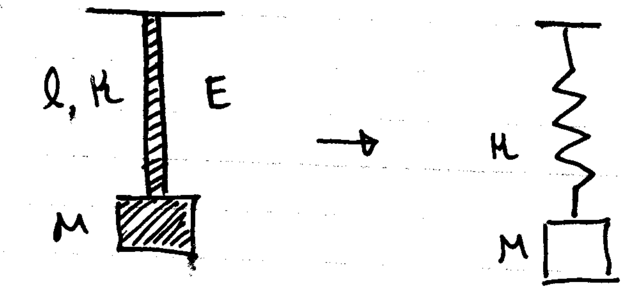
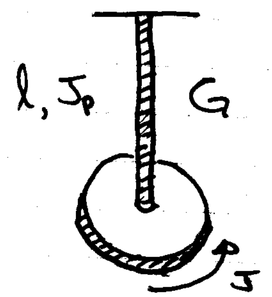


$$k = \frac{AE}{l}$$



$$k = \frac{GJ_p}{l}$$

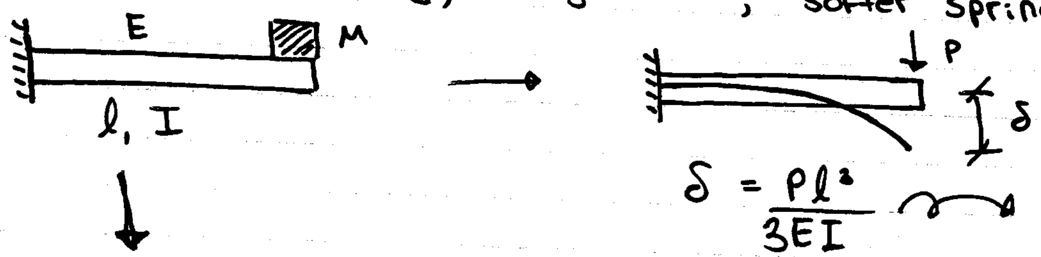
$$\omega_n = \sqrt{\frac{k}{J}}$$

natural frequency

moment of inertia of disk

number of turns

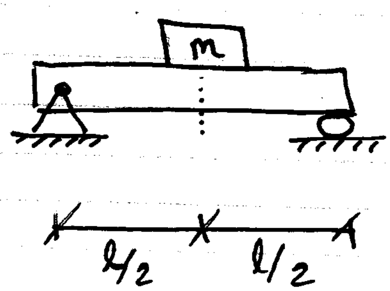
- shorter spring, smaller "n", stronger spring
- longer spring, larger "n", softer spring

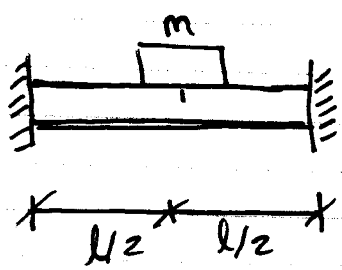


$$\delta = \frac{Pl^3}{3EI} \rightarrow P = \boxed{\frac{3EI}{l^3}}\,\delta$$

$$k = \frac{3EI}{l^3}$$

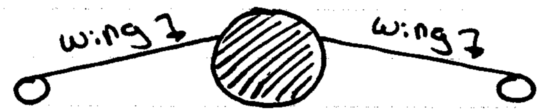
$$\omega_n = \sqrt{\frac{k}{m}}$$





→ Page 53, table 1.3

Example Consider front of airplane :



No fuel, $m = 10$ kg
Full fuel, $m = 1000$ kg
$I = 5.2 \times 10^{-5}$
$l = 2m$
$E = 6.9 \times 10^{9}$ Pa
→ Find the freq. of the wing for the two cases

Solution :
Full fuel, $\omega_n = \sqrt{\dfrac{k}{m}} = \sqrt{\dfrac{3EI}{ml^3}}$

$$= \sqrt{\frac{(3)\times(6.9\times10^{9})\times(5.2\times10^{-5})}{(1000)\times(2)^3}}$$

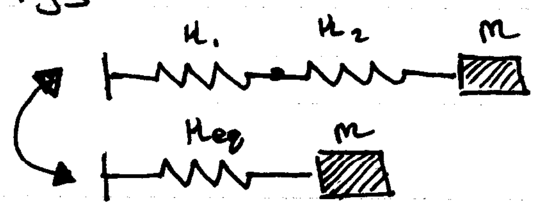
$\omega_n = 11.6$ rad/s

No fuel, $\omega_n = \sqrt{\dfrac{(3)(6.9\times10^{9})(5.2\times10^{-5})}{(10)(2)^3}}$

