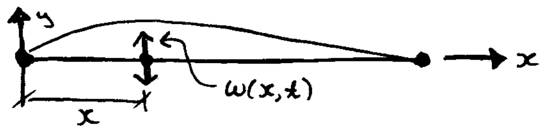
NOV. 26/19

$\boxed{\text{Example}}$ determine the frequencies of a string (G3)



Tension $\Rightarrow$ $T = \text{const.}$

mass density $\Rightarrow$ $\rho = \text{const.}$ (mass per unit length)

<u>Solution</u>:

$$c^2 \frac{\partial^2 w}{\partial x^2} = \frac{\partial^2 w}{\partial x^2} \quad ; \quad c = \sqrt{\frac{T}{\rho}}$$

tension

mass density

$x = 0, \quad w(0,t) = 0$
$x = l, \quad w(l,t) = 0$ } Boundary conditions

$w(x,t) = W(x)\, e^{i\omega t}$

$\rightarrow c^2 W'' = -\omega^2 W$

$W'' + (\omega^2/c^2) W = 0$

$\rightarrow W(x) = a_1 \sin(\omega/c)x + a_2 \cos(\omega/c)x$

@ $x = 0$, $w(0,t) = W(0)e^{i\omega t} = 0$

$W(0) = 0$

@ $x = l$, $w(l,t) = W(l)e^{i\omega t} = 0$

$W(l) = 0$

$$\rightarrow \begin{cases} a_1(0) + a_2(1) = 0 \longrightarrow a_2 = 0 \\ a_1 \sin(\omega l/c) + a_2 \cos(\omega l/c) = 0 \end{cases}$$

$\hookrightarrow a_1 \sin(\omega l/c)$

since $a_1 \neq 0$

$$\boxed{\sin(\omega l/c) = 0}$$

The solution:

$(\omega l/c) = \pi, 2\pi, 3\pi, \ldots$

or $(\omega_n l/c) = n\pi, \quad n = 1, 2, 3, \ldots$

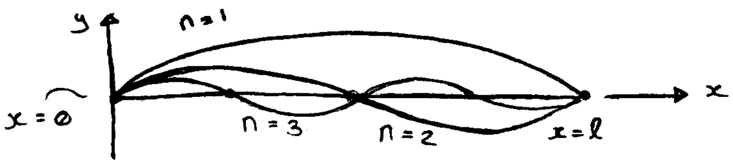
$\hookrightarrow \omega_n = \dfrac{n\pi l}{c}, \quad n = 1, 2, 3, \ldots$

The Fundamental Frequency is :

$$\boxed{\omega_1 = \frac{\pi c}{l}}$$

The mode shapes (eigenFunctions)

$$W_n(x) = a_1 \sin\left(\frac{\omega_n x}{c}\right) \quad ; n = 1, 2, 3, \dots$$
$$= a_1 \sin\left(\frac{n\pi x}{l}\right)$$



→ For the Frequency ω_n

$$W_n(x) \cdot \left(A_n \sin(\omega_n t) + B_n \cos(\omega_n t)\right)$$

→ For the response of the string

$$w(x,t) = \sum_{n=1}^{\infty} W_n(x)\left(A_n \sin(\omega_n t) + B_n \cos(\omega_n t)\right)$$

For G3 :

Tension : $T = 30$ lbs $= 133.447$ N

mass density : $\rho = 0.00011602$ lbs/in

$= 0.00207510$ kg/m

The wave (phase) velocity :

$$c = \sqrt{\frac{T}{\rho}} \rightarrow c = \sqrt{\frac{133.447}{0.00207510}} \rightsquigarrow c = 253.592 \text{ m/s}$$

The length of the string is :

