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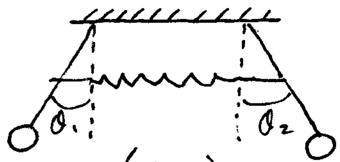
; where $\dfrac{\theta_1}{\theta_2} = 1$

(ω_1)



(ω_2)

beam bending

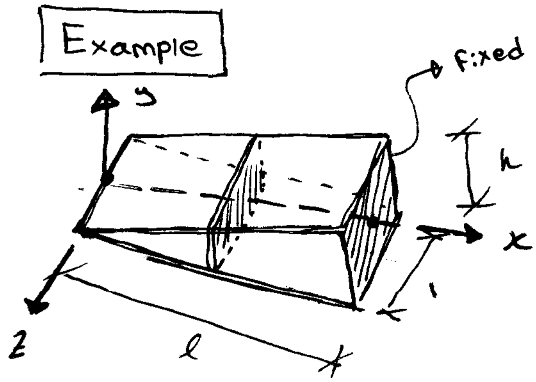
$$\omega^2 = \frac{\int_0^l EI(y'')^2\,dx}{\int_0^l \rho A y^2\,dx}$$

Here, $y = y(x)$: the assumed deflection

$y(x)$: the static deflection

ω^2 : an approximation of the fundamental freq. of the beam

Example



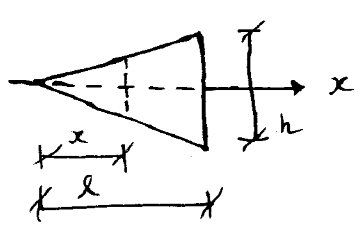
E is constant, estimate first natural frequency.

Solution : $y(x)$ trial function

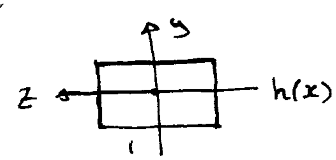
then $y(l) = 0$, $y'(l) = 0$

Let $y(x) = \left[1 - (x/l)\right]^2$

→ Side view



$h(x) = \left(\frac{x}{l}\right)h$



$I = (1/12)(1)(h(x)^3)$

$= (1/12)\left(\frac{x}{l}h\right)^3$

Since $y'' = 2/\ell^2$

$$\therefore \quad \omega^2 = \frac{\int_0^\ell E\cdot(1/12)(x/\ell\, h)^3 \cdot (2/\ell^2)^2\, dx}{\int_0^\ell \rho \frac{x}{\ell} h\left[(1-\frac{x}{\ell})^2\right]^2 dx} = 2.5\,\frac{Eh^2}{\rho \ell^4}$$

$$\omega = 1.5811\sqrt{\frac{Eh^2}{\rho\ell^4}} \quad \& \quad \omega_{exact} = 1.5343\sqrt{\frac{Eh^2}{\rho\ell^4}}$$

— Matrix iteration method

$$[M]\ddot{\vec{x}} + [K]\vec{x} = 0$$

The natural frequency and mode shape :

$$(-\omega^2[M] + [K])\vec{X} = 0$$

$$[K]^{-1}(-\omega^2[M] + [K])\vec{X} = 0$$

$$\Rightarrow (-\omega^2[K]^{-1}[M] + [I])\vec{X} = 0$$

Define : $[D] = [K]^{-1}[M]$; $\lambda = 1/\omega^2$

$$\Rightarrow ([D] - \lambda[I])\vec{X} = 0$$

then $[D]\vec{X} = \lambda\vec{X}$

1° : $\vec{X} = \vec{X}_1 \ (\neq 0)$

2° :

$$[D]\vec{X}_1 = \vec{X}_2$$

$$[D]\vec{X}_2 = \vec{X}_3$$

$$\vdots$$

$$[D]\vec{X}_r = \vec{X}_{r+1} \approx \lambda\vec{X}_r$$

$$\vec{X}_r = \begin{Bmatrix} X_{1r} \\ X_{2r} \\ \vdots \\ X_{nr} \end{Bmatrix} \qquad \vec{X}_{r+1} = \begin{Bmatrix} X_{1,r+1} \\ X_{2,r+1} \\ \vdots \\ X_{n,r+1} \end{Bmatrix}$$

$$\frac{X_{1,r+1}}{X_{1,r}} = \frac{X_{2,r+1}}{X_{2,r}} = \cdots = \frac{X_{n,r+1}}{X_{n,r}} \approx \frac{X_{n,r+1}}{X_{n,r}} \approx \lambda$$

