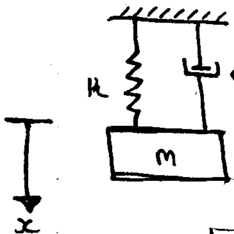


$m\ddot{x} + c\dot{x} + kx = 0$ (w/ no damping)

$\omega_n = \sqrt{\frac{k}{m}}$

damping ratio

$\hookrightarrow \zeta = c/c_r \quad ; \quad c_{cr} = 2\sqrt{mk}$

$\Rightarrow \boxed{\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = 0}$ (w/ damping)

Initial conditions:

$x(0) = x_0$

$\dot{x}(0) = v_0$

$\zeta > 1$ overdamped

$\zeta = 1$ critical

$\zeta < 1$ underdamped

Example:

$m = 49.2 \times 10^{-3}$ kg

$k = 857.8$ N/m

$c = 0.11$ kg/s

→ Determine the damping ratio..

Solution:

$c_{cr} = 2\sqrt{mk}$

$= 2\sqrt{(49.2\times10^{-3})(857.8)} = 12.99$ kg/s

Damping ratio $= \zeta = c/c_{cr} = \frac{0.11}{12.99}$

$\zeta = 0.0085$ ($\ll 1$)

∴ underdamped.

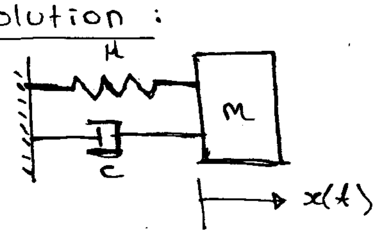
Example:

$\omega_n = 20$ Hz

$\zeta = 0.224$

→ Find the response of the tip if the initial velocity is $v_0 = 0.6$ m/s and initial displacement $x_0 = 0$. What is the maximum acceleration experienced by the leg? (Assuming no damping)

Solution:



cont'd :

Response :

$$x(t) = Ae^{-\zeta\omega_n t}\sin(\omega_d t + \phi)$$

Here,

damped natural frequency → $\omega_d = \sqrt{1-\zeta^2}\,\omega_n$ (always $< \omega_n$)

$$A = \frac{1}{\omega_d}\sqrt{(v_0 + \zeta\omega_n x_0)^2 + (x_0\omega_d)^2}$$

$$\phi = \tan^{-1}\left(\frac{x_0\omega_d}{v_0 + \zeta\omega_n x_0}\right)$$

$\omega_n = 20\ Hz = 20\ \frac{1}{s}$

↳ $\omega_n = 20(2\pi)$ rad/s

$= 40\pi$ rad/s

$\omega_d = (1-\zeta^2)^{1/2}\omega_n = (1-0.224^2)^{1/2}(125.66)$

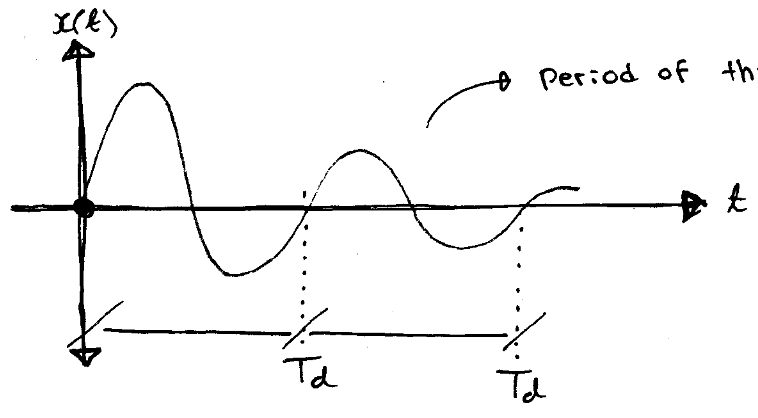
$= 122.467$ rad/s

$\Rightarrow \begin{cases} A = 0.005 \\ \phi = 0 \end{cases}$

↳ both terms +ve, less than 90°

both -ve, greater than 180°

$\Rightarrow x(t) = 0.005e^{-2\times 8.148\,t}\sin(122.46)t$



→ period of this system : $T = 2\pi/\omega_d$

Maximum acceleration (by assuming no damping)

$a_{max} = A\omega_n^2$

no damping :

