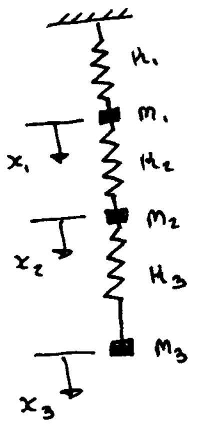
$\underline{\underline{k_{ij}}}$: the force at point i due to a unit displacement at point j. when all the other points have zero displacement.

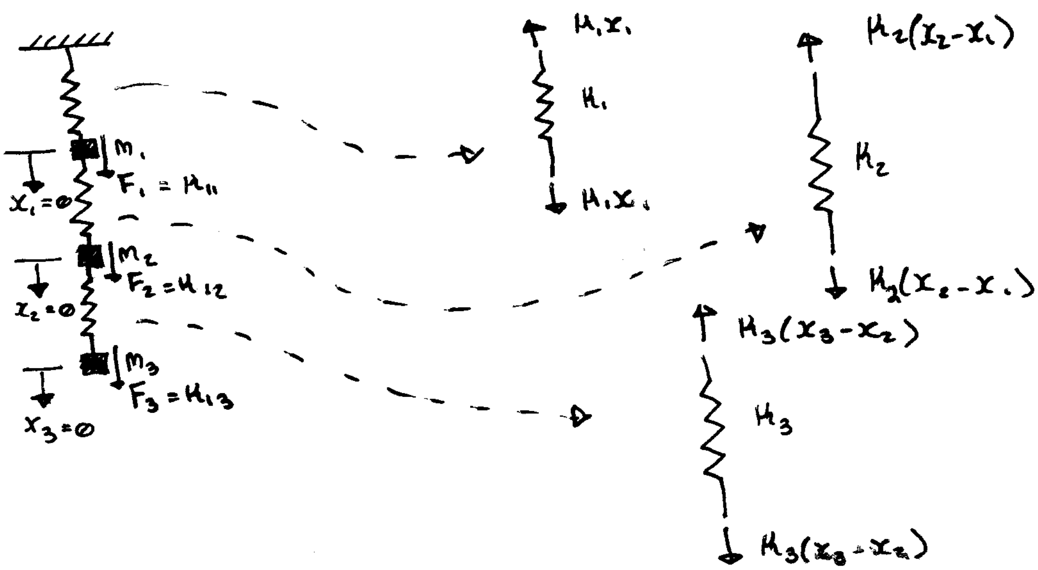
Example:

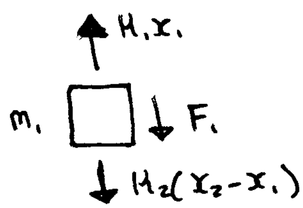


- Find the stiffness influence matrix.

$$K = \begin{bmatrix} k_{11} & k_{12} & k_{13} \\ k_{21} & k_{22} & k_{23} \\ k_{31} & k_{32} & k_{33} \end{bmatrix}$$

Solution: $x_1 = 0$; $x_2 = 0$; $x_3 = 0$

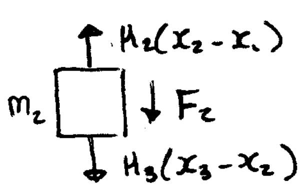




i.e. $\sum F = 0$

$$F_1 + k_2(x_2 - x_1) - k_1 x_1 = 0$$

$$\boxed{F_1 = (k_1 + k_2)x_1 - k_2 x_2}$$



$$\ldots \boxed{F_2 = -k_2 x_1 + (k_2 + k_3)x_2 - k_3 x_3}$$

$$\ldots \boxed{F_3 = -k_3 x_2 + k_3 x_3}$$

Set $x_1 = 1$; $x_2 = x_3 = \emptyset$

$F_1 = (k_1 + k_2)$ $\quad F_2 = -k_2$ $\quad F_3 = \emptyset$

$k_{11} = k_1 + k_2$ $\quad k_{21} = -k_2$ $\quad k_{31} = \emptyset$

Set $x_1 = x_3 = \emptyset$; $x_2 = 1$

$F_1 = -k_2$ $\quad F_2 = k_2 + k_3$ $\quad F_3 = -k_2$

$k_{21} = -k_2$ $\quad k_{22} = k_2 + k_3$ $\quad k_{32} = -k_3$

Set $x_1 = x_2 = \emptyset$; $x_3 = 1$

$F_1 = \emptyset$ $\quad F_2 = -k_3$ $\quad F_3 = k_3$

$k_{13} = \emptyset$ $\quad k_{23} = -k_3$ $\quad k_{33} = k_3$

$$\rightarrow [K] = \begin{bmatrix} k_1 + k_2 & -k_2 & \emptyset \\ -k_2 & k_2 + k_3 & -k_3 \\ \emptyset & -k_3 & k_3 \end{bmatrix}$$