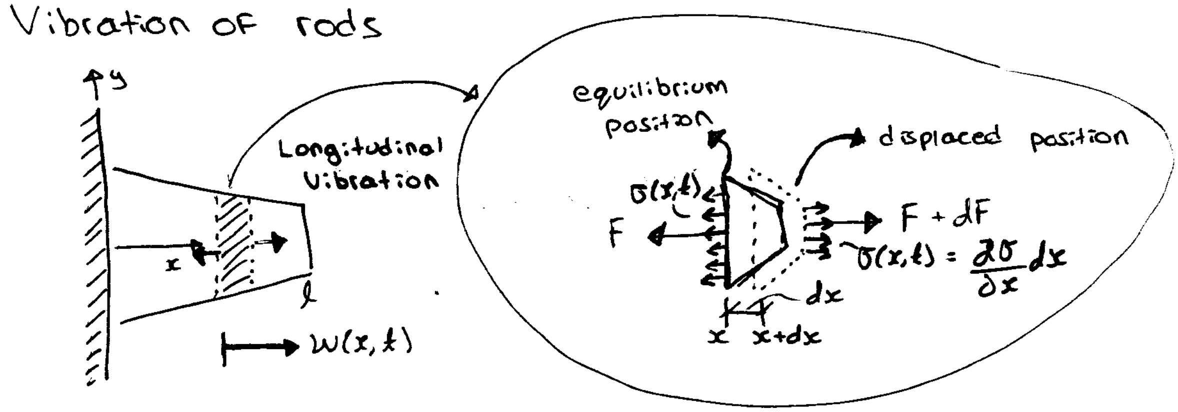
$l = 25.5$ in $= 0.65$ m

The Fundamental Freq :

$$\omega_1 = \frac{\pi c}{l} \quad \text{(but we use Hz)}$$

$$\text{or} \quad f_1 = \frac{\omega_1}{2\pi} = \frac{c}{2l} = \frac{253.592}{2(0.65)} = 195.1 \text{ Hz}$$

Vibration of rods



$$F + dF - F = ma$$
$$F + dF = A + (dA/dx)dx\left(\sigma + \frac{\partial \sigma}{\partial x}dx\right)$$
$$F = A\sigma$$
$$m = \rho A\, dx \qquad a = (\partial^2 w/\partial t^2)$$

$$\longrightarrow \quad (A + dA)\left(\sigma + (\partial\sigma/\partial x)dx\right) - A\sigma = \rho A\, dx \cdot (\partial^2 w/\partial t^2)$$
$$A(\partial\sigma/\partial x)dx + \sigma\, dA = \rho A(\partial^2 w/\partial t^2)dx$$
$$\partial/\partial x(A\sigma) = \rho A(\partial^2 w/\partial t^2)$$

From Hooke's Law: $\sigma = E\varepsilon = E(\partial w/\partial x)$

$$\longrightarrow \quad (\partial^2 w/\partial x^2)AE = \rho A(\partial^2 w/\partial t^2)$$

$$\Rightarrow \quad \boxed{\frac{\partial}{\partial x}\left(AE\frac{\partial w}{\partial x}\right) = \rho A\left(\frac{\partial^2 w}{\partial t^2}\right)}$$

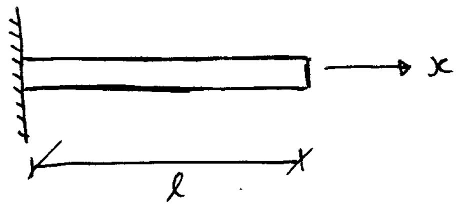
Consider: $A = \text{const.}$, $E = \text{const.}$

$$\frac{E}{\rho}\left(\frac{\partial^2 w}{\partial x^2}\right) = \left(\frac{\partial^2 w}{\partial t^2}\right)$$

Define: $c = \sqrt{\dfrac{E}{\rho}}$

$$\Rightarrow \quad \boxed{c^2\left(\frac{\partial^2 w}{\partial x^2}\right) = \left(\frac{\partial^2 w}{\partial t^2}\right)}$$

For the Fixed-Free bar :



E, ρ : constant

Let : $\quad w(x,t) = W(x)\, e^{i\omega t}$

$$\Rightarrow \quad \frac{\partial^2 w}{\partial x^2} + \frac{\omega^2}{c^2} W = 0$$

$$W(x) = a_1 \sin\left(\tfrac{\omega}{c} x\right) + a_2 \cos\left(\tfrac{\omega}{c} x\right)$$

Boundary conditions :

$x = 0 \;, \quad w(0,t) = 0 \quad \Rightarrow \quad W(0) = 0$

$x = l \;, \quad \sigma(l,t) = 0$

$$\sigma(x,t) = E\varepsilon(x,t) = E\left(\partial w / \partial x\right)$$

$$\sigma(l,t) = E\left(\partial w/\partial x \,\Big|_{x=l}\right) = E\left(\partial W/\partial x \big|_{x=l}\right) e^{i\omega t} = 0$$

$$\Rightarrow \quad \frac{\partial W}{\partial x} = 0$$

$$W(0) = a_1 \cdot 0 + a_2 \cdot 1 = a_2 = 0$$

$$\left.\frac{\partial W}{\partial x}\right|_{x=l} = a_1 \left(\frac{\omega}{c}\right) \cos\left(\frac{\omega l}{c}\right) = 0$$