$x = A\sin(\omega_n t + \phi)$

$\dot{x} = A\omega_n\cos(\omega_n t)$

$\ddot{x} = -A\omega_n^2\sin(\omega_n t)$

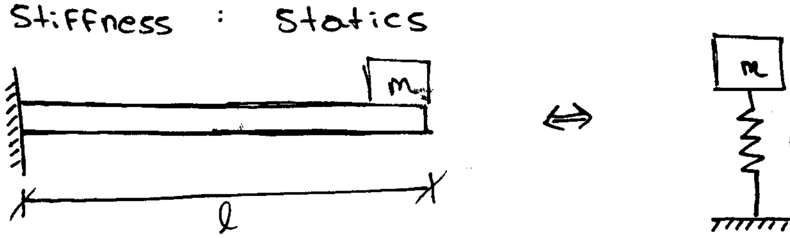
} Then the max is just the coefficients (when sin/cos = 1)

$a_{max} = 0.005(125.68)^2 = 75.396\ m/s^2$

Measurement

Mass :

Stiffness : Statics



$$k = \frac{3EI}{l^3}$$

Measure the period T.

$T = 2\pi/\omega_n$; $\omega_n = \sqrt{k/m}$

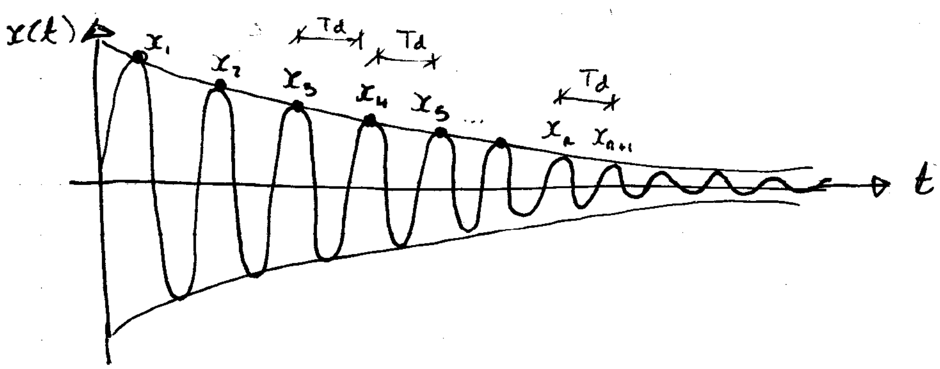
$\omega_n = 2\pi/T = \sqrt{k/m}$

$$\Rightarrow \frac{k}{m} = \frac{4\pi^2}{T^2}$$

$$\Rightarrow \frac{3EI}{l^3 m} = \frac{4\pi^2}{T^2}$$

$$\Rightarrow E = \left(\frac{4\pi^2}{T^2}\right)\left(\frac{m l^3}{3I}\right)$$

Damping (underdamped) : . . . . . $x(t) = Ae^{-\zeta\omega_n t}\sin(\omega_d t + \phi)$



At time $t + T_d$:

$x(t+T_d) = Ae^{-\zeta\omega_n(t+T_d)}\sin(\omega_d(t+T_d)+\phi)$

Ratio :

$$\frac{x(t)}{x(t+T_d)} = \frac{Ae^{-\zeta\omega_n t}\sin(\omega_d t+\phi)}{Ae^{-\zeta\omega_n(t+T_d)}\sin(\omega_d t+\omega_d T_d+\phi)}$$

$$= \left(\frac{1}{e^{-\zeta\omega_n T_d}}\right)\cdot\left(\frac{\sin(\omega_d t+\phi)}{\sin(\omega_d t+\phi)}\right)$$ . . . $\sin(2\pi + x) = \sin(x)$

$$\left(\begin{array}{l}\omega_d = 2\pi/T_d \\ T_d = 2\pi/\omega_d\end{array}\right)$$

$$= e^{\zeta\omega_n T_d}$$

Define the logarithmic decrement:

$$\delta = \ln\left[\frac{x(t)}{x(t+T_d)}\right]$$

$$\Rightarrow \delta = \ln e^{\zeta\omega_n T_d}$$

$$\delta = \zeta\omega_n T_d$$

Since,

$$T_d = \frac{2\pi}{\omega_d} = \frac{2\pi}{\omega_n\sqrt{1-\zeta^2}}$$

$$\Rightarrow \boxed{\delta = 2\pi\left(\frac{\zeta}{\sqrt{1-\zeta^2}}\right)}$$

If $\zeta << 1$; $\sqrt{1-\zeta^2} \approx 1$

$$\delta = 2\pi\zeta$$

$$\boxed{\zeta = \frac{\delta}{2\pi}}$$

If ζ is <u>not</u> small

$$\boxed{\zeta = \frac{\delta}{\sqrt{4\pi^2+\delta^2}}}$$

(From diagram...) $\frac{x_1}{x_2} = e^{\zeta\omega_n T_d}$

$$\frac{x_2}{x_3} = e^{\zeta\omega_n T_d} \quad \Rightarrow \quad \frac{x_1}{x_2}\cdot\frac{x_2}{x_3}\cdot\frac{x_3}{x_4}\cdots\frac{x_n}{x_{n+1}} = e^{(\zeta\omega_n T_d)^n}$$

