Sep. 3/19

Engineering vibration - 4th

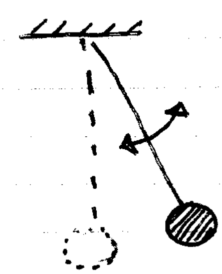
Office hours: Tues / Thurs → (2:30 ~ 3:30)

Tutorials to have questions marked in class (worth 20%)
↳ groups of 2/3

Midterm: October 10TH/19 during lecture time (75 min)
↳ Formula Sheet provided

Chapter 1 - Introduction to Vibration and Free Response
Fundamentals of Vibration
Vibration is a mechanical phenomenon whereby oscillations occur about an equilibrium point.

The oscillations of the vibrations may be periodic (like a pendulum):



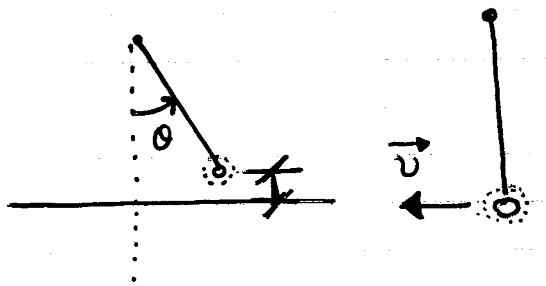
or random (the movement of a tire on gravel road):

Avoid vibrations
* Cyclic motion implies cyclic forces:
  - aircraft Frame and wings
  - imbalances in rotating parts
* even modest levels of vibration can cause discomfort
  - automobiles
* Vibrations generally lead to a loss of precision in controlling machinery

"Tacoma narrows bridge"
↳ opened July 1, 1940
↳ collapsed Nov. 7, 1940

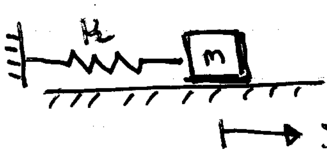
"<u>Vehicle suspension systems</u>"

Good uses of vibrations:
↳ music, guitar, speakers
↳ structural analysis (ultrasonic), detecting cracks

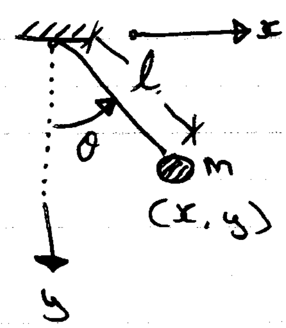


For any vibration system
1°: means for storing potential energy
 - spring / elasticity
2°: means for storing kinetic energy
 - mass / inertia
3°: means by which energy is gradually lost
 - damper

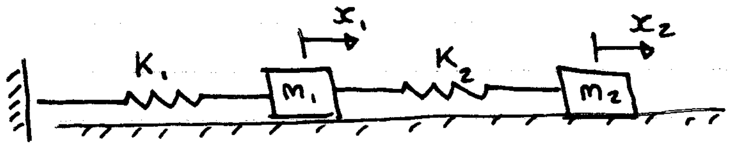
<u>Degree of freedom</u>:
The DOF of a system is defined as the minimum numbers of independent coordinates required to determine completely the positions of all parts of the system at any instant of time.



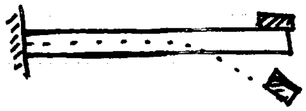
→ 1 DOF



→ $x^2 + y^2 = l^2$
two coordinates, but not independent, so still 1 DOF.



2 DOF



Infinite number of DOF

Discrete and continuous systems

→ Systems with finite number of DOF is called discrete or lumped parameters system
→ Systems with infinite number of DOF is called continuous system

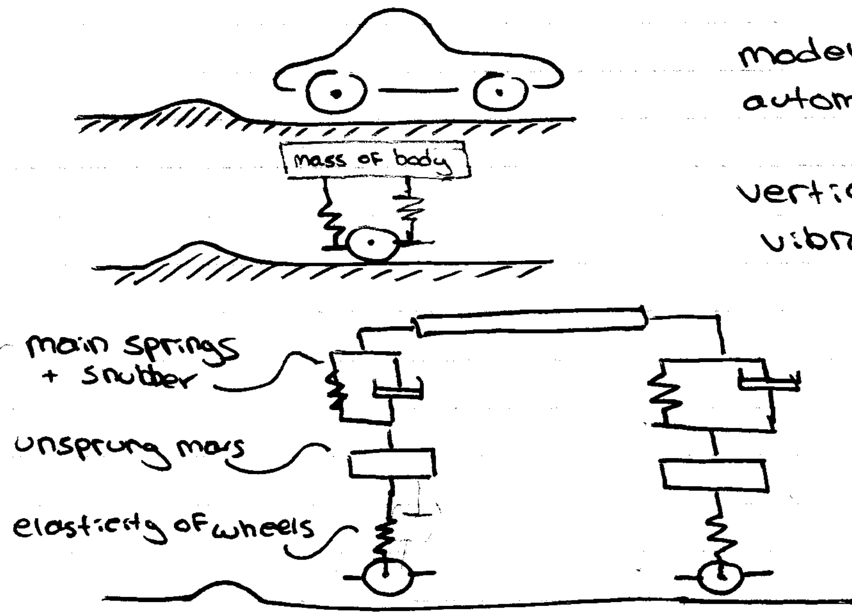
Vibration :