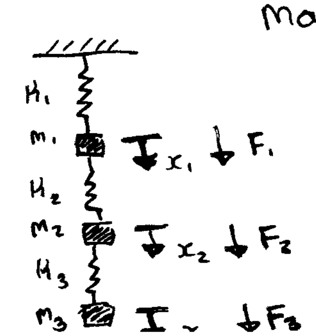
(stiffness matrix by statics)

a_{ij} : the deflection at point i due to a unit force at point j.

$$[A] = [a_{ij}] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix}$$

$$[A][K] = [I]$$

$\boxed{\text{Example}}$ Find the flexibility influence coefficient matrix



Solution:

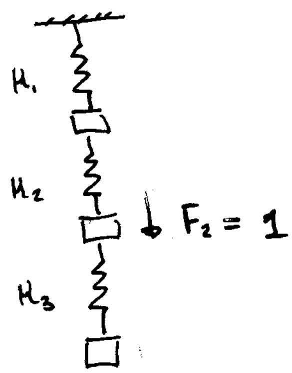
$F_1 = 1$; $F_2 = F_3 = \emptyset$

disp: $\quad x_1 = a_{11} \quad x_2 = a_{21} \quad x_3 = a_{31}$

Spring 1 : $F_1 = K_1 x_1 = 1$ (*)

$x_1 = 1/K_1 = x_2 = x_3$

Let $F_1 = F_3 = \emptyset$, $F_2 = 1$



$$K_{eq} = \frac{K_1 K_2}{K_1 + K_2} \quad ; \quad x_2 = \frac{F_2}{K_{eq}}$$

$$\therefore x_2 = \frac{1}{K_1} + \frac{1}{K_2} = x_3$$

$$a_{22} = x_2 = \frac{1}{K_1} + \frac{1}{K_2}$$

$$a_{32} = x_3 = \frac{1}{K_1} + \frac{1}{K_2}$$

$$\boxed{a_{12} = a_{21} = \frac{1}{K_1}}$$

$$\frac{1}{K_{eq}} = \frac{1}{K_1} + \frac{1}{K_2} + \frac{1}{K_3}$$

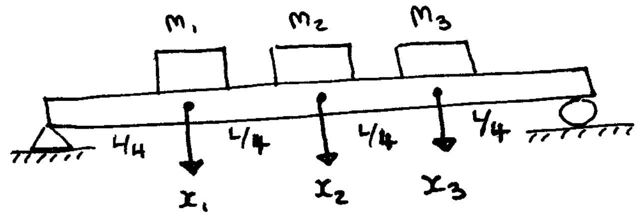
$$x_3 = \frac{F_3}{K_{eq}} = \frac{1}{K_{eq}} = \frac{1}{K_1} + \frac{1}{K_2} + \frac{1}{K_3}$$

$$\therefore a_{33} = x_3 = \frac{1}{K_1} + \frac{1}{K_2} + \frac{1}{K_3}$$

$$a_{13} = a_{31} = \frac{1}{K_1} \quad ; \quad a_{23} = a_{32} = \frac{1}{K_1} + \frac{1}{K_2}$$

