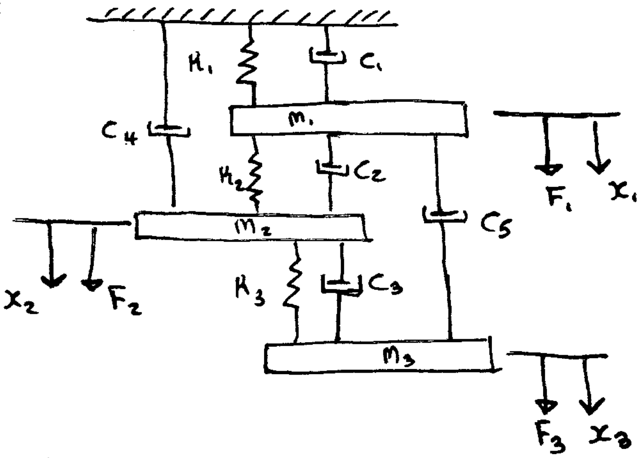
Nov. 12/19

Modal summation

Example:



Find the eqn's of motion and calculate the forced response.

Solution:

Kinetic energy: $T = (1/2)m_1\dot{x}_1^2 + (1/2)m_2\dot{x}_2^2 + (1/2)m_3\dot{x}_3^2$

Potential energy: $V = (1/2)k_1x_1^2 + (1/2)k_2(x_2 - x_1)^2 + (1/2)k_3(x_3 - x_2)^2$

Rayleigh's Formula: $R = (1/2)c_1\dot{x}_1^2 + (1/2)c_2(\dot{x}_2 - \dot{x}_1)^2 + (1/2)c_3(\dot{x}_3 - \dot{x}_2)^2 + \ldots$

$\ldots (1/2)c_4\dot{x}_2^2 + (1/2)c_5(\dot{x}_3 - \dot{x}_1)^2$

The generalized force

$$\vec{Q} = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix}$$

Equations of Motion:

$$(d/dt)(\partial T/\partial \dot{q}_1) + (\partial V/\partial q_1) - (\partial T/\partial q_1) + (\partial R/\partial \dot{q}_1) = Q$$

where: $q_1 = x_1$

then $(\partial V/\partial q_1) = k_1x_1 + k_2(x_1 - x_2)$

$(\partial R/\partial \dot{q}_1) = c_1\dot{x}_1 + c_2(\dot{x}_1 - \dot{x}_2) + c_5(\dot{x}_1 - \dot{x}_3)$

$$\therefore \quad m_1\ddot{x}_1 + c_1\dot{x}_1 + c_2(\dot{x}_1 - \dot{x}_2) + c_5(\dot{x}_1 - \dot{x}_3) + k_1 x_2 + k_2(x_1 - x_2) = F_1$$

. . . etc. For other equations

Matrix Form :

$$[M] = \begin{bmatrix} m_1 & \emptyset & \emptyset \\ \emptyset & m_2 & \emptyset \\ \emptyset & \emptyset & m_3 \end{bmatrix}$$

$$[C] = \begin{bmatrix} c_1 + c_2 + c_5 & -c_2 & -c_5 \\ -c_2 & c_2 + c_3 + c_4 & -c_3 \\ -c_5 & -c_3 & c_3 + c_5 \end{bmatrix}$$

$$[K] = \begin{bmatrix} k_1 + k_2 & -k_2 & \emptyset \\ -k_2 & k_2 + k_3 & -k_3 \\ \emptyset & -k_3 & k_3 \end{bmatrix}$$