Given $[D]$: $\lambda_1, \lambda_2, \ldots, \lambda_n$
$\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_n$

Then

$$\vec{X}_1 = c_1\vec{u}_1 + c_2\vec{u}_2 + \cdots + c_n\vec{u}_n \qquad (c_n \neq 0)$$

$$\vec{X}_2 = [D]\vec{X}_1 = c_1[D]\vec{u}_1 + c_2[D]\vec{u}_2 + \cdots + c_n[D]\vec{u}_n$$

$$\vec{X}_2 = c_1\lambda_1\vec{u}_1 + c_2\lambda_2\vec{u}_2 + \cdots + c_n\lambda_n\vec{u}_n$$

$$\vec{X}_3 = [D]\vec{X}_2 = c_1\lambda_1^2\vec{u}_1 + c_2\lambda_2^2\vec{u}_2 + \cdots + c_n\lambda_n^2\vec{u}_n$$

$\Downarrow$ (thus ...)

$$\vec{X}_r = c_1\lambda_1^{r-1}\vec{u}_1 + c_2\lambda_2^{r-1}\vec{u}_2 + \cdots + c_n\lambda_n^{r-n}\vec{u}_n$$

$$\vec{X}_{r+1} = c_1\lambda_1^{r}\vec{u}_1 + c_2\lambda_2^{r}\vec{u}_2 + \cdots + c_n\lambda_n^{r}\vec{u}_n$$

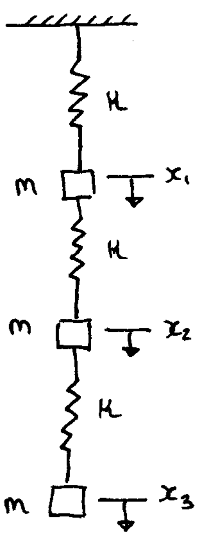
When $\vec{X}$ is large enough :

$$\vec{X}_r \approx c_n\lambda_n^{r-1}\vec{u}_n$$

$$\vec{X}_{r-1} \approx c_n\lambda_n^{r}\vec{u}_n$$

$$(\lambda_1 < \lambda_2 < \cdots < \lambda_n)$$

$\boxed{\text{Examples}}$ Find the natural Freq. using iteration method.



Solution :

$$[M] = \begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{bmatrix} = m\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$[K] = k\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

$$[D] = [K]^{-1}[M] = \frac{m}{k}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$$

Consider : $([D_1] - \lambda[I])x = 0$

Let : $\lambda = \dfrac{k}{m}\cdot\dfrac{1}{\omega^2}$ $\Rightarrow$ $[D_1] = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$

(for this example only)

Take $\vec{X_1} = \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix}$

Then $\vec{X_2} = [D]\vec{X_1} = \begin{Bmatrix} 3 \\ 5 \\ 6 \end{Bmatrix} = 3 \begin{Bmatrix} 1 \\ 1.666\overline{7} \\ 2 \end{Bmatrix}$

$$\vec{X_3} = [D]\vec{X_2} = [D] \begin{Bmatrix} 1 \\ 1.666\overline{7} \\ 2 \end{Bmatrix} = \begin{Bmatrix} 4.6667 \\ 8.3333 \\ 10.3333 \end{Bmatrix}$$

$$= (4.6667) \begin{Bmatrix} 1 \\ 1.7857 \\ 2.2143 \end{Bmatrix}$$

$$\vec{X_4} = [D]\vec{X_3} = [D] \begin{Bmatrix} 1 \\ 1.7857 \\ 2.2143 \end{Bmatrix} = \begin{Bmatrix} 5.0000 \\ 9.0000 \\ 11.2143 \end{Bmatrix}$$

$$= (5.0000) \begin{Bmatrix} 1 \\ 1.80000 \\ 2.24286 \end{Bmatrix}$$

After 9 iterations :

$$= (5.04892) \begin{Bmatrix} 1.00000 \\ 1.80194 \\ 2.24698 \end{Bmatrix}$$

$$\therefore \lambda = 5.04892, \quad \vec{u} = \begin{Bmatrix} 1 \\ 1.80194 \\ 2.24698 \end{Bmatrix}$$

$$\therefore \omega = \sqrt{\frac{1}{\lambda}} \cdot \sqrt{\frac{k}{m}} = 0.44504 \sqrt{k/m}$$

