Legrange's Method for deriving equations of motion.

For a conservative system:

Kinetic energy $T = (1/2) m \dot{x}^2$

Potential energy $V = (1/2) k x^2$

Define the Legrangian L:

$$L = T - V = \underbrace{(1/2) m \dot{x}^2}_{\text{velocity}} - \underbrace{(1/2) k x^2}_{\text{displacement}}$$

$L = L(x, \dot{x}, t)$ $\quad (\theta, \dot{\theta}, t)$

The equation of motion:

$$\rightarrow \boxed{\frac{d}{dx}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0}$$

Since $L = T - V$:

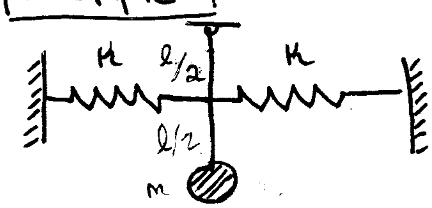
$$\begin{cases} \dfrac{\partial L}{\partial \dot{x}} = \dfrac{\partial T}{\partial \dot{x}} & \left(\text{since } \dfrac{\partial V}{\partial \dot{x}} = 0\right) \\[2ex] \dfrac{\partial L}{\partial x} = \dfrac{\partial T}{\partial x} - \dfrac{\partial V}{\partial x} & \end{cases}$$

$$\Rightarrow \boxed{\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}}\right) - \frac{\partial T}{\partial x} + \frac{\partial V}{\partial x} = 0}$$

Let q be the generalized coordinate,

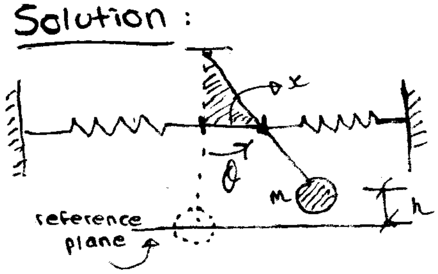
$$\Rightarrow \boxed{\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}}\right) - \frac{\partial T}{\partial q} + \frac{\partial V}{\partial q} = 0}$$

Example



→ Derive the equation of motion.

Solution:



θ: the generalized coordinate

Kinetic Energy:

$$T = (1/2) J \omega^2 = (1/2)(m l^2)\dot{\theta}^2 \quad \left(\frac{1}{2} m (l\dot{\theta})^2\right)$$

Potential Energy:

$$V = (1/2)kx^2 + (1/2)k(-x)^2 + mgh$$ (should be expressed in terms of θ)

where $x = (l/2)\sin\theta$

$h = l - l\cos\theta = l(1-\cos\theta)$

$$\therefore \quad V = (1/2)k\left[(l/2)\sin\theta\right]^2 + (1/2)k\left[(l/2)\sin\theta\right]^2 + mgl(1-\cos\theta)$$

$$V = (1/4)kl^2\sin^2\theta + mgl(1-\cos\theta)$$

$$\frac{\partial T}{\partial \dot\theta} = \left(\frac{1}{2}\right)ml^2 \cdot 2\dot\theta = ml^2\dot\theta$$

$$\frac{\partial T}{\partial \theta} = 0$$

$$\frac{\partial V}{\partial \theta} = \left(\frac{1}{4}\right)kl^2 \cdot 2\sin\theta\cos\theta + mgl(0-(-\sin\theta))$$

$$\frac{\partial V}{\partial \theta} = \left(\frac{1}{2}\right)kl^2\sin\theta\cos\theta + mgl\sin\theta$$

$$\Rightarrow \quad \frac{d}{dt}\left(ml^2\dot\theta\right) - 0 + \left(\frac{1}{2}\right)kl^2\sin\theta\cos\theta + mgl\sin\theta = 0$$

$$ml^2\ddot\theta + (1/2)kl^2\sin\theta\cos\theta + mgl\sin\theta = 0$$

Linearization: θ small,

then $\sin\theta \approx \theta$, $\quad \cos\theta \approx 1$

$$\Rightarrow \quad ml^2\ddot\theta + (1/2)kl^2\theta + mgl\theta = 0$$

$$ml^2\ddot\theta + \left[(1/2)kl^2 + mgl\right]\theta = 0$$

$$\Rightarrow \quad \ddot\theta + \left[\frac{(1/2)kl^2 + mgl}{ml^2}\right]\theta = 0$$

The natural frequency:

$$\omega_n = \sqrt{\frac{(1/2)kl^2 + mgl}{ml^2}}$$

$$\omega_n = \sqrt{\frac{kl + 2mg}{2ml}}$$

Taylor series exp.