Given:

$$m_1 = m_2 = m_3 = m$$

$$k_1 = k_2 = k_3 = k$$

$$c_4 = c_5 = \emptyset \; ; \; \boxed{\zeta_1 = \zeta_2 = \zeta_3 = \emptyset.\emptyset 1}$$

$$F_1 = F_2 = F_3 = F_0 \cos(\omega t)$$

$$\hookrightarrow \omega = 1.75\sqrt{k/m}$$

Step 1: The natural frequencies and modal shape (w/o damping)

$$[M] = m\begin{bmatrix} 1 & \emptyset & \emptyset \\ \emptyset & 1 & \emptyset \\ \emptyset & \emptyset & 1 \end{bmatrix}$$

$$[K] = k\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ \emptyset & -1 & 1 \end{bmatrix}$$

$$\Rightarrow \left(-\omega^2 m \begin{bmatrix} 1 & \emptyset & \emptyset \\ \emptyset & 1 & \emptyset \\ \emptyset & \emptyset & 1 \end{bmatrix} + k \begin{bmatrix} 2 & -1 & \emptyset \\ -1 & 2 & -1 \\ \emptyset & -1 & 1 \end{bmatrix} \right) \vec{u} = \emptyset$$

After solving:

$\omega_1 = 0.44504 \times \sqrt{K/m}$

$\omega_2 = 1.2470 \times \sqrt{K/m}$

$\omega_3 = 1.8019 \times \sqrt{K/m}$

$$\vec{u}_1 = \frac{1}{\sqrt{m}} \begin{Bmatrix} 0.32799 \\ 0.59101 \\ 0.73698 \end{Bmatrix}$$

$$\vec{u}_2 = \frac{1}{\sqrt{m}} \begin{Bmatrix} 0.73698 \\ 0.32799 \\ -0.59101 \end{Bmatrix}$$

$$\vec{u}_3 = \frac{1}{\sqrt{m}} \begin{Bmatrix} 0.59101 \\ -0.73688 \\ 0.32799 \end{Bmatrix}$$

Step 2 : $\vec{U} = [\vec{u}_1 \ \vec{u}_2 \ \vec{u}_3]_{3\times3}$

Verify : $[U]^T[M][U] = [I]$

$[U]^T[K][U] = \mathrm{diag}(\omega_1^2, \omega_2^2, \omega_3^2)$

Define : $\vec{x} = [U]\vec{q}$

The equations of motion

$$[M]\ddot{\vec{x}} + [C]\dot{\vec{x}} + [K]\vec{x} = \vec{Q}$$

$$[M][U]\ddot{\vec{q}} + [C][U]\dot{\vec{q}} + [K][U]\vec{q} = \vec{Q}$$

$$[U]^T[M][U]\ddot{\vec{q}} + [U]^T[C][U]\dot{\vec{q}} + [U]^T[K][U]\vec{q} = [U]^T\vec{Q}$$

$$\ddot{\vec{q}} + [U]^T[C][U]\dot{\vec{q}} + \mathrm{diag}(\omega_1^2 \omega_2^2 \omega_3^3)\vec{q} = [U]^T\vec{Q}$$

Proportional damping :

$$[U]^T[C][U] = \mathrm{diag}(2\zeta_1\omega_1, \ 2\zeta_2\omega_2, \ 2\zeta_3\omega_3)$$

$$\begin{Bmatrix} Q_{10} \\ Q_{20} \\ Q_{30} \end{Bmatrix} = [U]^T\vec{Q} = [U]^T \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix}$$

$$= [U]^T F_0 \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix} \cos(\omega t) = \frac{F_0}{\sqrt{m}} \begin{Bmatrix} 1.6560 \\ 0.47395 \\ 0.18202 \end{Bmatrix} \cos(\omega t)$$

Equations in the Modal Coordinates

$$\begin{cases} \ddot{q}_1 + 2\zeta_1\omega_1\dot{q}_1 + \omega_1^2 q_1 = Q_{10} = (F_0/\sqrt{m})(1.6560)\cos(\omega t) \\ \ddot{q}_2 + 2\zeta_2\omega_2\dot{q}_2 + \omega_2^2 q_2 = Q_{20} = (F_0/\sqrt{m})(0.47395)\cos(\omega t) \\ \ddot{q}_3 + 2\zeta_3\omega_3\dot{q}_3 + \omega_3^2 q_3 = Q_{30} = (F_0/\sqrt{m})(0.18202)\cos(\omega t) \end{cases}$$