The largest Freq:

$$(-\omega^2[M] + [K])\vec{X} = 0$$

$$\Rightarrow (-\omega^2[I] + [M]^{-1}[K])\vec{X} = 0$$

$$\Rightarrow [M]^{-1}[K]\vec{X} = \omega^2\vec{X}$$

Define : $[E] = [M]^{-1}[K]$

$$[E]\vec{X} = \omega^2\vec{X} = \lambda\vec{X}$$

Iteration :

$$X_2 = [E]\vec{X_1}$$

$$X_3 = [E]\vec{X_2}$$

$$\vdots$$

$$X_{r+1} = [E]\vec{X_r}$$

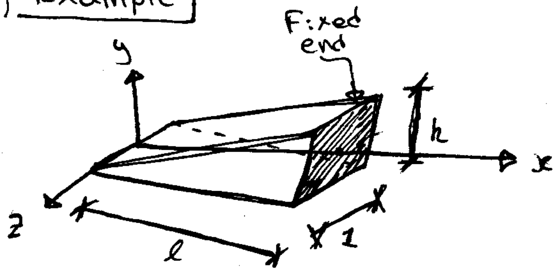
$$\downarrow$$

ω_n and $\vec{u_n}$

Rayleigh-Ritz Method :

$$\omega^2 = \frac{\int_0^l EI(y'')^2\,dx}{\int_0^l \rho A\, y^2\,dx}$$

$\boxed{\text{Example}}$



Solution :

$y_1 = (1 - x/l)^2$

$y_2 = (1 - x/l)^2(x/l)$

Let $y(x) = c_1 y_1 + c_2 y_2$

Then $$\omega^2 = \frac{\int_0^l EI(c_1 y_1'' + c_2 y_2'')^2\,dx}{\int_0^l \rho A(c_1 y_1 + c_2 y_2)^2\,dx}$$

$$\longrightarrow \quad \omega^2(c_1, c_2) = \frac{P(c_1, c_2)}{Q(c_1, c_2)}$$

where $\omega^2(c_1, c_2)$: stationary value

$$\frac{\partial \omega^2}{\partial c_1} = \emptyset \quad ; \quad \frac{\partial \omega^2}{\partial c_2} = \emptyset$$

$$\frac{\partial \omega^2}{\partial c_1} = \frac{\partial}{\partial c_1}\left(\frac{P}{Q}\right) = \frac{\partial P}{\partial c_1}\cdot\frac{1}{Q} + P\cdot\left(\frac{-1}{Q^2}\right)\frac{\partial Q}{\partial c_1} = \emptyset$$

$$\Rightarrow \quad \frac{\partial P}{\partial c_1} - \frac{P}{Q}\frac{\partial Q}{\partial c_1} = \emptyset$$

$$\frac{\partial P}{\partial c_1} - \omega^2\frac{\partial Q}{\partial c_1} = \emptyset$$

$$P = \int_0^l EI(c_1 y_1'' + c_2 y_2'')^2\,dx$$

$$= \int_0^l EI(c_1^2 y_1''^2 + 2c_1c_2 y_1'' y_2'' + c_2^2 y_2''^2)\,dx$$

$$= c_1^2\int_0^l EI y_1''^2\,dx + 2c_1c_2\int_0^l EI y_1'' y_2''\,dx + c_2^2\int_0^l EI\, y_2''\,dx$$

$$\longrightarrow \frac{\partial P}{\partial c_1} = 2c_1\int_0^l EI y_1''^2\,dx + 2c_2\int_0^l EI y_1'' y_2''\,dx + \emptyset$$

$$Q = \int_0^l \rho A(c_1 y_1 + c_2 y_2)^2\,dx$$

$$= c_1^2\int_0^l \rho A y_1^2\,dx + 2c_1c_2\int_0^l \rho A y_1 y_2\,dx + c_2\int_0^l \rho A y_2^2\,dx$$

$$\longrightarrow \frac{\partial Q}{\partial c_1} = 2c_1\int_0^l \rho A y_1^2\,dx + 2c_2\int_0^l \rho A\, y_1 y_2\,dx + \emptyset$$