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \ldots = \theta$$

$$\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} + \ldots = 1 - \frac{\theta^2}{2}$$

to linearize earlier...

$$\Rightarrow \quad V = (1/4)kl^2\theta^2 + mgl(\theta^2/2)$$

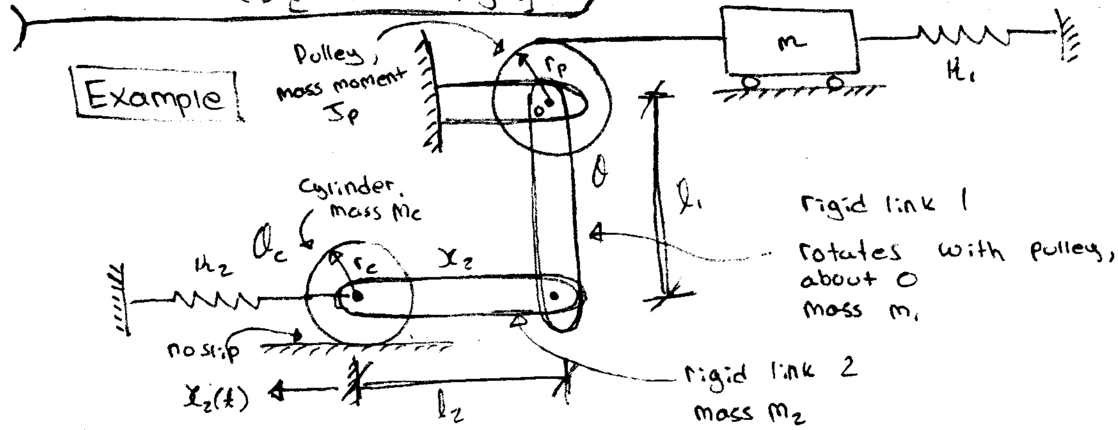
cont'd From prev...

$$\Rightarrow \quad V = (1/4)\, k l^2 \theta^2 + mgl(\theta^2/2)$$

$$\downarrow \; = \left[(1/4)\, k l^2 + (1/2)\, mgl\right]\theta^2$$

$$V = (1/2)\left[(1/2)\, k l^2 + mgl\right]\theta^2$$

Example



Kinetic Energy :

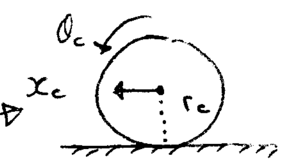
$$T = (1/2) m\dot{x}^2 + (1/2) J_p \dot{\theta}^2 + (1/2)\left[(1/3) m_1 l_1^2\right]\dot{\theta}^2 + (1/2) m_2 \dot{x}_2^2 \;\cdots$$

$$\cdots + (1/2) m_c \dot{x}_2^2 + (1/2)\left[(1/2) m_c r_c^2\right]\dot{\theta}_c^2$$

Pulley : $\quad x = r_p\theta \;\rightarrow\; \theta = x/r_p$

Bars : $\quad x_2 = l_1\theta \;\rightarrow\; x_2 = \dfrac{l_1 x}{r_p} = \dfrac{l_1}{r_p}\, x$

Cylinders :