$\vdots$

$$\frac{x_n}{x_{n+1}} = e^{\zeta\omega_n T_d}$$

$$\Rightarrow \frac{x_1}{x_{n+1}} = \left(e^{\zeta\omega_n T_d}\right)^n$$

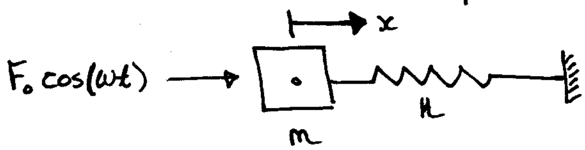
$$\Rightarrow \boxed{\ln\left(\frac{x_1}{x_{n+1}}\right) = n\zeta\omega_n T_d}$$

$$\delta = \ln(x_1/x_2) = \ln(x_2/x_3) = \dots = \ln(x_n/x_{n+1})$$

$$\boxed{\delta = \left(\frac{1}{n}\right)\ln(x_1/x_{n+1})}$$

Chapter 2 : Response to Harmonic Excitation

(2.1) Undamped System



ω : driving frequency

F_o : magnitude of force

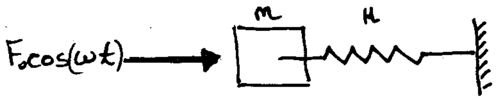
$$m\ddot{x} + kx = F_o \cos(\omega t)$$

$$\Rightarrow \ddot{x} + \omega_n^2 x = \frac{F_o}{m}\cos(\omega t) = f_o \cos(\omega t)$$

$$f_o = \frac{F_o}{m} \quad ; \quad \omega_n = \sqrt{\frac{k}{m}}$$

Ch. 2 - Response to Harmonic Motion Excitation

2.1 underdamped system



ω : driving force
F_o : magnitude

$$m\ddot{x} + kx = F_o\cos(\omega t)$$
$$\ddot{x} + (k/m)x = (F_o/m)\cos(\omega t)$$

where $f_o = \dfrac{F_o}{m}$; $\omega_n = \sqrt{\dfrac{k}{m}}$

Particular solution:

$x_p(t) = X\cos(\omega t)$ ← unknown const.

Since $\dot{x}_p(t) = -\omega X\sin(\omega t)$
$\ddot{x}_p(t) = -\omega^2 X\cos(\omega t)$

$$-\omega^2 X\cos(\omega t) + \omega_n^2 X\cos(\omega t) = f_o\cos(\omega t)$$
$$-\omega^2 X = \omega_n^2 X = f_o$$
$$\omega \neq \omega_n \longrightarrow X = \frac{f_o}{(\omega_n^2 - \omega^2)}$$
$$x_p(t) = \frac{f_o}{(\omega_n^2 - \omega^2)}\cos(\omega t) \quad (\text{where } \omega \neq \omega_n)$$

The general solution of the forced vibration:

$$x(t) = A_1\sin(\omega_n t) + A_2\cos(\omega_n t) + \frac{f_o}{(\omega_n^2 - \omega^2)}\cos(\omega t)$$

Initial conditions:

$x(0) = x_o$; $\dot{x}(0) = v_o$

Since $x(0) = 0 + A_2 + \dfrac{f_o}{(\omega_n^2 - \omega^2)} = x_o$

$$A_2 = x_o - [f_o / [\omega_n^2 - \omega^2]]$$
$$\dot{x}(t) = \omega_n A_1\cos(\omega_n t) - \omega_n A_2\sin(\omega_n t) - [f_o/(\omega_n^2 - \omega^2)]\sin(\omega t)$$
$$\dot{x}(0) = \omega_n A_1 = v_o$$
$$A_1 = v_o/\omega_n$$

$$\therefore x(t) = \left(\frac{v_0}{\omega_n}\right)\sin(\omega_n t) + \left(x_0 - \frac{f_0}{\omega_n^2-\omega^2}\right)\cos(\omega_n t) + \ldots$$

$$\ldots\left(\frac{f_0}{\omega_n^2-\omega^2}\right)\cos(\omega t)$$

Example:

$\omega_n = 1$ rad/s
$\omega = 2$ rad/s
$x_0 = 0.01$
$v_0 = 0.01$
$f_0 = 0.1$

$$x(t) = (0.01)\sin(t) + 0.0433\cos(t) + (-0.0333\cos(2t))$$

... → see text - becomes periodic, but no longer harmonic

Example:

$m = 10$ kg
$k = 1000$ N/m
$x_0 = 0$
$v_0 = 0.2$ m/s
$F = 23$ N

$\omega = 2\omega_n$ ← excitation frequency

Find the response

Solution : $\omega_n = \sqrt{k/m}$; $\omega_n = 10$ rad/s
$\omega = 2\omega_n \Rightarrow \omega = 20$ rad/s
$f_0 = F/m = 23/10 = 2.3$

$$\therefore x(t) = \left(\frac{v_0}{\omega_n}\right)\sin(\omega_n t) + \left(x_0 - \frac{f_0}{\omega_n^2-\omega}\right)\cos(\omega_n t) + \left(\frac{f_0}{\omega_n^2-\omega^2}\right)\cos(\omega t)$$

$$= \left(\frac{0.2}{10}\right)\sin(10t) + \left(0 - \frac{2.3}{10^2-20^2}\right)\cos(10t) + \left(\frac{2.3}{10^2-20^2}\right)\cos(20t)$$

$$= 0.02\sin(10t) + (7.9667\times10^{-3})\cos(10t) - 7.9667\times10^{-3}\cos(20t)$$

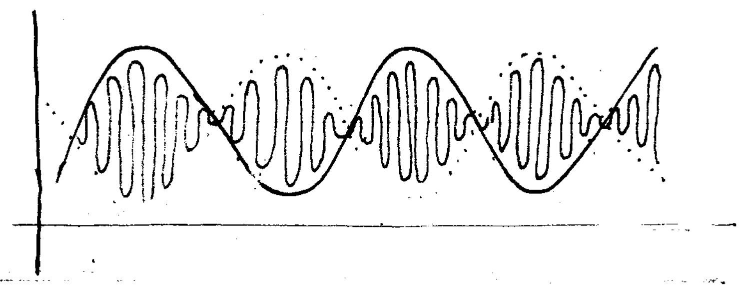
→ When ω is near ω_n, what will happen?
Consider : $f_0 = 1$, $\omega_n = 2\pi$ rad/s, $x_0 = v_0 = 0$

$$x(t) = \left(\frac{f_0}{\omega_n^2-\omega^2}\right)\left[\cos(\omega t) - \cos(\omega_n t)\right]$$

$$\left(\omega_{slow} = \frac{\omega_n - \omega}{2} \quad ; \quad \omega_{fast} = \frac{\omega_n + \omega}{2} \right)$$

when $\omega \rightarrow \omega_n$, becomes a beat (or a beating freq. occurs)

Beat: $\omega_{beat} = |\omega_n - \omega|$



(fix from textbook.)

What if $\omega = \omega_n$?

Particular solution $\rightarrow x_p(t) = x \cdot t \cdot \sin \omega t$

$\Rightarrow \dot{x}_p(t) = X \cdot \sin\omega t + X\omega t \cos\omega t$

$\Rightarrow \ddot{x}_p(t) = X\omega\cos\omega t + X\omega\cos\omega t + (-X t\omega^2 \sin\omega t)$

$\therefore \ddot{x}_p + \omega_n^2 x_p = 2X\omega\cos\omega t$

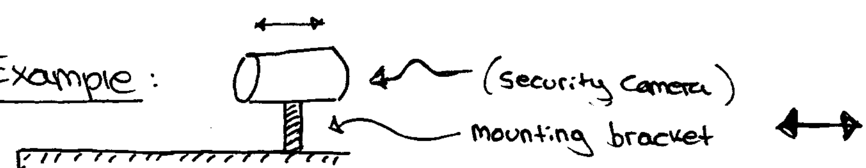
$\ddot{x}_p + \omega_n^2 x_p = \boxed{f_0 \cos\omega t}$

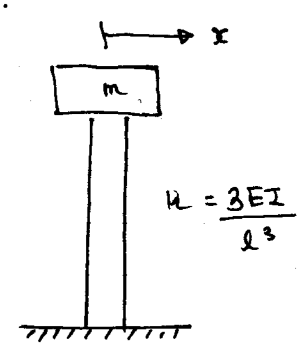
$\rightarrow 2X\omega = f_0 \quad ; \quad X = \dfrac{f_0}{2\omega}$

$\rightarrow x(t) = A_1 \sin(\omega t) + A_2 \cos(\omega t) + (f_0/2\omega)\, t \sin(\omega t)$

$\rightarrow$ the amplitude of the vibration grows without bounds, this is known as a resonance condition.

Example:





Want to design $l > 0.2\,m$

The maximum disp. of the camera is $\leq 0.01\,m$

With load: $F = 15\,N$, $\omega = 10\,Hz$

Camera: $m = 3\,kg$

beam: $0.02 \times 0.02\,m$

Find the length.