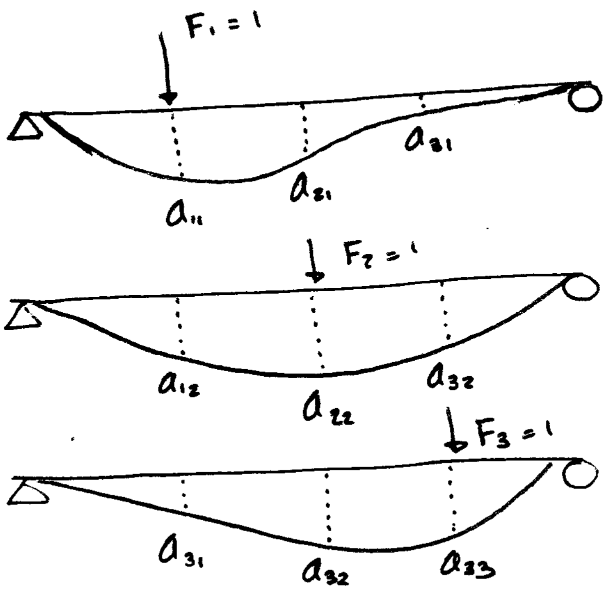
$$\therefore [A] = \begin{bmatrix} 1/K_1 & 1/K_1 & 1/K_1 \\ 1/K_1 & 1/K_1 + 1/K_2 & 1/K_1 + 1/K_2 \\ 1/K_1 & 1/K_1 + 1/K_2 & 1/K_1 + 1/K_2 + 1/K_3 \end{bmatrix}$$

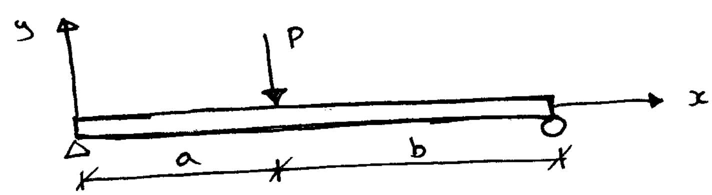
Example Find the flexibility matrix of the weightless beam shown:



EI = const.

Solution:





$L = a + b$

$$y = \begin{cases} \dfrac{Pbx}{6EIL}\left(L^2 - b^2 - x^2\right) & ; \quad 0 \le x \le a \\[2ex] -\dfrac{Pa(L-x)}{6EIL}\left(a^2 + x^2 - 2Lx\right) & ; \quad a \le x \le L \end{cases}$$

a_{11} : $a = L/4$; $b = 3/4\,L$

At $x = L/4$

$$a_{11} = \frac{P(3/4\,L)(1/4\,L)}{6EIL}\left(L^2 - (3L/4)^2 - (L/4)^2\right)$$

$$a_{11} = \left(\frac{9}{[illegible]}\right)\left(\frac{L^3}{EI}\right)$$

At $x = L/2$

$$a_{21} = \left(\frac{11}{768}\right)\left(\frac{L^3}{EI}\right)$$

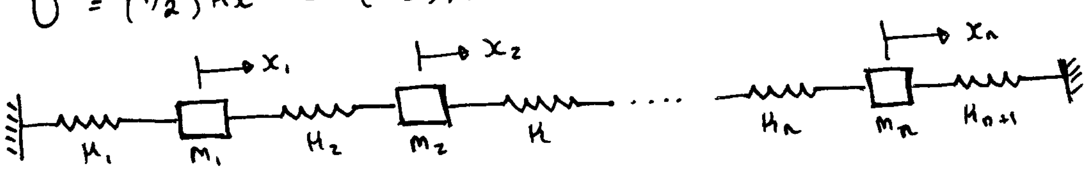
At $x = 3L/4$

$$a_{31} = \left(\frac{7}{768}\right)\left(\frac{L^3}{EI}\right)$$

$$[A] = \frac{L^3}{768EI}\begin{bmatrix} 9 & 11 & 7 \\ 11 & 16 & 11 \\ 7 & 11 & 9 \end{bmatrix}$$

Potential and Kinetic Energy:

$$U = (1/2)Kx^2 = (1/2)Fx$$



$$\{\vec{F}\} = [K]\{\vec{x}\}$$

Here: $\{\vec{F}\} = (F_1, F_2, \ldots, F_n)^T$

$\{\vec{x}\} = (x_1, x_2, \ldots, x_n)^T$

The potential energy:

$$U = (1/2)F_1x_1 + (1/2)F_2x_2 + \ldots + (1/2)F_nx_n$$

$$= (1/2)(F_1 \; F_2 \; \ldots \; F_n)\begin{Bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{Bmatrix}$$