Step 3 : Steady-state response of each modal coordinate

$$\boxed{q_i(t) = q_{i0}\cos(\omega t - \phi_i)} \quad ; \quad i = 1, 2, 3$$

and

$$q_{i0} = \frac{Q_{i0}}{\omega_i^2} \cdot \frac{1}{\sqrt{(1-(\omega/\omega_i)^2)^2 + (2\zeta_i\,\omega/\omega_i)^2}}$$

$$\phi_i = \tan^{-1}\left(\frac{2\zeta(\omega/\omega_i)}{1-(\omega/\omega_i)^2}\right)$$

$i = 1$: $\dfrac{\omega}{\omega_1} = \dfrac{1.75\sqrt{K/m}}{0.44504\sqrt{K/m}} = 3.9322$

$q_{10} = 0.57811\,(F_0\sqrt{m}/K)$

$\phi_1 = 3.1367$

$i = 2$: $\dfrac{\omega}{\omega_2} = \dfrac{1.75}{1.2470} = 1.4034$

$q_{20} = 1.0980$

$\phi_2 = 3.1187$

$i = 3$: $\dfrac{\omega}{\omega_3} = \dfrac{1.75}{1.809} = 0.97118$

$q_{30} = 8.4938$

$\phi_3 = 0.32941$

$$\vec{x} = [U]\,\vec{q} = [U]\begin{Bmatrix} q_1(t) \\ q_2(t) \\ q_3(t) \end{Bmatrix} = [U]\begin{Bmatrix} q_{10}\cos(\omega t - \phi_1) \\ q_{20}\cos(\omega t - \phi_2) \\ q_{30}\cos(\omega t - \phi_3) \end{Bmatrix}$$

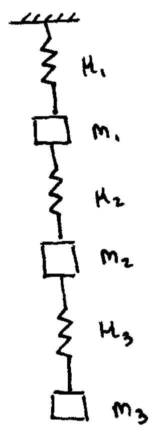
$$= \frac{1}{\sqrt{m}} \begin{bmatrix} 0.32799 & 0.73098 & 0.59101 \\ 0.59101 & 0.32799 & -0.73098 \\ 0.73098 & -0.59101 & 0.32799 \end{bmatrix} \begin{Bmatrix} 0.57811 \cos(\omega t - 3.1362) \\ 10.980 \cos(\omega t - 3.112) \\ 8.4930 \cos(\omega t - 0.32944) \end{Bmatrix} \cdot \frac{F_0 \sqrt{m}}{k}$$

$$= \frac{F_0}{k} \begin{Bmatrix} 3.7515 \cdot \cos \omega t + 1.6483 \sin(\omega t) \\ -6.6248 \cdot \cos \omega t - 2.0127 \sin(\omega t) \\ 2.8587 \cdot \cos \omega t + 0.88472 \sin(\omega t) \end{Bmatrix}$$

Determination of natural Frequencies

Dunkerley's Formula :

the Fundamental natural freq.



$$\left(-\omega^2 [M] + [K]\right) \vec{u} = 0$$

The Flexibility matrix

$$[A] = [K]^{-1}$$
$$[K] = [A]^{-1}$$

$$\longrightarrow [A]\left(-\omega^2 [M] + [K]\right) \vec{u} = 0$$
$$\left(-\omega^2 [A][M] + [A][K]\right) \vec{u} = 0$$

$$\longrightarrow \left(-[A][M] + (1/\omega^2)[I]\right) \vec{u} = 0$$

Eigenvalues :

$$\left| -[A][M] + (1/\omega^2)[I] \right| = 0$$

$$[A] = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \quad ; \quad [M] = \begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix}$$

$$[A][M] = \begin{bmatrix} a_{11}m_1 & a_{12}m_2 & a_{13}m_3 \\ a_{21}m_1 & a_{22}m_2 & a_{23}m_3 \\ a_{31}m_1 & a_{32}m_2 & a_{33}m_3 \end{bmatrix}$$