→ Free vibration : the system, after an initial disturbance, is left to vibrate on its own.
→ Forced vibration : the system is subjected to an external force.
→ undamped vibration : no energy lost
→ damped vibration : energy lost
→ linear vibration : if all the basic components of a system is linear, the principle of superposition holds, and the differential equation is linear.
(beyond scope of class) → Non-linear vibration
→ Deterministic vibration : the values of the exciting forces is known at all times
→ Random vibration

- Vibration analysis

The vibration analysis of an engineering System involves the following four steps ;

1- mathematical modeling : the mathematical model is simplified, keeping in view the purpose of the analysis



modeling of automobile

vertical vibration

(Expanded look...)

2 - Governing Equation

3 - Solution

(*) concentration of course

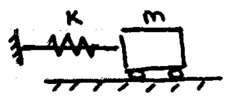
4 - Interpretation of Results

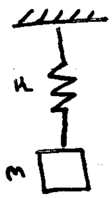
Spring element 

mass element 

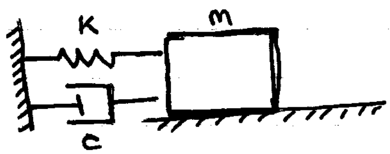
damper 

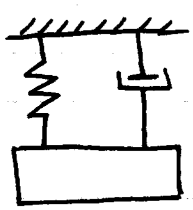
Example: spring-mass

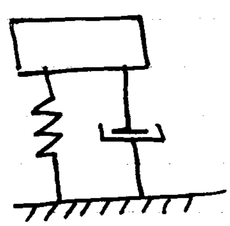




Example: spring-mass-damper

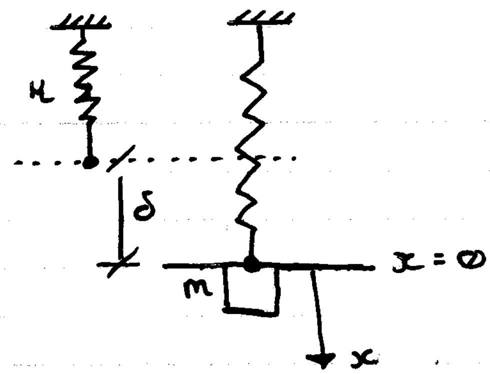


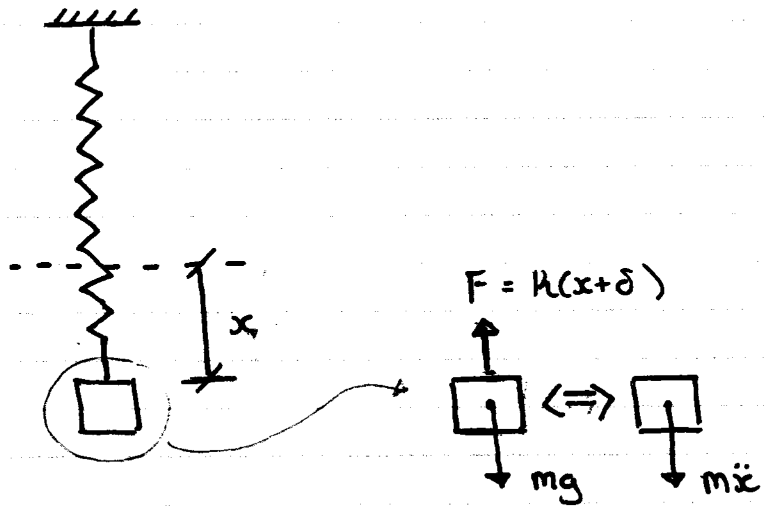




All represent
1 DOF

Modeling a single DOF





$$\Sigma F = ma$$

$$mg - k(x+\delta) = m\ddot{x}$$

Since $mg = k\delta$ (Hooke's Law)

$$\boxed{m\ddot{x} + kx = 0}$$

The solution:

$$x(t) = A\sin(\omega_n t + \phi) \quad (*)$$

(in radians)

Since

$$\dot{x}(t) = \omega_n A\cos(\omega_n t + \phi)$$

Then

$$\ddot{x}(t) = -A\omega_n^2\sin(\omega_n t + \phi) = -\omega_n^2 x(t) \quad (*)$$

