \sqrt{A_s^2 + B_s^2},\quad \tan\theta = \frac{B_s}{A_s}$$

Here,

$$\left.\begin{aligned}X &= \frac{f_0}{\sqrt{(\omega_n^2-\omega^2)^2 + (2\zeta\omega_n\omega)^2}}\\ \theta &= \tan^{-1}\left[\frac{2\zeta\omega_n\omega}{\omega_n^2-\omega^2}\right]\end{aligned}\right\}$$

$\therefore$ the response:

$$\boxed{\begin{aligned}x(t) &= x_n(t) + x_p(t)\\ &= Ae^{-\zeta\omega_n t}\sin(\omega_d t + \phi) + X\cos(\omega t - \phi)\end{aligned}}$$

free vibration only

Forced vibration added

Example: Find the response of the system.

$\omega_n = 10$ rad/s

$\omega = 5$ rad/s

$\zeta = 0.01$

$F_o = 1000$ N

$m = 100$ kg

$x_o = 0.05$ m

$v_o = 0$

Solution: $f_o = \dfrac{F_o}{m} = \dfrac{1000}{100} = 10$ N/kg

$$X = \frac{f_o}{\sqrt{(\omega_n^2 - \omega^2)^2 + (2\zeta\omega_n\omega)^2}} \quad 0.13332$$

$$\theta = \tan^{-1}\left[\frac{2\zeta\omega_n\omega}{\omega_n^2 - \omega^2}\right] = 0.013333 \text{ rad}$$

$$\omega_d = \omega_n\sqrt{1-\zeta^2} = 9.9995 \text{ rad/s}$$

$$\therefore \quad x(t) = Ae^{-(0.01)(10)t}\sin(9.9995t + \phi) + 0.13332\cos(5t - 0.0133333)$$

Velocity

$$\dot{x} = -0.1\,Ae^{-0.1t}\sin(9.9995t + \phi) + (9.9995)Ae^{-0.1t}\cos(9.9995t + \phi) - \cdots$$

$$\cdots \quad (0.13332)(5)\sin(5t - 0.0133333)$$

At $t = 0$:

$$x(0) = A\sin\phi + 0.13332\cos(-0.013333) = 0.05$$

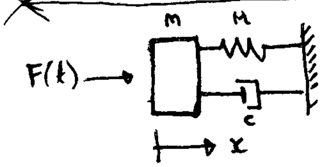
$$v(0) = -(0.1)A\sin\phi + (9.9995)A\cos\phi - (0.13332)(5)\sin(-0.013333) = 0$$

$$\Rightarrow \left.\begin{aligned} A &= 0.083327 \\ \phi &= 1.5501 \text{ (rad)} \end{aligned}\right\}$$

$$\therefore \quad x(t) = 0.083327\,e^{-0.1t}\sin(9.9995t + 1.5501) + \cdots$$

$$\cdots (0.13332)\cos(5t - 0.013333) \qquad (\text{in m})$$

- Midterm location to be emailed
  - Sections: 1.1 → 1.6
  - 3 questions
  - Formula sheet provided



$$m\ddot{x} + c\dot{x} + kx = F_0 \cos(\omega t)$$

$$x = x_h(t) + x_p(t)$$

$$x_h(t) = Ae^{-\zeta\omega_n t}\sin(\omega_d t + \phi) \quad \leftarrow \text{transient response}$$

$$x_p(t) = X\cos(\omega t - \theta) \quad \leftarrow \text{steady-state response}$$

when $t \to \infty$, $x_h(t) \to 0$

$$X = \frac{f_0}{\sqrt{(\omega_n^2 - \omega^2)^2 + (2\zeta\omega_n\omega)^2}}$$

$$\theta = \arctan\left(\frac{2\zeta\omega_n\omega}{\omega_n^2 - \omega^2}\right)$$

$$f_0 = \frac{F}{m}$$

Define $r = \frac{\omega}{\omega_n}$, $\omega = r\omega_n$

frequency ratio

$$\therefore X = \frac{f_0}{\sqrt{(\omega_n^2 - \omega^2)^2 + (2\zeta\omega_n\omega)^2}}$$

$$= \frac{f_0}{\omega_n^2}\frac{1}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}}$$

$$\frac{f_0}{\omega_n^2} = \frac{F_0/m}{k/m} = \frac{F_0}{k} = \delta_{st.}$$

$$\Rightarrow \boxed{\frac{X}{\delta_{st.}} = \frac{1}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}}}$$

amplitude factor

(ratio between dynamic response and static response)

$$\boxed{\theta = \arctan\left(\frac{2\zeta\omega_n\omega}{\omega_n^2 - \omega^2}\right) = \tan^{-1}\left(\frac{2\zeta r}{1-r^2}\right)}$$