$\omega_n = 115$ rad/s
↳ this doesn't consider mass of wing

Combining Springs

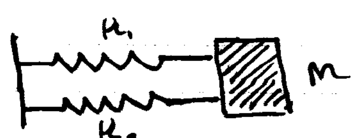
→ series



Each spring has the same force.

$$k_{eq} = \frac{1}{1/k_1 + 1/k_2} = \frac{k_1 k_2}{k_1 + k_2}$$

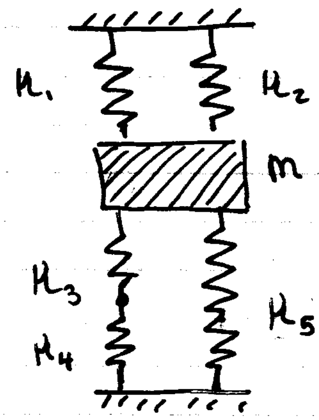
→ Parallel



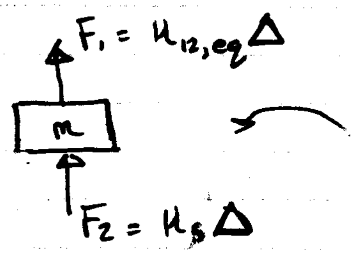
Each spring has the same deformation
$k_{eq} = k_1 + k_2$

Example :

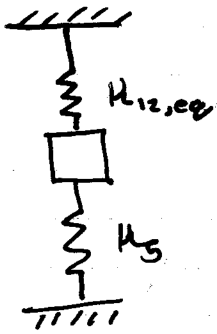
Find the equivalent stiffness (k_{eq}) :

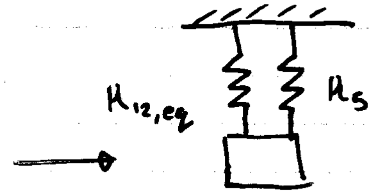


Solution :



consider :



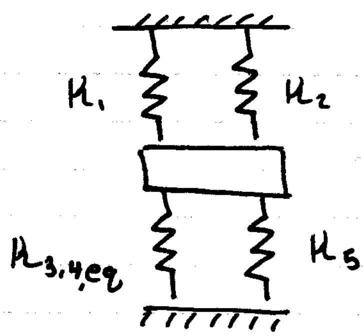


Thus, <u>Same deformation</u>

$$F = F_1 + F_2 = (k_{12,eq} + k_5)\Delta = k_{eq}$$

Springs k_3 and k_4 are in series.

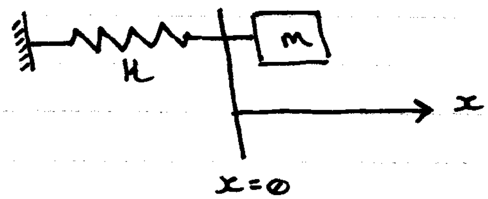
$$k_{34,eq} = \frac{k_3 k_4}{k_3 + k_4}$$



k_1 k_2 $k_{34,eq}$ and k_5 are in parallel.

$$k_{eq} = k_1 + k_2 + k_5 + k_{34,eq}$$
$$= k_1 + k_2 + k_5 + \frac{k_3 k_4}{k_3 + k_4}$$

Harmonic motion



$$m\ddot{x} + kx = 0$$

(where: $\omega_n = \sqrt{\frac{k}{m}}$

(Divide by m) $\ddot{x} + \frac{k}{m}x = 0$

(Replace) $\boxed{\ddot{x} + \omega_n^2 x = 0}$

Displacement : $x = A\sin(\omega_n t + \phi)$

Velocity : $\dot{x} = \omega_n A\cos(\omega_n t + \phi)$

Acceleration : $\ddot{x} = -\omega_n^2 \underbrace{A\sin(\omega_n t + \phi)}_{x} = -\omega_n^2 x$

Max displacement : $x_{max} = A$

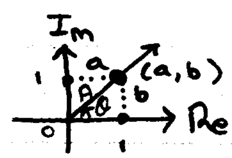
Max velocity : $v_{max} = \omega_n A$ (or when $\cos(\omega_n t + \phi) = 1$)