$a_1 \neq 0 \,, \quad \omega \neq 0$

$$\boxed{\cos\left(\frac{\omega l}{c}\right) = 0}$$

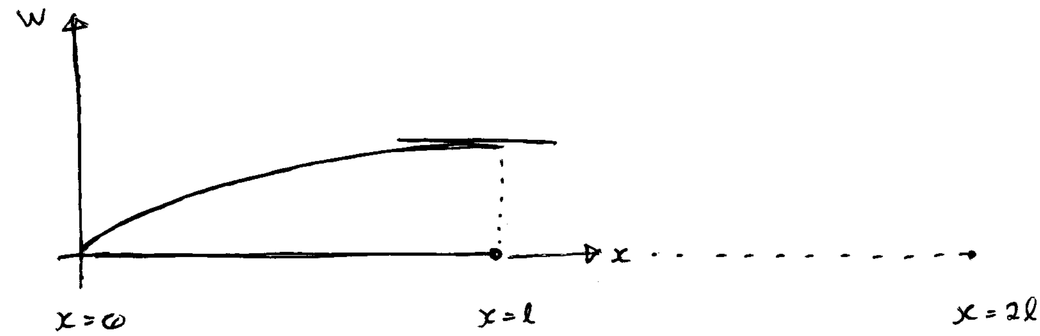
$$\Rightarrow \quad \frac{\omega l}{c} = \frac{\pi}{2} \,, \quad \pi + \frac{\pi}{2} \,, \quad 2\pi + \frac{\pi}{2} \,, \quad \cdots\cdots \,, \quad n\pi + \frac{\pi}{2}$$

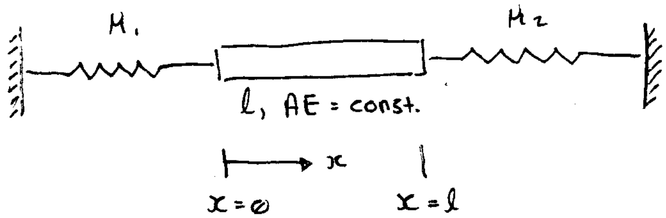
$$\frac{\omega_n l}{c} = (n-1)\pi + \frac{\pi}{2} = (n - 1/2)\pi \,, \quad n = 1, 2, 3 \ldots$$

$$\omega_n = \left(\frac{c}{l}\right)\left(\frac{2n-1}{2}\right)\pi \,, \quad n = 1, 2, 3, \ldots$$

$$W_n(x) = a_1 \sin\left(\frac{\omega_n x}{c}\right) = a_1 \sin\left[\frac{(2n-1)\pi}{2} \cdot \frac{x}{l}\right]$$

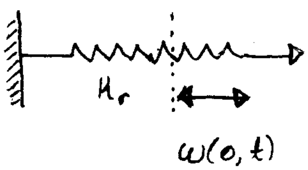
$n = 1 : \quad \omega_1 = \dfrac{c\pi}{2l} \quad ; \quad$ and $\quad W_1 = a_1 \sin\left(\pi x / 2l\right)$


w
x
x = 0
x = l
x = 2l



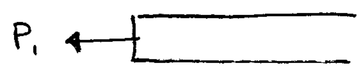
$$c^2 \left(\frac{\partial^2 w}{\partial x^2}\right) = \left(\frac{\partial^2 w(x,t)}{\partial x^2}\right) \qquad 0 < x < l$$

Left end has a displacement $w(0,t)$



$P_1 = k_1 w(0,t)$

(Equivalent)



$$P_1 = A\sigma(0,t) = AE\varepsilon(0,t) = AE\,\frac{\partial w(0,t)}{\partial x}$$

$$\Rightarrow \quad k_1 w(0,t) = AE\left(\frac{\partial w(0,t)}{\partial x}\right)$$