$$= (1/2)\{\vec{F}\}^T\{x\}$$

$$U = (1/2)\left([K]\{\vec{x}\}\right)^T\{\vec{x}\}$$

$$= (1/2)\{\vec{x}\}^T[K]\{\vec{x}\}$$

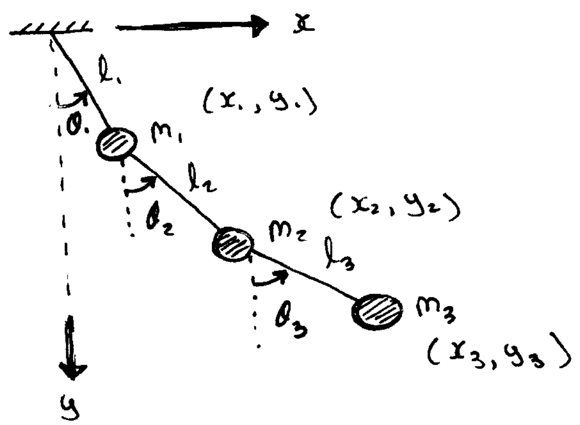
Stiffness matrix

The kinetic energy:

$$T = (1/2) m_1 \dot{x}_1^2 + (1/2) m_2 \dot{x}_2^2 + \cdots + (1/2) m_n \dot{x}_n^2$$
$$= (1/2) \{\dot{\vec{x}}\}^T [M] \{\dot{\vec{x}}\}$$

Here
$$[M] = \begin{bmatrix} m_1 & & & \emptyset \\ & m_2 & & \\ & & \ddots & \\ \emptyset & & & m_n \end{bmatrix}$$

Generalized coordinates:



$$x_1^2 + y_1^2 = l_1^2$$
$$(x_2 - x_1)^2 + (y_2 - y_1)^2 = l_2^2$$
$$(x_3 - x_2)^2 + (y_3 - y_2)^2 = l_3^2$$

6: x_1 y_1 x_2 y_2 x_3 y_3

3 constrained eq'ns

only $6 - 3 = 3$ are independent.

* θ_1 θ_2 θ_3 are three independent generalized coordinates.

Generalized coordinates :

$q_1 = \theta_1 \; ; \; q_2 = \theta_2 \; ; \; q_3 = \theta_3$

$$\begin{cases} x_1 = l_1 \sin(\theta_1) & ; \quad y_1 = l_1 \cos(\theta_1) \\ x_2 = x_1 + l_2 \sin(\theta_2) & ; \quad y_2 = y_1 + l_2 \cos(\theta_2) \\ x_3 = x_2 + l_3 \sin(\theta_3) & ; \quad y_3 + y_2 + l_3 \cos(\theta_3) \end{cases}$$

$$\begin{cases} x_i = x_i(q_1, q_2, q_3) \\ y_i = y_i(q_1, q_2, q_3) \end{cases} \qquad i = 1, 2, 3$$

Virtual displacement :

$q_1, q_2, \ldots, q_n \longrightarrow \delta q_1, \delta q_2, \ldots, \delta q_n$

The work done : $\delta w_1, \delta w_2, \ldots, \delta w_n$

The generalized Force :

$$Q_1 = \frac{\delta w_1}{\delta q_1}, \quad Q_2 = \frac{\delta w_2}{\delta q_2}, \quad \ldots, \quad Q_n = \frac{\delta w_n}{\delta q_n}$$

Define the Lagrangian

$L = T - V$

Then the equations of motion

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} + \frac{\partial V}{\partial q_i} = Q_i \quad ; \quad i = 1, 2, \ldots, n$$

If F_{xk}, F_{yk}, F_{zk} are the external forces acting on the k^{th} mass in the x, y, z-directions,

suddenly changing from i to j, but same thing

$$Q_j = \sum_k \left[F_{xk}\left(\frac{\partial x_k}{\partial \dot{q}_j}\right) + F_{yk}\left(\frac{\partial y_k}{\partial \dot{q}_j}\right) + F_{zk}\left(\frac{\partial z_k}{\partial \dot{q}_j}\right) \right] \quad ; \quad j = 1, 2, \ldots, n$$

Viscously damped Systems

Rayleigh's dissipation function :

$$R = (1/2)\, \dot{\vec{x}}^T [c]\, \dot{\vec{x}}$$

here $[c]$ is the damping matrix

The equations of motion

$$[M]\{\ddot{\vec{x}}\} + [C]\{\dot{\vec{x}}\} + [K]\{\vec{x}\} = \{\vec{F}\}$$

Proportional damping matrix

$$[C] = \alpha[M] + \beta[K]$$

The generalized force of the viscously damping

$$Q_i = \frac{-\partial R}{\partial \dot{x}_i}$$

In the generalized coordinates,

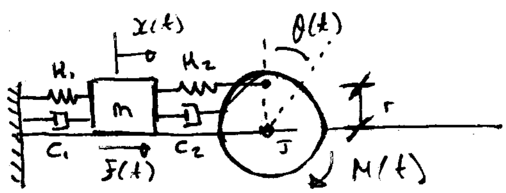
$$Q_j = \frac{-\partial R}{\partial \dot{q}_i}$$