→ From kinematics

$$\begin{cases} x_c = r_c\theta_c \\ x_c = x_2 \end{cases}$$

$$\theta_c = \frac{x_2}{r_c} = \frac{l_1}{r_c r_p}\, x$$

$$\Rightarrow \quad \dot{\theta} = \frac{\dot{x}}{r_p} \;;\quad \dot{x}_2 = \frac{l_1}{r_p}\,\dot{x} \;;\quad \dot{\theta}_c = \frac{l_1}{r_c r_p}\,\dot{x}$$

$$\therefore \quad T = (1/2) m\dot{x}^2 + (1/2) J_p (\dot{x}/J_p)^2 + (1/2)(1/3\, m_1 l_1^2)(\dot{x}/r_p)^2 \;\cdots$$

$$\cdots + (1/2) m_2 \left[(l_1/r_p)\,\dot{x}\right]^2 + (1/2) m_c \left[(l_1/r_p)\,\dot{x}\right]^2 \;\cdots$$

$$\cdots + (1/2)(1/2\, m_c r_c^2)\left(l_1/r_c r_p\;\dot{x}\right)$$

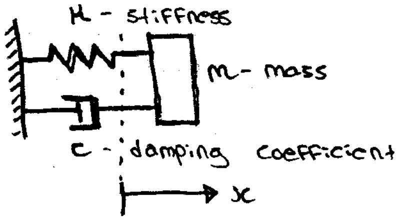
$$= (1/2)\, m_{eq}\,\dot{x}^2$$

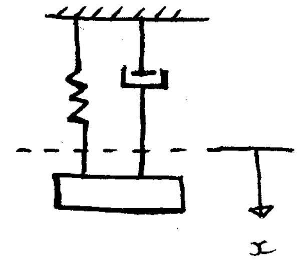
Here

$$m_{eq} = m + J_p/r_p^2 + 1/3\, m_1\, l_1^2/r_p^2 + m_2\, l_1^2/r_p^2 + m_c(l_1/r_p)^2 + (1/2) m_c (l_1/r_p)^2$$

$$= m + J_p/r_p^2 + (1/3) m_1 (l_1^2/r_p^2) + m_2 (l_1^2/r_p^2) + (3/2) m_c (l_1^2/r_p^2)$$

Free response with viscous damping





damping Force

$f_c = -cv = -c\dot{x}$

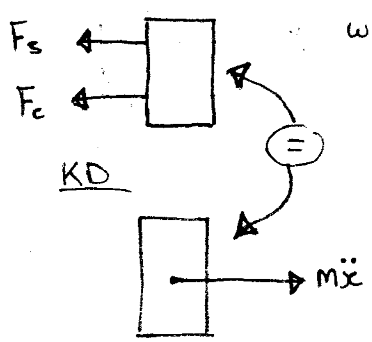
(f_c in N, v in m/s)

unit of the damping constants $[c]$

$$[c] = \frac{[f]}{[v]} = \frac{N}{m/s} = \frac{N \cdot s}{m}$$

$$[c] = \frac{kg \cdot m}{s^2} \cdot \frac{s}{m} = kg/s$$

FBD



where $F_s = kx$

$F_c = c\dot{x}$

$\Rightarrow -kx - c\dot{x} = m\ddot{x}$

$\Rightarrow \boxed{m\ddot{x} + c\dot{x} + kx = 0}$

Equation of Motion (ODE)

Assume $x = a e^{\lambda t}$ (a const., λ const.)

Then $\dot{x} = a\lambda e^{\lambda t} = \lambda x$ (*)

$\ddot{x} = \lambda \dot{x} = \lambda^2 x$ (*)

$\Rightarrow m\lambda^2 \underline{x} + c\lambda \underline{x} + k\underline{x} = 0$

$\Rightarrow \boxed{(m\lambda^2 + c\lambda + k)x = 0}$

Since $x \neq 0$

$$\boxed{m\lambda^2 + c\lambda + k = 0}$$

$$\lambda^2 + \frac{c}{m}\lambda + \frac{k}{m} = 0$$

The roots:

$$\lambda_{1,2} = \frac{1}{2}\left(-\frac{c}{m} \pm \sqrt{\left(\frac{c}{m}\right)^2 - \frac{4k}{m}}\right)$$