Max acceleration : $a_{max} = \omega_n^2 A$ (or when $x = A$)

Complex number

$C = a + ib$

$i = \sqrt{-1}$



where $a = A\cos\theta$

$b = A\sin\theta$

then $C = A\cos\theta + iA\sin\theta = A(\cos\theta + i\sin\theta)$

$C = Ae^{i\theta}$ $\quad \boxed{e^{i\theta} = \cos\theta + i\sin\theta}$

Differential equation

$m\ddot{x} + kx = 0$

Solution of the ODE

$x(t) \doteq ae^{\lambda t}$ (*)

→ $\dot{x} = \lambda a e^{\lambda t} = \lambda x(t)$

$\ddot{x} = \lambda\dot{x} \doteq \lambda^2 x(t)$ (*)

Substitution: $m\lambda^2 x(t) + kx(t) = 0$

$(m\lambda^2 + k)x(t) = 0$

Since $x(t) \neq 0$

$$m\lambda^2 + K = 0$$

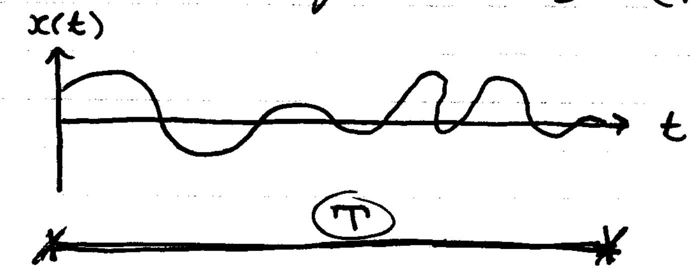
$$\rightarrow \lambda = \pm\sqrt{-K/m} = \pm i\sqrt{K/m}$$

$$\lambda = \pm i\omega_n$$

$x(t) = a_1 e^{i\omega_n t} + a_2 e^{-i\omega_n t}$; a_1 and a_2 are constant

$$x(t) = A\sin(\omega_n t + \phi)$$

Root mean square values (RMS):



$$\lim_{T\to\infty} \frac{1}{T}\int_0^T x(t)\,dt = \bar{x}$$

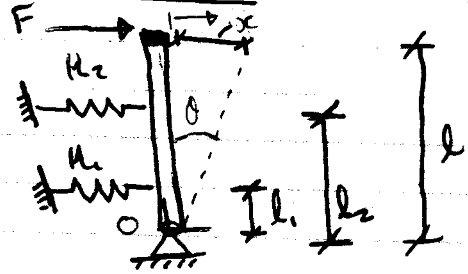
average value

$$\bar{x}^2 = \lim_{T\to\infty} \frac{1}{T}\int_0^T x^2(t)\,dt$$

mean square value

$$x_{rms} = \sqrt{\bar{x}^2}$$

Example



Find the equivalent stiffness of the system that relates the applied force to the resulting displacement x

$$F = k_{eq} x$$

Solution potential energy in the real system equals to the energy stored in the equivalent spring:

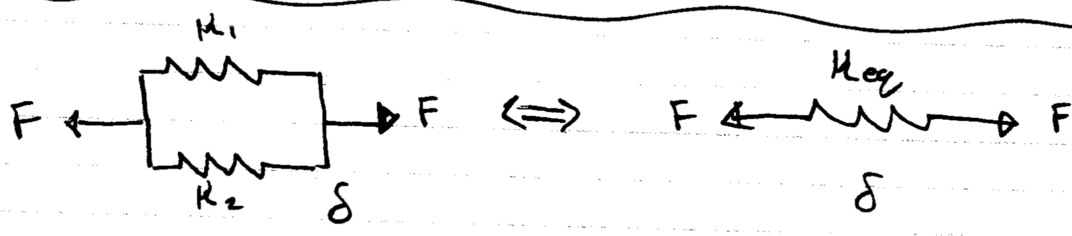
For small disp. x, angle θ is small as well.
$x_1 = l_1\theta$; $x_2 = l_2\theta$ (only when ↻)

$$(1/2)k_1 x_1^2 + (1/2)k_2 x_2^2 = (1/2)k_{eq} x^2$$

Since $x = l\theta$:

$$(1/2)k(l_1\theta)^2 + (1/2)k_2(l_2\theta)^2 = (1/2)(l\theta)^2$$

$$\boxed{k_{eq} = k_1(l_1/l)^2 + k_2(l_2/l)^2}$$



$$(1/2)k_1\delta^2 + (1/2)k_2\delta^2 = (1/2)k_{eq}\delta^2$$

The decibel dB scale

measure the vibration relative to some reference value:

$$10 \log_{10}\left(\frac{x}{x_0}\right)^2 \leftarrow \text{dB}$$

Reference power P_0, the sound produces twice as much as the reference.