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$$\cancel{2}C_1\int_0^l EI y_1''^2\,dx + \cancel{2}C_2\int_0^l EI y_1'' y_2''\,dx - \omega^2\left(\cancel{2}C_1\int_0^l \rho A y_1^2\,dx + \cancel{2}C_2\int_0^l \rho A y_1 y_2\,dx\right) = 0$$

where $\dfrac{\partial \omega^2}{\partial C_2} = 0 \rightarrow \dfrac{\partial P}{\partial C_2} - \omega^2 \dfrac{\partial Q}{\partial C_2} = 0$

$$C_1\int_0^l EI y_1'' y_2''\,dx + C_2\int_0^l EI y_2''^2\,dx \cdots$$
$$\cdots - \omega^2\left(C_1\int_0^l \rho A y_1 y_2\,dx + C_2\int_0^l \rho A y_2^2\,dx\right) = 0$$

Matrix Form:

$$\left([K] - \omega^2[M]\right)\begin{Bmatrix} C_1 \\ C_2 \end{Bmatrix} = 0$$

Here:

$$[K] = \begin{bmatrix} \int_0^l EI y_1''^2\,dx & \int_0^l EI y_1'' y_2''\,dx \\ \int_0^l EI y_1'' y_2''\,dx & \int_0^l EI y_2''^2\,dx \end{bmatrix}$$

same

$$[M] = \begin{bmatrix} \int_0^l \rho A y_1^2\,dx & \int_0^l \rho A y_1 y_2\,dx \\ \int_0^l \rho A y_1 y_2\,dx & \int_0^l \rho A y_2^2\,dx \end{bmatrix}$$

same

Since $A = \left(\dfrac{h}{l}\right)x$; $I = \dfrac{1}{12}\left(\dfrac{hx}{l}\right)^3$

$$\Rightarrow [K] = \begin{bmatrix} 0.0833333 & 0.0333333 \\ 0.0333333 & 0.0333333 \end{bmatrix}\frac{Eh^3}{l}$$

$$[M] = \begin{bmatrix} 0.0333333 & 0.00952381 \\ 0.00952381 & 0.00357143 \end{bmatrix}\rho h$$

$\Rightarrow$ Solution:

$$\omega_1^2 = 2.35741\,\frac{Eh^2}{\rho l^4}$$

$$\omega_2^2 = 24.9426\,\frac{Eh^2}{\rho l^4}$$

The First natural Frequency:

$$\omega_1 = 1.5353\sqrt{Eh^2/\rho l^4}$$

For one term, $C_2 = 0$

$$\omega_1^2 = 2.50000\,\frac{Eh^2}{\rho l^4}$$

$$\omega_1 = 1.5811\sqrt{Eh^2/\rho l^4}$$

The exact solution

$$\omega_{1,exact} = 1.5343\sqrt{Eh^2/\rho l^4}$$

Take more terms

$y_3 = (1-x/\ell)^2(x/\ell)^2$; $y_4 = (1-x/\ell)^2(x/\ell)^3$

$y_5 = (1-x/\ell)^2(x/\ell)^4$

$y(x) = C_1y_1 + C_2y_2 + C_3y_3 + C_4y_4 + C_5y_5$

$\Rightarrow \quad \omega_1^2 = 2.354190 \ Eh/\rho\ell^4$

$\omega_1 = 1.53434 \sqrt{Eh/\rho\ell^4}$

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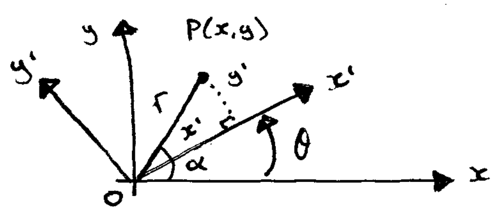
(*) Correction: (For last lecture) units of $[K]$: $\frac{Eh^3}{l} \rightarrow \frac{Eh^3}{l^3}$

$[M]$: $\rho h \rightarrow \rho h l$

Jacobi's Method:

Coordinate transformation



$$x = r\cos\alpha$$
$$y = r\sin\alpha$$
$$P(x,y) \rightarrow P(x',y')$$
$$x' = r\cos(\alpha-\theta)$$
$$y' = r\sin(\alpha-\theta)$$

$$\Rightarrow \quad x' = r\cos\alpha\cos\theta + r\sin\alpha\cdot\sin\theta = x\cos\theta + y\sin\theta$$
$$y' = r\sin\alpha\cos\theta - r\cos\alpha\cdot\sin\theta = -x\sin\theta + y\cos\theta$$

$$\Rightarrow \quad \begin{Bmatrix} x' \\ y' \end{Bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{Bmatrix} x \\ y \end{Bmatrix}$$

$$\begin{Bmatrix} x \\ y \end{Bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{Bmatrix} x' \\ y' \end{Bmatrix}$$