$$\Rightarrow \begin{vmatrix} a_{11}m_1 - (1/\omega^2) & a_{12}m_2 & a_{13}m_3 \\ a_{21}m_1 & a_{22}m_2 - (1/\omega^2) & a_{23}m_3 \\ a_{31}m_1 & a_{32}m_2 & a_{33}m_3 - (1/\omega^2) \end{vmatrix} = \emptyset$$

$$\Rightarrow \left[a_{11}m_1 - (1/\omega^2)\right]\left[a_{22}m_2 - (1/\omega^2)\right]\left[a_{33}m_3 - (1/\omega^2)\right] \cdots$$

$$\cdots + a_{12}m_2 a_{23}m_3 + a_{31}m_1 + a_{21}m_1 a_{32}m_2 a_{13}m_3 \cdots$$

$$\cdots - a_{13}m_3 a_{31}m_1 (a_{22}m_2 - 1/\omega^2) \cdots$$

$$\cdots - a_{12}m_2 a_{21}m_2 (a_{33}m_3 - 1/\omega^2) \cdots$$

$$\cdots - a_{22}m_3 a_{32}m_2 (a_{11}m_1 - 1/\omega^2) = \emptyset$$

Cubic eq. w.r.t. $1/\omega^2$

$$(1/\omega^2)^3 - (a_{11}m_1 + a_{22}m_2 + a_{33}m_3)(1/\omega^2)^2 + (\cdots)(1/\omega^2) + (\cdots) = \emptyset$$

The three roots : $1/\omega_1^2$; $1/\omega_2^2$; $1/\omega_3^2$

$$\boxed{1/\omega_1^2 + 1/\omega_2^2 + 1/\omega_3^2 = a_{11}m_1 + a_{22}m_2 + a_{33}m_3}$$

$$(1/\omega^2 - 1/\omega_1^2)(1/\omega^2 - 1/\omega_2^2)(1/\omega^2 - 1/\omega_3^2) = \emptyset$$

When $\omega_2 \gg \omega_1$, $\omega_3 \gg \omega_1$

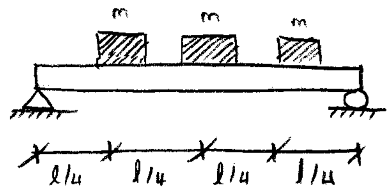
$$1/\omega_1^2 \approx 1/\omega_1^2 + 1/\omega_2^2 + 1/\omega_3^2 = a_{11}m_1 + a_{22}m_2 + a_{33}m_3$$

$$\therefore \quad \omega_1 = \frac{1}{\sqrt{a_{11}m_1 + a_{22}m_2 + a_{33}m_3}}$$

$$1/\omega_1^2 + 1/\omega_2^2 + \dots + 1/\omega_n^2 = a_{11}m_1 + a_{22}m_2 + \dots + a_{nn}m_n$$

$$\boxed{1/\omega_1^2 \approx a_{11}m_1 + \dots + a_{nn}m_n}$$

Example



$EI = \text{const.}$

Estimate the Fundamental Frequency.

Solution: $a_{11} = a_{33} = \left(\frac{3}{256}\right)(l^3/EI)$ — by symmetry

$a_{22} = (1/48)(l^3/EI)$

$$\therefore \quad 1/\omega_1^2 = a_{11}m_1 + a_{22}m_2 + a_{33}m_3$$

$$= (3/256)(l^3/EI)m_1 + (1/48)(l^3/EI)m_2 + (3/256)(l^3/EI)m_3$$

$$1/\omega_1^2 = 0.04427\,(l^3/EI)m$$

$$\omega_1 = 4.754\sqrt{EI/ml^3}$$

The exact solution:

$$\omega_{1,\text{exact}} = 4.734\sqrt{EI/ml^3} \qquad ???$$

(should always be larger than ω_1)

$$1/\omega_{1,d}^2 = 1/\omega_{1,\text{exact}}^2 + 1/\omega_2^2 + \dots + 1/\omega_n^2 > 1/\omega_{1,\text{exact}}^2$$

Rayleigh's Method

$$[M]\ddot{\vec{x}} + [K]\vec{x} = 0$$

The motion:

$$\vec{x} = e^{i\omega t}\vec{u}$$

ω: the natural freq.