If $P = 2P_0$:

$$10 \log_{10}(P/P_0) = 10\log_{10} 2 = 3\,\text{dB}$$

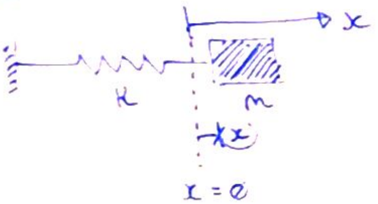
If $P = 10P_0$:

$$10 \log_{10} 10 = 10\,\text{dB}$$

If $P = 10^6 P_0$:

$$10 \log_{10}(10^6) = 60\,\text{dB}$$

Modeling and Energy Methods



Potential energy

$\rightarrow U = (1/2)kx^2$

Kinetic energy

$\rightarrow T = (1/2)m\dot{x}^2$

For rotating about a fixed axis:

$\rightarrow T = (1/2)J\dot{\theta}^2$

Conservation of energy:

$\rightarrow T + U = \text{const.}$

$T_1 + U_1 = T_2 + U_2$

$T_{max} = U_{max}$

$d/dt(T + U) = 0$

Spring-mass:

$T = (1/2)m\dot{x}^2$

$U = (1/2)kx^2$

$d/dt(T+U) = 0$

$\rightarrow d/dt((1/2)m\dot{x}^2 + (1/2)kx^2) = 0$

$1/2\, m \cdot 2\dot{x}\, d\dot{x}/dt + 1/2\, k \cdot 2x\, dx/dt = 0$

$m\dot{x}\ddot{x} + kx\dot{x} = 0$

$\dot{x}(m\ddot{x} + kx) = 0$

Since $\dot{x}$ cannot be zero all the time,

$\rightarrow m\ddot{x} + kx = 0$

$\boxed{\text{Example}}$ Find the natural frequency from the energy:

The displacement : $x = A\sin(\omega_n t + \phi)$

$x_{max} = A$

Velocity : $\dot{x} = A\omega_n \cos(\omega_n t + \phi)$

$\dot{x}_{max} = A\omega_n$

$T_{max} : (1/2)m(\dot{x}_{max})^2 = (1/2)m(A\omega_n)^2$

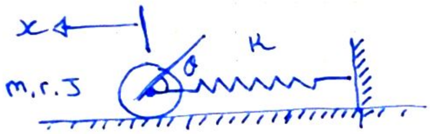
$U_{max} : (1/2)k(x_{max})^2 = (1/2)kA^2$

Since $T_{max} = U_{max}$;

$(1/2)m(A\omega_n)^2 = (1/2)kA^2$

$\omega_n = \sqrt{k/m}$

$\boxed{\text{Example}}$



Assume it is a conservative system and rolls without slipping
Find the natural frequency of the disk.

Solution : rolling w/o slipping

$x = r\theta$

$(x = A\sin(\omega_n t + \phi))$

$\dot{\theta} = \dot{x}/r$

The kinetic energy

$$T = (1/2)J\dot{\theta}^2 + (1/2)m\dot{x}^2$$

$$= (1/2)J(\dot{x}/r)^2 + (1/2)m\dot{x}^2$$

$$= (1/2)\Big[\underbrace{(J/r^2) + m}_{\text{equivalent mass}}\Big]\dot{x}^2$$

$$\therefore T_{max} = (1/2)\{(J/r^2) + m\}(A\omega_n)$$

Potential energy

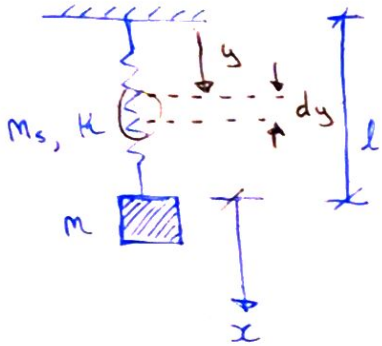
$U = (1/2) K x^2$

$U_{max} = (1/2) K A^2$

$\Rightarrow (1/2)\left[(J/r^2) + m\right](A\omega_n)^2 = (1/2) K A^2$

$\Rightarrow \omega_n = \sqrt{\dfrac{K}{(J/r^2)+m}}$

$\boxed{\text{Example}}$ The effect of including the mass of the spring on the value of the frequency