Define : $[Q] = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

$$[Q]^{-1} = [Q]^T$$

Given $[D] = \begin{bmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{bmatrix}$; where $d_{12} = d_{21}$

$$\Rightarrow \quad [Q]^T[D][Q] = [D_1]$$

$$[D_1] = \begin{bmatrix} d_{11}^{(1)} & d_{12}^{(1)} \\ d_{21}^{(1)} & d_{22}^{(1)} \end{bmatrix}$$

Here :

$$d_{12}^{(1)} = (\cos^2\theta - \sin^2\theta)\, d_{12} + (d_{22} - d_{11})\sin\theta\cos\theta$$
$$d_{11}^{(1)} = d_{11}\cos^2\theta + 2d_{12}\cos\theta\sin\theta + d_{22}\sin^2\theta$$
$$d_{22}^{(1)} = d_{11}\sin^2\theta - 2d_{12}\cos\theta\sin\theta + d_{22}\cos^2\theta$$

Let $d_{12}^{①} = 0$

$$\Rightarrow \tan(2\theta) = \frac{2 d_{12}}{d_{11} - d_{22}}$$

$$[D] = \begin{bmatrix} d_{11} & d_{12} & d_{13} \\ d_{21} & d_{22} & d_{23} \\ d_{31} & d_{32} & d_{33} \end{bmatrix}$$

$$[Q_1] = \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 & 0 \\ \sin\theta_1 & \cos\theta_1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$[Q_1]^T [D][Q_1] = [D_1] = \begin{bmatrix} d_{11}^{①} & 0 & d_{13}^{①} \\ 0 & d_{22}^{①} & d_{23}^{①} \\ d_{31}^{①} & d_{32}^{①} & d_{33}^{①} \end{bmatrix}$$

$$[Q_2] = \begin{bmatrix} \cos\theta_2 & 0 & -\sin\theta_2 \\ 0 & 1 & 0 \\ \sin\theta_2 & 0 & \cos\theta_2 \end{bmatrix}$$

$$[Q_2]^T [D_1][Q_2] = \begin{bmatrix} d_{11}^{②} & d_{12}^{②} & 0 \\ d_{21}^{②} & d_{22}^{②} & d_{23}^{②} \\ 0 & d_{32}^{②} & d_{33}^{②} \end{bmatrix} = [D_2]$$

$\rightarrow$ Repeat many times ...

$$[Q_m]^T \cdots [Q_2]^T [Q_1]^T [D][Q_1][Q_2] \cdots [Q_m]$$

$$= \begin{bmatrix} d_{11}^{m} & 0 & 0 \\ 0 & d_{22}^{m} & 0 \\ 0 & 0 & d_{33}^{m} \end{bmatrix}$$

Define: $[U] = [Q_1][Q_2] \cdots [Q_M]$

$$\rightarrow \quad [D][U] = [U][\Lambda]$$

$$[\Lambda] = \mathrm{diag}(d_{11}^{ⓜ}, d_{22}^{ⓜ}, d_{33}^{ⓜ})$$

$\boxed{\text{Example}}$ Find the eigenvalues by using Jacobi's method

$$[D] = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$$

Solution: $\rightarrow \tan(2\theta) = \dfrac{2d_{12}}{d_{11} - d_{22}} = \dfrac{2 \times 1}{1-2} = -2$

$$\theta = -0.553574$$

$$[Q_1] = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0.8506 & 0.5257 & 0 \\ -0.5257 & 0.8506 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$[Q_1]^T[D][Q_1] = \begin{bmatrix} 0.3820 & 0 & -0.2008 \\ 0 & 2.618 & 2.227 \\ -0.2008 & 2.227 & 3 \end{bmatrix} = D_1$$

$\rightarrow \tan(2\theta) = \dfrac{2d_{23}}{d_{22} - d_{33}} = \dfrac{2 \times 2.227}{2.618 - 3}$

$$\theta = -0.7426$$

$$[Q_2] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.7367 & -0.6762 \\ 0 & -0.6762 & 0.7367 \end{bmatrix}$$

$$[Q_2]^T[D_1][Q_2] = \begin{bmatrix} 0.3820 & 0.1358 & -0.1479 \\ & 0.5738 & 0 \\ & & 5.044 \end{bmatrix} = D_2$$