$\vec{u}$: the mode shape

$$\Rightarrow \quad ([K] - \omega^2[M])\vec{u} = 0$$

$$\Rightarrow \quad \vec{u}^T([K] - \omega^2[M])\vec{u} = 0$$

$$\Rightarrow \quad \vec{u}^T[K]\vec{u} - \omega^2\vec{u}^T[M]\vec{u} = 0$$

$$\Rightarrow \quad \omega^2 = \frac{\vec{u}^T[K]\vec{u}}{\vec{u}^T[M]\vec{u}}$$

Define: $R(\vec{x}) = \dfrac{\vec{x}^T[K]\vec{x}}{\vec{x}^T[M]\vec{x}}$

} If x is close to modal shape, then it approximates the natural frequency

$T = (1/2)\,\dot{\vec{x}}^T [M] \dot{\vec{x}}$

$V = (1/2)\,\vec{x}^T [K] \vec{x}$

→ Harmonic motion:

$\vec{x} = \vec{X} e^{i\omega t}$

$\dot{\vec{x}} = \vec{X} \cdot i\omega e^{i\omega t}$

→ The max kinetic energy:

$T_{max} = (1/2)\,\omega^2 \vec{X}^T [M] \vec{X}$

→ The max potential energy:

$V_{max} = (1/2)\,\vec{X}^T [K] \vec{X}$

Conservation:

$T_{max} = V_{max}$

$(1/2)\,\omega^2 \vec{X}^T [M] \vec{X} = (1/2)\,\vec{X}^T [K] \vec{X}$

→ $$\omega^2 = \frac{\vec{X}^T [K] \vec{X}}{\vec{X}^T [M] \vec{X}}$$

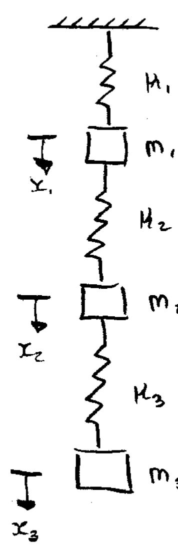
Verify:

$\omega_1^2 \le R(\vec{x}) \le \omega_n^2$

$\vec{X}$: the static deflection (disp.) of the system

Then $R(\vec{x})$: approximation of the Fundamental Freq.

$\boxed{\text{Example}}$



Let $m_1 = m_2 = m_3 = m$

$k_1 = k_2 = k_3 = k$

Estimate the Fundamental Freq. of the system.

Solution : $[M] = \begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{bmatrix} = m \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$[K] = k \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$

Take $\vec{x} = \begin{Bmatrix} 1 \\ 2 \\ 3 \end{Bmatrix} = \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}$

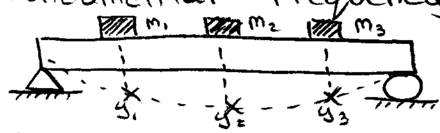
$$R(\vec{x}) = \frac{\vec{x}^T [K] \vec{x}}{\vec{x}^T [M] \vec{x}} = \frac{(1,2,3)\begin{bmatrix} -2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}\begin{Bmatrix} 1 \\ 2 \\ 3 \end{Bmatrix}}{(1,2,3)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{Bmatrix} 1 \\ 2 \\ 3 \end{Bmatrix}} \frac{k}{m}$$

$$= 0.2143 \left(k/m\right)$$

$$\therefore \quad \omega_1^2 \simeq R(x) = 0.2143 \frac{k}{m}$$

$$\omega_1 = 0.4629 \sqrt{k/m} \qquad \left(\omega_{1,exact} = 0.4450 \sqrt{k/m}\right)$$