Define the critical damping constant C_{cr} :

$$\left(\frac{C_{cR}}{m}\right)^2 - \frac{4K}{m} = \emptyset$$

$$\Rightarrow \quad C_{cR} = 2\sqrt{Km}$$

Define the damping ratio :

$$\boxed{\zeta = \frac{c}{C_{cR}}}$$

$$c = \zeta \cdot C_{cR} = \zeta \cdot 2\sqrt{Km}$$

$$\Rightarrow \lambda^2 + \frac{\zeta \cdot 2\sqrt{Km}}{m}\lambda + \frac{K}{m} = \emptyset$$

$$\lambda^2 + 2\zeta\sqrt{K/m}\,\lambda + K/m = \emptyset$$

$\omega_n = \sqrt{K/m}$ ← natural frequency

$$\Rightarrow \boxed{\lambda^2 + 2\zeta\omega_n\lambda + \omega_n^2 = \emptyset}$$

$$\Rightarrow \lambda_{1,2} = -\zeta\omega_n \pm \sqrt{\zeta^2 - 1}\,\omega_n$$

Case #1 Critically Damped Motion : $\zeta = 1$

$\lambda_1 = \lambda_2 = -\zeta\omega_n$

The solution of the system :

$$x = a_1 e^{-\omega_n t} + a_2 t e^{-\omega_n t} = (a_1 + a_2 t)e^{-\omega_n t}$$

The initial conditions:

$x(\emptyset) = x_0$ ← given ; $\dot{x}(\emptyset) = v_0$ ← given

Since $x(\emptyset) = (a_1 + a_2 t)e^{-\omega_n t}\Big|_{t=\emptyset} = a_1 = x_0$ (*)

$$\dot{x}(\emptyset) = a_2 e^{-\omega_n t} + (a_1 + a_2 t)(-\omega_n e^{-\omega_n t})$$

$$= (a_2 - [a_1 + a_2 t]\omega_n)e^{-\omega_n t}$$

$$\dot{x}(\emptyset) = a_2 - a_1\omega_n = v_0$$

(*) $\Rightarrow a_2 = v_0 + x_0\omega_n$

$\therefore x = [x_0 + (v_0 + x_0\omega_n)t]e^{-\omega_n t}$ ← response of system w/ critical damping

$t \to +\infty, \quad x \to \emptyset$

$\boxed{\text{Example}}$ $\quad m = 100 \text{ kg}$

$k = 225 \text{ N/m}$

$\zeta = 1$

Find the disp of the system for different initial conditions.

1°: $x_0 = 0.4 \text{mm}$; $v_0 = 1 \text{ mm/s}$

2°: $x_0 = 0.4 \text{mm}$; $v_0 = 0$

3°: $x_0 = 0.4 \text{mm}$; $v_0 = -1 \text{ mm/s}$

Solution : $\omega_n = \sqrt{k/m}$

$\omega_n = \sqrt{225/100} = 1.5 \text{ rad/s}$

$$\left.\begin{aligned} 1°:\quad & x(t) = (0.4 + 1.6t)e^{-1.5t} \\ 2°:\quad & x(t) = (0.4 + 0.6t)e^{-1.5t} \\ 3°:\quad & x(t) = (0.4 - 0.4t)e^{-1.5t} \end{aligned}\right\} \text{(See picture for graph)}$$

Case #2: Overdamped motion $(\zeta > 1)$

$$\lambda_{1,2} = -\zeta\omega_n \pm \sqrt{\zeta^2-1}\,\omega_n$$

The disp. :

$$x = a_1 e^{\lambda_1 t} + a_2 e^{\lambda_2 t}$$

$$= a_1 e^{-\zeta\omega_n t + \sqrt{\zeta^2-1}\,\omega_n} + a_2 e^{-\zeta\omega_n t - \sqrt{\zeta^2-1}\,\omega_n}$$

(∗) → $$x = e^{-\zeta\omega_n t}\left(a_1 e^{\sqrt{\zeta^2-1}\,\omega_n t} + a_2 e^{\sqrt{\zeta^2-1}\,\omega_n t}\right)$$