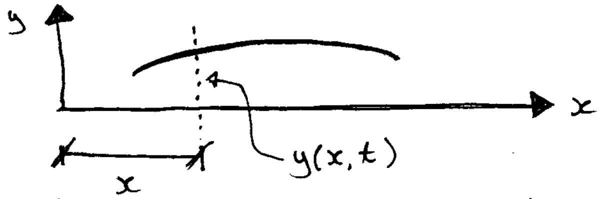
$$\rightarrow [Q_7]^T \cdots [Q_1]^T [D][Q_1] \cdots [Q_7] = \begin{bmatrix} 0.3080 & 0.1762 \times 10^{-6} & 0 \\ & 0.6431 & 7.02 \times 10^{-6} \\ \text{Sym.} & & 5.049 \end{bmatrix}$$

and $[U] = [Q_1][Q_2] \cdots [Q_7]$

$$= \begin{bmatrix} 0.5910 & 0.7370 & 0.3280 \\ -0.7370 & 0.3280 & 0.5910 \\ 0.3280 & -0.5910 & 0.7370 \end{bmatrix}$$

(Column Corresponds to): $\uparrow 0.3820$ $\quad \uparrow 0.6431$ $\quad \uparrow 5.049$

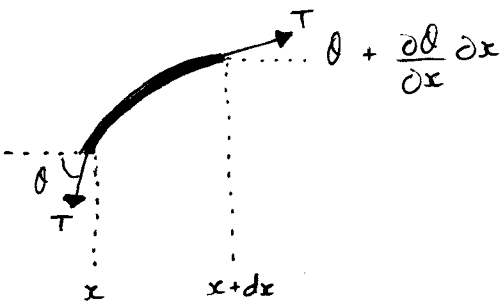
Vibrating string :



: the displacement is a function of both space and time.

Mass density per unit length : ρ

small vibration, the tension : T = const.



$$T \sin\left(\theta + \frac{\partial \theta}{\partial x} \partial x\right)$$

$$T \sin\left(\theta + \frac{\partial \theta}{\partial x} dx\right) - T \sin\theta = \rho \, dx \frac{\partial^2 y}{\partial t^2}$$

$\theta \ll 1$, then $\sin\theta \approx \theta$

$$T\left(\theta + \frac{\partial \theta}{\partial x} dx\right) - T\theta = \rho \frac{\partial^2 y}{\partial t^2} \partial x$$

$$T\left(\partial \theta / \partial x\right) = \rho \left(\partial^2 y / \partial t^2\right)$$

$$\theta \approx \tan\theta = \partial y / \partial x$$

$$\rightarrow \quad T\left(\frac{\partial^2 y}{\partial x^2}\right) = \rho\left(\partial^2 y / \partial t^2\right)$$

$$\rightarrow \quad \boxed{\left(\partial^2 y / \partial x^2\right) = \left(1/c^2\right)\left(\partial^2 y / \partial t^2\right) \; ; \; c = \sqrt{T/\rho}}$$

wave speed

$$\underset{\longrightarrow}{f(x - ct)} \, , \quad \underset{\longleftarrow}{g(x + ct)}$$

$\rightarrow$ $y(x,t) = Y(x) \cdot T(t)$

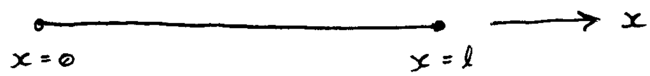
$\rightarrow$ Harmonic solution of time t

$$y(x,t) = Y(x) e^{i\omega t}$$

$$\frac{\partial^2 y}{\partial x^2} = Y''(x) e^{i\omega t}$$

$$\frac{\partial^2 y}{\partial x^2} = -\omega^2 Y(x) e^{i\omega t}$$

$$Y'' e^{i\omega t} = -\frac{\omega^2}{c^2} Y(x) e^{i\omega t} \quad \rightarrow \quad Y'' + \frac{\omega^2}{c^2} Y = 0 \qquad \text{I}$$



At $x = 0$, $\quad y(0, t) = Y(0)\,e^{i\omega t} = 0$

$$Y(0) = 0 \qquad \text{II}$$

At $x = l$, $\quad y(l, t) = Y(l)\,e^{i\omega t} = 0$

$$Y(l) = 0 \qquad \text{III}$$

Boundary value problem.