Fundamental Frequency of beams and shafts



beam / shaft : weightless

The max potential energy = the max strain energy

= the work done by all the forces

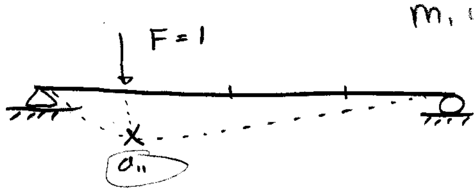
$$V_{max} = (1/2)(m_1 g y_1 + m_2 g y_2 + m_3 g y_3)$$

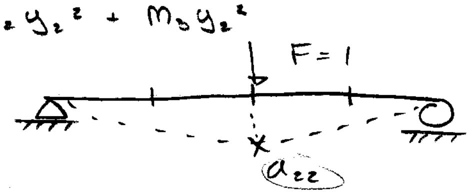
$$T_{max} = (1/2)\omega^2(m_1 y_1^2 + m_2 y_2^2 + m_3 y_3^2)$$

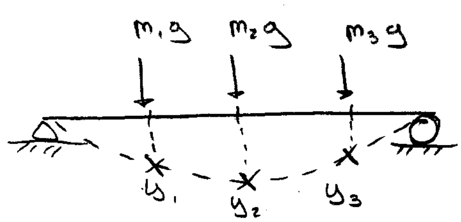
$$\Rightarrow (1/2)(m_1 g y_1 + m_2 g y_2 + m_3 g y_3)$$

$$= (1/2)\omega^2(m_1 y_1^2 + m_2 y_2^2 + m_3 y_3^2)$$

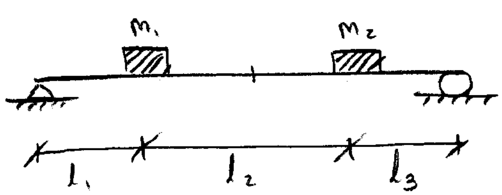
$$\rightarrow \omega^2 = \frac{m_1 g y_1 + m_2 g y_2 + m_3 g y_3}{m_1 y_1^2 + m_2 y_2^2 + m_3 y_3^2}$$







$\boxed{\text{Example:}}$ Estimate the fundamental freq. of the beam as shown.

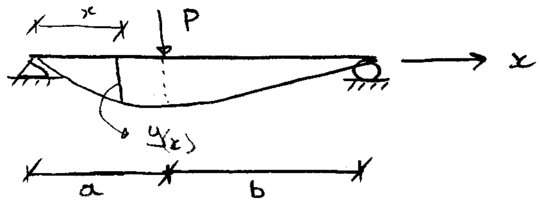


where $l_1 = 1\,m$; $l_2 = 3\,m$; $l_3 = 2\,m$

$m_1 = 20\,kg$; $m_2 = 50\,kg$

$EI = const.$

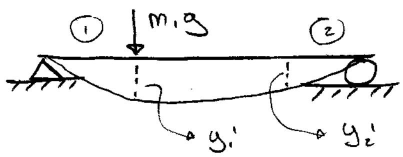
Solution :



The deflection :

$$y(x) = \begin{cases} \dfrac{Pbx}{6EIl}\left(l^2 - b^2 - x^2\right) & ; \quad 0 \le x \le a \\ -\dfrac{Pa(l-x)}{6EIl}\left(a^2 + x^2 - 2lx\right) & ; \quad a \le x \le 6 \end{cases}$$

→ The deflection due to $P = m_1 g$



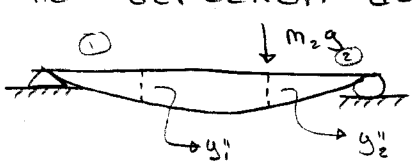
$l = l_1 + l_2 + l_3 = 6\,m$

$$y_1' = \frac{Pbx}{6EIl}\left(l^2 - b^2 - x^2\right)\Big|_{x=1} = \frac{(20)(9.81)(5)(1)}{6EI(6)}\left(6^2 - 5^2 - 1^2\right)$$

$$= \frac{272.5}{EI}$$

$$y_2' = -\frac{(20)(9.81)(1)(6-4)}{6EI(6)}\left(1^2 + 4^2 - 2\times 6\times 4\right) = \frac{337.9}{EI}$$