The final equations of motion:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} + \frac{\partial V}{\partial q_j} = -\frac{\partial R}{\partial \dot{q}_j} + Q_j$$

$$\boxed{\Rightarrow \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} + \frac{\partial R}{\partial \dot{q}_i} + \frac{\partial V}{\partial q_i} = Q_j}$$

Example:



Derive the equations of motion.

Solution: $q_1 = x(t)$; $q_2 = \theta(t)$

→ generalized coordinates

Kinetic:

$$T = (1/2)m\dot{x}^2 + (1/2)J\dot{\theta}^2$$

$$T = (1/2)m\dot{q}_1^2 + (1/2)J\dot{q}_2^2$$

Potential:

$$V = (1/2)k_1 x^2 + (1/2)k_2(r\theta - x)^2$$

$$V = (1/2)k_1 q_1^2 + (1/2)k_2(rq_2 - q_1)^2$$

Rayleigh's dissipation function

$$R = (1/2)c_1\dot{x}^2 + (1/2)c_2(r\dot{\theta} - \dot{x})^2$$

$$= (1/2)c_1\dot{q}_1^2 + (1/2)c_2(r\dot{q}_1 - \dot{q}_2)^2$$

For q_1 :

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_1}\right) - \frac{\partial T}{\partial q_1} + \frac{\partial R}{\partial \dot{q}_1} + \frac{\partial V}{\partial q_1} = Q_1$$

$$\rightarrow \frac{\partial T}{\partial \dot{q}_1} = m\dot{q}_1 \quad ; \quad \rightarrow \frac{\partial T}{\partial q_1} = 0$$

$$\rightarrow \frac{\partial R}{\partial \dot{q}_1} = c\dot{q}_1 + C_2(r\dot{q}_2 - \dot{q}_1)\cdot(-1)$$

$$\rightarrow \frac{\partial V}{\partial q_1} = K_1 q_1 + K_2(r q_2 - q_1)(-1)$$

$$\rightarrow Q_1 = F(t)$$

(1) $$m\ddot{q}_1 + C_1\dot{q}_1 + C_2(\dot{q}_1 - r\dot{q}_2) + K_1 q_1 + K_2(q_1 - r q_2) = F(t)$$

For q_2 :

$$\rightarrow \frac{\partial T}{\partial \dot{q}_2} = J\dot{q}_2 \quad ; \quad \rightarrow \frac{\partial T}{\partial q_2} = 0$$

$$\rightarrow \frac{\partial R}{\partial \dot{q}_2} = C_2(r\dot{q}_2 - \dot{q}_1)(r) = C_2 r(r\dot{q}_2 - \dot{q}_1)$$

$$\rightarrow \frac{\partial V}{\partial q_2} = K_2 r(r q_2 - q_1)$$

$$\rightarrow Q_2 = M(t)$$

(2) $$J\ddot{q}_2 + C_2 r(r\dot{q}_2 - \dot{q}_1) + K_2 r(r q_2 - q_1) = M(t)$$

Putting (1) and (2) into matrix form :

$$\begin{bmatrix} m & 0 \\ 0 & J \end{bmatrix}\begin{Bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{Bmatrix} + \begin{bmatrix} C_1 + C_2 & -C_2 r \\ -C_2 r & C_2 r^2 \end{bmatrix}\begin{Bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{Bmatrix} + \begin{bmatrix} K_1 + K_2 & -K_2 r \\ -K_2 r & K_2 r^2 \end{bmatrix}\begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix} = \begin{Bmatrix} F(t) \\ M(t) \end{Bmatrix}$$

$$T = (1/2)m\dot{q}_1^2 + (1/2)J\dot{q}_2^2 = (1/2)(\dot{q}_1 \; \dot{q}_2)\begin{bmatrix} m & 0 \\ 0 & J \end{bmatrix}\begin{Bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{Bmatrix}$$