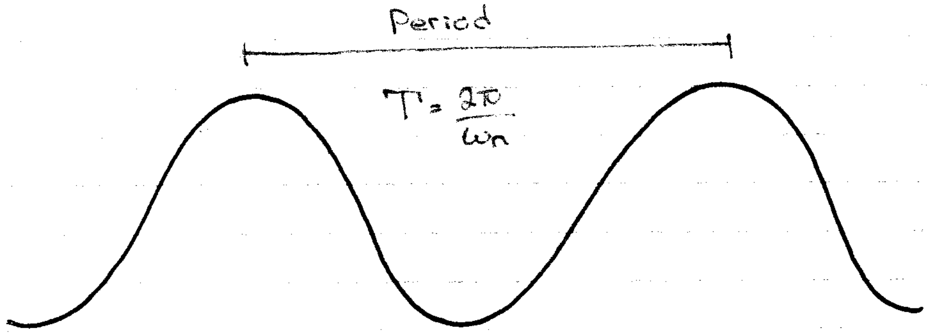
$$\Rightarrow m(-\omega_n^2 x) + kx = 0$$

$$\Rightarrow (-m\omega_n^2 + k)x = 0$$

$$\Rightarrow -m\omega_n^2 + k = 0$$

The "natural frequency" $\omega_n = \sqrt{\frac{k}{m}}$ → unit of ω_n: rad/s

$$x(t) = A\sin(\omega t + \phi)$$



Period : $\omega_n T = 2\pi$

$$T = \frac{2\pi}{\omega_n}$$

Frequency (f_n) : $f_n = \frac{1}{T} = \frac{\omega_n}{2\pi}$ (measured in Hz)

$$\omega_n = 2\pi f_n$$

Given initial distance x_0 and initial velocity v_0 :

$$\begin{cases} x_0 = x(t)\big|_{t=0} = A\sin\phi \\ v_0 = \dot{x}(t)\big|_{t=0} = A\omega_n\cos\phi \end{cases}$$

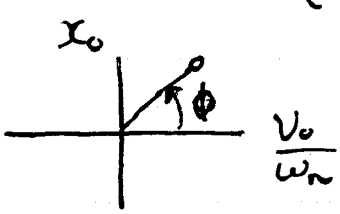
$$\frac{v_0}{\omega_n} = A\cos\phi$$

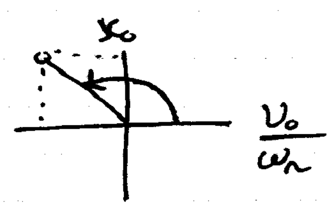
$$x_0^2 + \left(\frac{v_0^2}{\omega_n}\right) = A^2\sin^2\phi + A^2\cos^2\phi = A^2$$

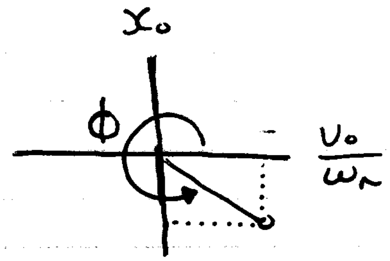
$$\rightarrow A = \sqrt{x_0^2 + \left(\frac{v_0}{\omega_n}\right)^2} = \frac{1}{\omega_n}\sqrt{\omega_n^2 x_0^2 + v_0^2}$$

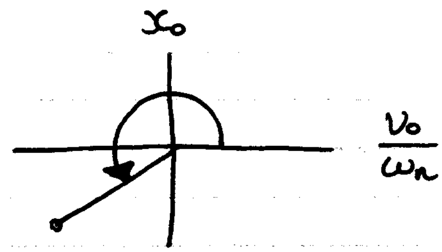
$$\frac{\omega_n x_0}{v_0} = \tan\phi$$

$$\phi = \tan^{-1}\left(\frac{\omega_n x_0}{v_0}\right)$$



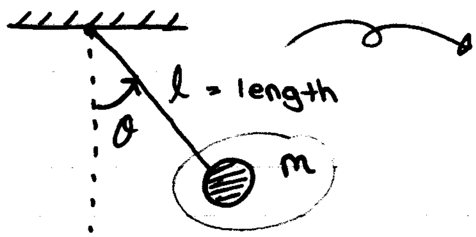






$$\boxed{x(t) = \frac{\sqrt{\omega_n^2 x_0^2 + v_0^2}}{\omega_n} \sin\left(\omega_n t + \tan^{-1} \frac{\omega_n x_0}{v_0}\right)}$$