Solution: The mass per unit length of the spring

m_s / l

Assumptions

The mass element dy:

$\dfrac{m_s}{l} dy$

the velocity:

$\dfrac{y}{l} \dot{x}$

$$T_s = \int_0^l (1/2)\left[(m_s/l)\,dy\right]\left[(y/l)\,\dot{x}\right]^2$$

$$= \left(\frac{1}{2}\right)\frac{m_s}{l}\left(\frac{\dot{x}}{l}\right)^2 \int_0^l y^2\,dy$$

$$= (1/2)(m_s/3)\,\dot{x}^2$$

The total kinetic energy:

$$T = (1/2)(m_s/3)\,\dot{x}^2 + (1/2)m\dot{x}^2$$

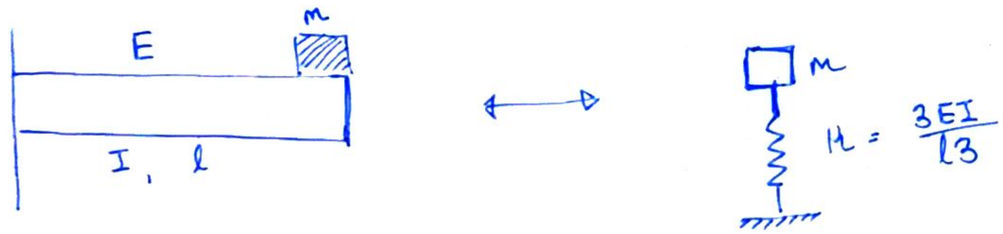
$$T = (1/2)\left[(m_s/3) + m\right]\dot{x}^2$$

$\longrightarrow T_{max} = (1/2)(m_s/3 + m)(A\omega_n)^2$

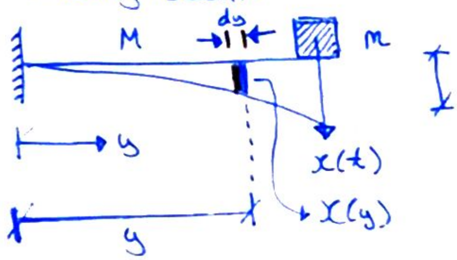
$U_{max} = (1/2) K A^2$

$\longrightarrow T_{max} = U_{max}$

$$\omega_n = \sqrt{\frac{K}{\left(\frac{m_s}{3} + m\right)}}$$



"Heavy beam"



The deflection at position y is:

$$x(y) = \frac{Py^2}{6EI}(3l - y)$$

The maximum deflection occurs at $y = l$

$$x_{max} = \frac{Pl^3}{3EI} \quad ; \quad P = \left(\frac{3EI}{l^3}\right) x_{max}$$

$$\rightarrow x(y) = \frac{3EI}{l^3} x_{max} \cdot \frac{y^2}{6EI}(3l - y)$$

$$= \frac{y^2}{2l^3}(3l - y) \cdot x_{max}$$

$$\dot{x}(y) = \frac{y^2(3l - y)}{2l^3} \dot{x}_{max}$$

For a small beam segment dy,

$$T_{beam} = \int_0^l \left(\frac{1}{2}\right)\left(\frac{M}{l} dy\right)\left(\dot{x}(y)\right)^2$$

$$\Rightarrow \int_0^l \left(\frac{1}{2}\right)\left(\frac{M}{l} dy\right)\left(\frac{y^2(3l - y)}{2l^3} \dot{x}_{max}\right)^2$$

$$= \left(\frac{1}{2}\right)\left(\frac{33}{140} M\right) \dot{x}_{max}^2$$

The total kinetic energy:

$$\rightarrow T = (1/2)\left[(33/140)M + m\right] \dot{x}_{max}^2$$

The equivalent mass of the system is:

$$\rightarrow m_{eq} = (33/140)M + m$$

$$\therefore \omega_n = \sqrt{\frac{\frac{3EI}{l^3}}{\frac{33}{140}M + m}}$$