At right end, $x = l$

$$k_2\, w(l,t) = -AE\left(\frac{\partial w(l,t)}{\partial x}\right)$$

Free-vibration:

$$w(x,t) = W(x)\,e^{i\omega t}$$

Eq. of motion:

$$c^2 \frac{d^2 W(x)}{dx^2} = -\omega^2 W$$

$$\Rightarrow \quad \frac{d^2 W(x)}{dx^2} + \frac{\omega^2}{c^2} W = 0$$

$$\Rightarrow \quad W(x) = a_1 \sin(\omega/c)x + a_2 \cos(\omega/c)x$$

Boundary conditions

$x = 0: \quad K_1 w(0,t) = \dfrac{AE\,\partial w(0,t)}{\partial x}$

$K_1 W(0) = AE\,W'(0)$

$x = l: \quad K_2 W(l) = -AE\,W'(l)$

Since $W'(x) = a_1 \,\omega/c \cos \omega/c\, x - a_2\, \omega/c \sin \omega/c\, x$

$x = 0: \quad K_1 a_2 = AE \cdot a_1 \,\omega/c$

$x = l: \quad K_2 (a_1 \sin(\omega l/c) + a_2 \cos(\omega l/c)) = AE(\omega/c)(a_1 \cos(\omega l/c) - a_2 \sin(\omega l/c))$

$$\rightarrow \begin{bmatrix} \times & \times \\ \times & \times \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix} = 0$$

$$\rightarrow \det \begin{bmatrix} \times & \times \\ \times & \times \end{bmatrix} = 0$$

$$a_1 = a_2 \frac{K_1}{AE(\omega/c)}$$

Sub $K_2\left(a_2 \dfrac{K_1}{AE(\omega/c)} \sin(\omega l/c) + a_2 \cos(\omega l/c)\right)$

$= -AE(\omega/c)\left(a_2 \dfrac{K_1}{AE(\omega/c)} \sin(\omega l/c) + a_2 \cos(\omega l/c)\right)$

Since $a_2 \neq 0$

$$\tan(\omega l/c) = \frac{K_1 + K_2}{(AE/l)(\omega l/c) - \dfrac{(K_1 K_2)}{(AE/l)\cdot(\omega l/c)}}$$

Define $\alpha = \dfrac{\omega l}{c}$; $K = \dfrac{AE}{l}$

$$\tan\alpha = \frac{K_1 + K_2}{K\alpha - \dfrac{K_1 K_2}{K\alpha}} = \frac{(K_1 + K_2)K\alpha}{K^2\alpha^2 - K_1 K_2}$$

$$\tan\alpha = \frac{(K_1/K + K_2/K)\alpha}{\alpha^2 - (K_1/K \cdot K_2/K)}$$

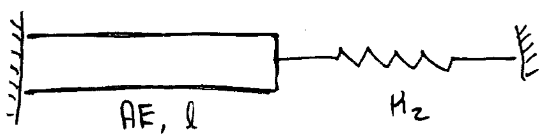
$$\boxed{\tan\alpha = \frac{\left(\frac{H_1}{H} + \frac{H_2}{H}\right)\alpha}{\alpha^2 - \frac{H_1}{H}\cdot\frac{H_2}{H}}}$$