→ The deflection due to $P = m_2 g$



$$y_1'' = \frac{844.75}{EI}$$

$$y_2'' = \frac{1744.0}{EI}$$

The total displacement :

$$y_1 = y_1' + y_1'' = (272.5/EI) + (844.75/EI) = (1117.25/EI)$$

$$y_2 = y_2' + y_2'' = (337.9/EI) + (1744.0/EI) = (2081.9/EI)$$

$$\therefore \quad \omega_1^2 = \frac{(m_1 y_1 + m_2 y_2)\, g}{m_1 y_1^2 + m_2 y_2^2}$$

$$\Rightarrow \omega_1 = 0.07166 \sqrt{EI}$$

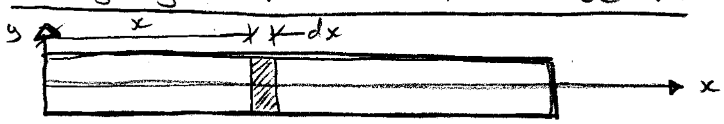
Dunkerley's Formula :

$$a_{11} = \frac{y_1'}{m_1 g} \quad ; \quad a_{22} = \frac{y_2''}{m_2 g}$$

$$\begin{aligned} 1/\omega_1^2 &= a_{11} m_1 + a_{22} m_2 \\ &= \frac{y_1'}{m_1 g} m_1 + \frac{y_2''}{m_2 g} m_2 \\ &= \frac{1}{9.81} \left(\frac{272.5}{EI} + \frac{1744.0}{EI} \right) \end{aligned}$$

$$\Rightarrow \omega_1 = 0.06974 \sqrt{EI}$$

Rayleigh's Method For a beam



mass density $\rho(x)$

area of cross-section $A(x)$

bending stiffness $EI(x)$

$y(x,t) = Y(x) e^{i\omega t}$

Max Kinetic energy :

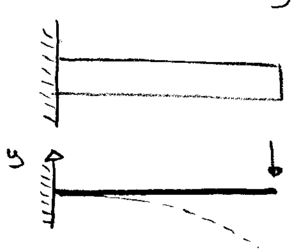
$$\begin{aligned} T_{max} &= (1/2)\, \omega^2 \int_0^l \rho A\, dx \cdot Y(x)^2 \\ &= (1/2)\, \omega^2 \int_0^l \rho A Y^2\, dx \end{aligned}$$

Max potential energy :

$$V_{max} = (1/2) \int_0^l EI (Y'')^2\, dx$$

$$(Y'' = d^2Y/dx^2)$$

$$\therefore \omega^2 = \frac{\int_0^l EI (Y'')^2\, dx}{\int_0^l \rho A Y^2\, dx}$$



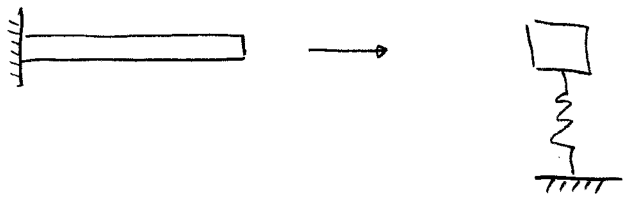
ρA , EI const.

$$y = \frac{-Px^2}{6EIl}(3l - x)$$

$$\omega^2 = (140/11) \ldots$$

$$\omega = 3.56753\sqrt{EI/\rho A l^4}$$

$$\omega_{1,exact} = 3.51602\sqrt{EI/\rho A l^4}$$



$$M = \frac{33}{140}\rho A l$$

$$k = \frac{3EI}{l^3}$$