When $x(0) = x_0$

$\dot{x}(0) = v_0$

$$a_2 = \frac{-v_0 + (-\zeta + \sqrt{\zeta^2-1})\,\omega_n x_0}{2\omega_n\sqrt{\zeta^2-1}}$$

$$a_1 = \frac{v_0 + (\zeta + \sqrt{\zeta^2-1})\,\omega_n x_0}{2\omega_n\sqrt{\zeta^2-1}}$$

Case #3: Underdamped motion $(\zeta < 1) \rightarrow (0 < \zeta < 1)$

$\zeta^2 - 1 < 0$

$$\lambda_{1,2} = -\zeta\omega_n \pm \sqrt{\zeta^2-1}\,\omega_n$$

$$= -\zeta\omega_n \pm j\sqrt{1-\zeta^2}\,\omega_n \qquad (j = \sqrt{-1})$$

$\lambda_2 = \bar{\lambda}_1$ (conjugate)

$$x = a_1 e^{\lambda_1 t} + a_2 e^{\lambda_2 t}$$

$$= a_1 e^{-\zeta\omega_n t + j\sqrt{1-\zeta^2}\,\omega_n t} + a_2 e^{-\zeta\omega_n t - j\sqrt{1-\zeta^2}\,\omega_n t}$$

$$= e^{-\zeta\omega_n t}\left(a_1 e^{j\sqrt{1-\zeta^2}\,\omega_n t} + a_2 e^{-j\sqrt{1-\zeta^2}\,\omega_n t}\right)$$

where $e^{j\alpha} = \cos\alpha + j\sin\alpha$

Define ω_d :

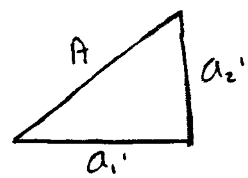
$$\boxed{\omega_d = \sqrt{1-\zeta^2}\,\omega_n}$$

damped natural freq.

$$x = e^{-\zeta\omega_n t}\left(a_1 e^{j\omega_d t} + a_2 e^{-j\omega_d t}\right)$$

$$= e^{-\zeta\omega_n t}\left(a_1\cos(\omega_d t) + j a_1 \sin(\omega_d t) + a_2\cos(\omega_d t) + j a_2 \sin(-\omega_d t)\right)$$

$$= e^{-\zeta\omega_n t}\left(\underbrace{(a_1+a_2)}_{a_1'}\cos\omega_d t + j\underbrace{(a_1-a_2)}_{a_2'}\sin\omega_d t\right).$$



$$\boxed{x = e^{-\zeta\omega_n t}\left(a_1'\cos(\omega_d t) + a_2'\sin(\omega_d t)\right)}$$

$$\boxed{x = Ae^{-\zeta\omega_n t}\sin(\omega_d t + \phi)}$$

Phase angle

$x(0) = x_0$; $\dot{x}(0) = v_0$

Since :

$$x(0) = A\sin(\phi) = x_0$$

$$\dot{x}(0) = \left(-\zeta\omega_n A e^{-\zeta\omega_n t}\right)\sin(\omega_d t + \phi) + Ae^{-\zeta\omega_n t}\cos(\omega_d t + \phi)(\omega_d)\Big|_{t=0}$$

$$= -\zeta\omega_n A\sin(\phi) + A\omega_d\cos(\phi) = v_0$$

$$= -\zeta\omega_n x_0 + A\omega_d\cos(\phi) = v_0$$

$$\Rightarrow A\cos(\phi) = \frac{v_0 + \zeta\omega_n x_0}{\omega_d}$$

$$\Rightarrow A\sin(\phi) = x_0$$

$$\Rightarrow A = \sqrt{x_0^2 + \left(\frac{v_0 + \zeta\omega_n x_0}{\omega_d}\right)^2}$$

$$\phi = \tan^{-1}\left(\frac{x_0\omega_d}{v_0 + \zeta\omega_n x_n}\right)$$