Define $\vec{q} = \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix}$

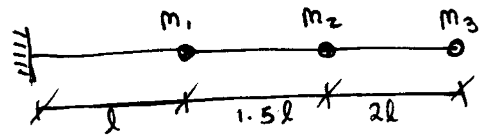
$$T = (1/2)\{\dot{\vec{q}}\}^T [M] \{\dot{\vec{q}}\}$$

$$V = (1/2)K_1 q_1^2 + (1/2)K_2(r^2 q_2^2 - 2r q_1 q_2 + q_1^2)$$

$$= (1/2)\left[(K_1 + K_2)q_1^2 - 2K_2 r q_1 q_2 + K_2 r^2 q_2^2\right]$$

$$= (1/2)\{\vec{q}\}^T \begin{bmatrix} K_1 + K_2 & -K_2 r \\ -K_2 r & K_2 r^2 \end{bmatrix} \{\vec{q}\}$$

Example



$m_1 = 3m$; $m_2 = 2m$; $m_3 = m$; EI = const.

Find the natural frequencies and mode shapes :

<u>Solution</u> :

$$[A] = \frac{l^3}{24EI}\begin{bmatrix} 8 & 26 & 50 \\ 26 & 125 & 275 \\ 50 & 275 & 729 \end{bmatrix}$$

$$= \frac{l^3}{EI}\begin{bmatrix} 0.33333 & 1.083333 & 2.08333 \\ & 5.20833 & 11.4583 \\ \text{sym.} & & 3.03750 \end{bmatrix}$$

The stiffness matrix

$$[K] = [A]^{-1}$$

$$[K] = \frac{EI}{l^3}\begin{bmatrix} 11.6399 & -3.70655 & 0.641026 \\ & 2.37322 & -0.641026 \\ \text{sym.} & & 0.230769 \end{bmatrix}$$

Mass matrix :

$$[M] = \begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix} = m\begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Natural Freq. and mode shape :

$$(-\omega^2 [M] + [K])\{\vec{u}\} = 0$$

$$\Rightarrow \left(-m\omega^2 \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \frac{EI}{l^3}\begin{bmatrix} \times & \times & \times \\ \times & \times & \times \\ \times & \times & \times \end{bmatrix} \right) \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = 0$$

$$\Rightarrow \left(\begin{bmatrix} \times & \times & \times \\ \times & \times & \times \\ \times & \times & \times \end{bmatrix} - \frac{ml^3}{EI}\omega^2 \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = 0$$

Define : $\lambda = \frac{m\ell^3}{EI}\omega^2$

$$\Rightarrow \left(\begin{bmatrix} \times & \times & \times \\ \times & \times & \times \\ \times & \times & \times \end{bmatrix} - \lambda \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = 0$$

$\Rightarrow \lambda_1 = 0.02502413$; $\omega_1 = \sqrt{EI/m\ell^3} \cdot \sqrt{\lambda_1}$

$\lambda_2 = 0.612216$

$\lambda_3 = 4.46008$

$\Rightarrow \omega_1 = 0.158224 \sqrt{EI/m\ell^3}$

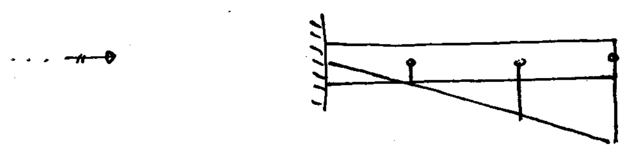
$\omega_2 = 0.782442 \sqrt{EI/m\ell^3}$

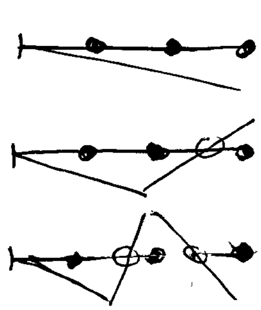
$\omega_3 = 2.11189 \sqrt{EI/m\ell^3}$

The modal shapes :

$$\{\vec{u}_i\} = \begin{Bmatrix} 0.0654970 \\ 0.343615 \\ 0.866597 \end{Bmatrix} ; \begin{Bmatrix} 0.25073 \\ 0.537842 \\ -0.483263 \end{Bmatrix} ; \begin{Bmatrix} 0.51646 \\ -0.304393 \\ 0.121367 \end{Bmatrix}$$

0 sign change ; 1 sign change ; 2 sign change





mode 1

mode 2

mode 3