(*) For the case $\frac{H_1}{H} = 1$, $\frac{H_2}{H} = 1$

$$\boxed{\tan\alpha = \frac{2\alpha}{\alpha^2 - 1}}$$

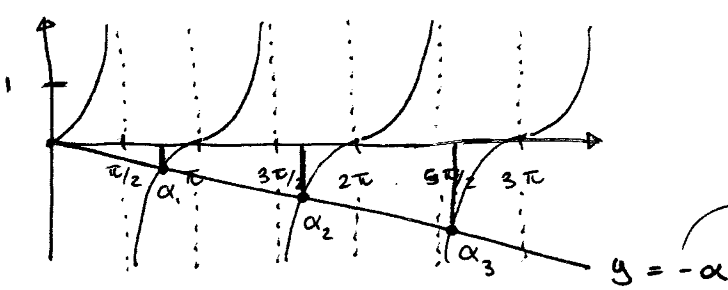
(*) For the case $H_1 \to \infty$

$$\boxed{\tan\alpha = \left(\frac{-H}{H_2}\right)\alpha}$$



considering $H_2 = H = \frac{AE}{l}$

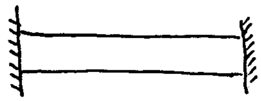
$$\tan\alpha = -\alpha$$



$(2n-1)\cdot\pi/2$
$n \to \infty$

(*) $M_2 \rightarrow \infty$

$\tan \alpha = 0 \Rightarrow \quad \sin \alpha = 0$

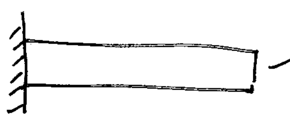


(*) $M_2 \rightarrow 0$

$M_2 \sin \alpha = -K \alpha \cos \alpha$

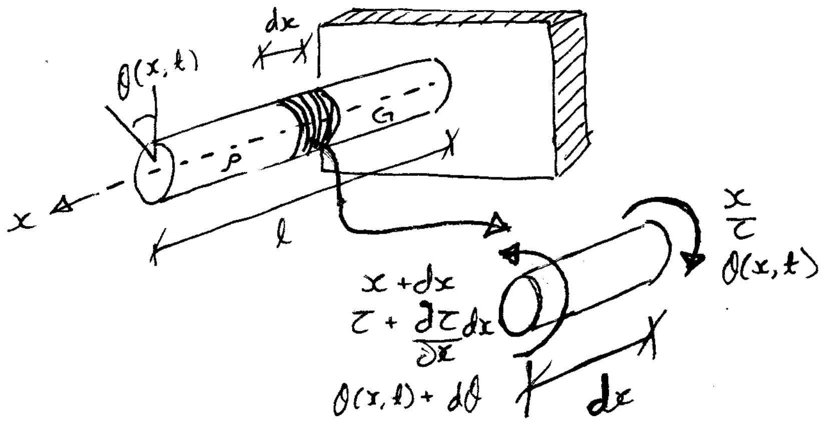
$\alpha \cos \alpha = 0$

$\alpha \neq 0$



$\cos \alpha = 0$

Torsional Vibration



→ $\boxed{\Sigma M_x = I_x \alpha}$ (Here τ is torsion)

$$\left(\tau + \frac{\partial \tau}{\partial x} dx\right) - \tau = (\rho J dx)\left(\frac{\partial^2 \theta}{\partial t^2}\right)$$

$$\Rightarrow \frac{\partial \tau}{\partial x} = \rho J \frac{\partial^2 \theta}{\partial t^2}$$

Mechanics of Materials :

$$\tau = GJ(\partial \theta / \partial x)$$

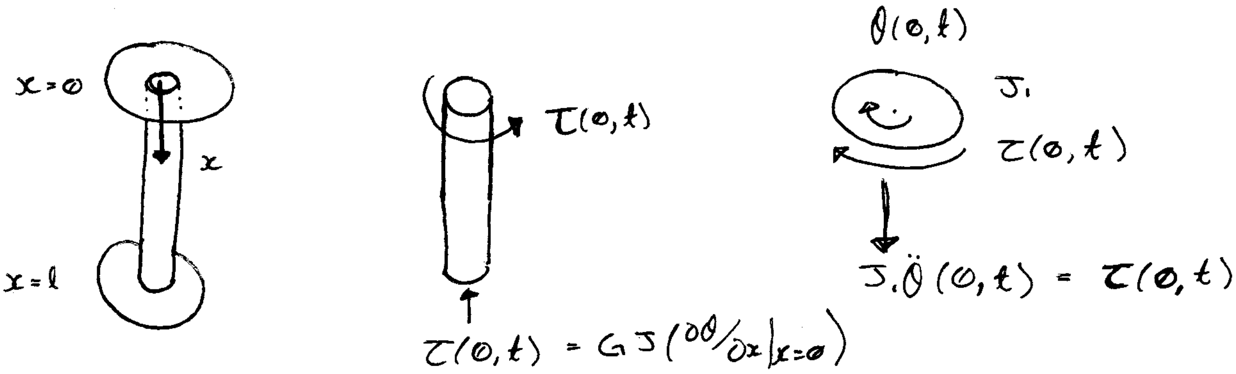
$$\Rightarrow GJ \frac{\partial^2 \theta}{\partial t^2} = \rho J \frac{\partial^2 \theta}{\partial t^2}$$

$$\Rightarrow \frac{G}{\rho} \frac{\partial^2 \theta}{\partial x^2} = \frac{\partial^2 \theta}{\partial t^2}$$

OR $C = \sqrt{\dfrac{G}{\rho}}$

$$\boxed{C^2 \frac{\partial^2 \theta}{\partial x^2} = \frac{\partial^2 \theta}{\partial t^2}}$$