Amplitude Factor :

(amplitude)

→ 1^{ST} $\zeta = 0$, $r = 1$ (resonance)

→ 2^{ND} Any amount of damping reduces the magnification Factor for all the Forcing Frequency

→ 3^{RD} Resonance Frequency : $\omega = \omega_r$

→ 4^{TH} $\frac{d}{dr}\left(\frac{X}{\delta_{st}}\right) = 0$

$0 < \zeta < (1/\sqrt{2}) \Rightarrow 0.707$

at $r = \sqrt{1-2\zeta^2}$, $X \rightarrow max$

$\zeta > 1/\sqrt{2}$ $(= 0.707)$; X_{max} occurs at $r = 0$

→ 5^{TH} $0 < \zeta < 1/\sqrt{2}$

at $r = \sqrt{1-2\zeta^2}$; $\left(\frac{X}{\delta_{st}}\right)_{max} = \frac{1}{2\zeta\sqrt{1-\zeta^2}}$

at $r = 1$; $\frac{X}{\delta_{st}} = \frac{1}{2\zeta}$

Phase $\theta = \tan^{-1}\left[\frac{2\zeta r}{1-r^2}\right]$

Section 2.3

$m\ddot{x} + c\dot{x} + kx = F_0 \cos(\omega t)$

$e^{i\omega t} = \underbrace{\cos(\omega t)}_{\text{real part}} + \underbrace{i\sin(\omega t)}_{\text{imaginary part}}$

$\rightarrow m\ddot{x} + c\dot{x} + kx = F_0 e^{i\omega t}$

Assume: $x(t) = Xe^{i\omega t}$

$\dot{x} = i\omega X e^{i\omega t}$

$\ddot{x} = (i\omega)^2 X e^{i\omega t} = -\omega^2 X e^{i\omega t}$

$\Rightarrow (-m\omega^2 + i\omega c + k) X e^{i\omega t} = F_0 e^{i\omega t}$

$\Rightarrow X = \frac{F_0}{-m\omega^2 + i\omega c + k}$

Define: $\frac{X}{F_0} = H(\omega) = \frac{1}{k - m\omega^2 + i\omega c}$

$H(i\omega)$: Frequency response function.

$$m\ddot{x} + c\dot{x} + kx = F_0 \cos(\omega t)$$

Laplace Transform

$$X(s) = \int_0^\infty x(t)\, e^{-st}\, dt$$

$\Rightarrow \int_0^\infty (m\ddot{x} + c\dot{x} + kx)\, e^{-st}\, dt$

$\Rightarrow \int_0^\infty F_0 \cos(\omega t) e^{-st}\, dt$

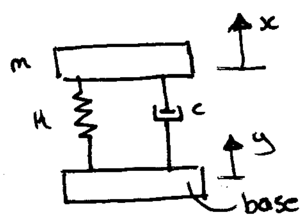
$\Rightarrow (ms^2 + cs + k)\, X(s) = \boxed{\dfrac{F_0 \cdot s}{s^2 + \omega^2}}$

$\Rightarrow$ Transfer Function

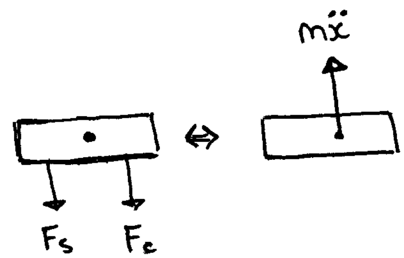
$$\boxed{H(s) = \frac{1}{ms^2 + cs + k}}$$

($s \to i\omega$ gives Freq. Function)

2.4 Base Extraction



FBD



where $F_s = k(x-y)$
$F_c = c(\dot{x} - \dot{y})$

Newton's 2nd Law:

$$m\ddot{x} = -k(x-y) - c(\dot{x} - \dot{y})$$

$$\Rightarrow m\ddot{x} + c\dot{x} + kx = ky + c\dot{y}$$

Assume: $y(t) = Y \sin(\omega t)$

$$m\ddot{x} + c\dot{x} + kx = kY \sin(\omega t) + cY \cos(\omega t)\omega$$

$$\Rightarrow \ddot{x} + 2\zeta\omega_n \dot{x} + \omega_n^2 x = (k/m)\, Y \sin(\omega t) + (c/m)\,\omega Y \cos(\omega t)$$

$$\begin{cases} f_{0s} = (k/m)Y = \omega_n^2 Y \\ f_{0c} = (c/m)\omega Y = 2\zeta\omega_n \omega Y \end{cases}$$