→ From textbook, Figure 6.8



$$J_1\ddot{\theta}(0,t) = \tau(0,t)$$

$$\tau(0,t) = GJ\left(\partial\theta/\partial x\big|_{x=0}\right)$$

At $x = 0$:

$$\Rightarrow \quad GJ\left(\frac{\partial\theta}{\partial x}\Big|_{x=0}\right) = J_1\left(\frac{\partial^2\theta}{\partial t^2}\Big|_{x=0}\right)$$

At $x = l$:

$$\Rightarrow \quad GJ\left(\partial\theta/\partial x\big|_{x=l}\right) = -J_2\left(\frac{\partial^2\theta}{\partial t^2}\Big|_{x=l}\right)$$

Assume

$$\theta(x,t) = \Theta(x)\, e^{i\omega t}$$

$$\frac{\partial^2\Theta}{\partial x^2} + \frac{\omega^2}{c^2}\Theta = 0 \qquad (0 < x < l) \qquad (1)$$

$x = 0$:

$$\Rightarrow \quad GJ\Theta' + J_1\omega^2\Theta = 0 \qquad (2)$$

$x = l$:

$$\Rightarrow \quad GJ\Theta' - J_2\omega^2\Theta = 0 \qquad (3)$$

$$(1) \rightarrow \quad \Theta(\omega) = a_1\sin(\omega/c\, x) + a_2\cos(\omega/c\, x) \qquad (4)$$

Since $\Theta'(x) = a_1(\omega/c)\cos(\omega/c\, x) - a_2(\omega/c)\sin(\omega/c\, x)$ (5)

Sub (4) & (5) into (2) & (3):

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix} = 0$$

$$\longrightarrow \det\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = 0$$

$$\longrightarrow \boxed{\tan\left(\frac{\omega}{c}l\right) = \frac{\frac{\omega}{c}l}{b\left(\frac{\omega}{c}l\right)^2 - a}}$$

$$\boxed{a = \rho l \frac{J}{J_1 + J_2} \qquad b = \frac{J_1 J_2}{\rho J l (J_1 + J_2)}}$$

J : Polar moment of inertia of shaft

Given : $J_1 = 10 \text{ kg}\cdot\text{m}^2$

$J_2 = 10 \text{ kg}\cdot\text{m}^2$

$J = 5 \text{ m}^4$

$l = 0.425 \text{ m}$

$\rho = 7870 \text{ kg/m}^3$

Then : $a = \dfrac{7870 \times 0.425}{10 + 10} \longrightarrow a = 836.1875$

$b = \dfrac{10 \times 10}{7870 \times 5 \times 0.425 \times (10 + 10)} \longrightarrow b = 0.000298976$

Freq. Eqn. :

$$\tan\left(\frac{\omega}{c} l\right) = \frac{\left(\frac{\omega}{c} l\right)}{0.000298976\left(\frac{\omega l}{c}\right)^2 - 836.1875}$$

1^{ST} : $\frac{\omega l}{c} = 0 \longrightarrow \omega = 0$ is a Freq.

2^{ND} : $f_1 = \dfrac{\omega_1}{2\pi} = 0$

$f_2 = \dfrac{\omega_2}{2\pi} = 3813 \text{ Hz}$

$f_3 = 76026 \text{ Hz}$

Summary of Concepts

Vibration

* Potential energy , spring, elasticity [K]
* Kinetic energy: mass / inertia [M]
* energy lost : damper [C]

Modeling :

- Newton's law

  Influence coefficients — Flexibility / Stiffness

- Energy Method — only for conservative (no damping)
- Lagrange Method

Free-vibration : (Free-response)

- 1 DOF : Natural freq. $\omega_n = \sqrt{k/m}$
- multiple DOF : Natural Frequencies / modal shapes (*)
  (eigenvalue) (eigenvector)

  Modal expansion

  → decouple eq. of motion

* response due to initial conditions (for single DOF)
* Find the natural freq / mode shapes
  * eigenvalue / eigenvector
  * approximate method
    - Dunkerley's Method
    - Rayleigh's Method
    - Matrix iteration (Power method)
    - Jacobi's Method

Forced Response : 1 DOF

* resonance *
* beat (2 DOF)
* base excitation
* rotating unbalance

disp. forced response

- Forced response
  ω_n multiple DOF - approx Method
  (never exceed 3)
  Free response
  3 x 3 for sure (for other Q's)
  5 Q's (roughly)
  closed book