Pendulum

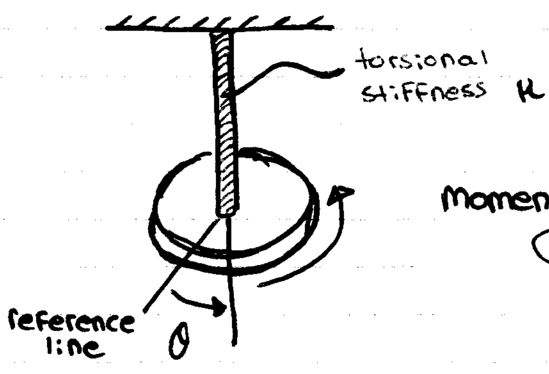


consider mass of bar (compared to mass of weight) as zero.

$$\ddot{\theta} + g/l\,\theta = 0$$

$$\omega_n = \sqrt{\frac{g}{l}}$$

Shaft and disk



moment of inertia J

$$J\ddot{\theta} + K\theta = 0$$

$$\omega_n = \sqrt{\frac{K}{J}}$$

Example: The total mass of the model is $m = 30$ kg, the frequency of the model is $f_n = 10$ Hz, what is the k?

Solution: Since $\omega_n = \sqrt{k/m}$

$$\Rightarrow \quad k = m\omega_n^2$$
$$= (30)(2\pi f_n)^2$$
$$= (30)(2\pi(10))^2$$
$$= 1.184 \times 10^5 \text{ N/m}$$

standard unit for spring constant

Example: $m = 2$ kg
$k = 200$ N/m

For the following initial conditions

a) $x_0 = 2$mm, $v_0 = 1$ mm/s
b) $x_0 = -2$mm, $v_0 = 1$ mm/s
c) $x_0 = 2$mm, $v_0 = -1$ mm/s

Find the response of the system.

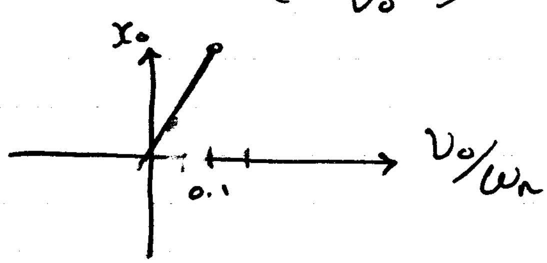
Solution: $\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{200}{2}} = 10$

The amplitude:

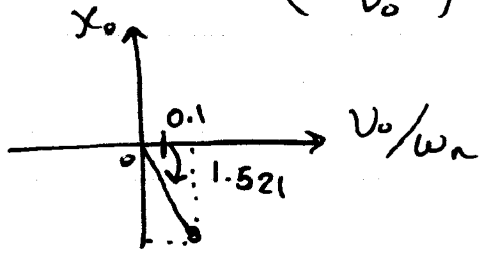
$$A = \frac{\sqrt{\omega_n^2 x_0^2 + v_0^2}}{\omega_n} = \frac{\sqrt{10^2(\pm 2)^2 + (\pm 1)^2}}{10} = 2.0025 \text{ mm}$$

Phase:

a) $\phi = \tan^{-1}\left(\frac{\omega_n x_0}{v_0}\right) = \tan^{-1}\left(\frac{(10)(2)}{(1)}\right) = 1.521$ rad $(87.147°)$

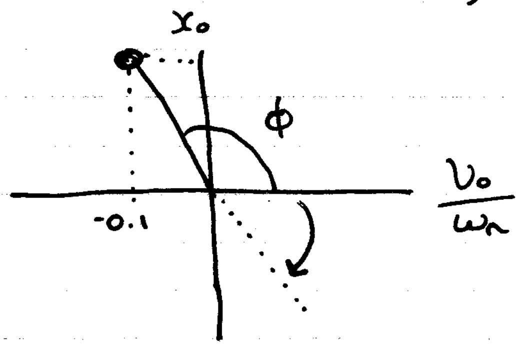


b) $\phi = \tan^{-1}\left(\frac{\omega_n x_0}{v_0}\right) = \tan^{-1}\left(\frac{(10)(-2)}{1}\right) = -1.521$ rad



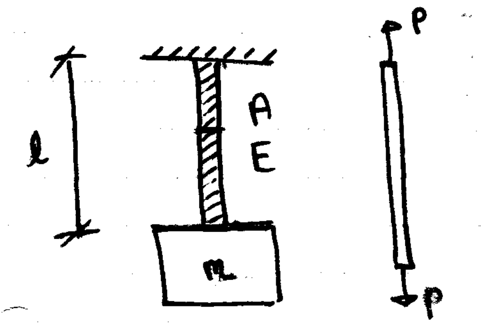
c) $\phi = \tan^{-1}\left(\frac{\omega_n x_0}{v_0}\right) = \tan^{-1}\left(\frac{(10)(2)}{(-1)}\right) = -1.521 + \pi \text{ rad}$

$= 1.621 \text{ rad}$



(2nd + 4th quadrant, add π to get correct value.)

More on springs and stiffness

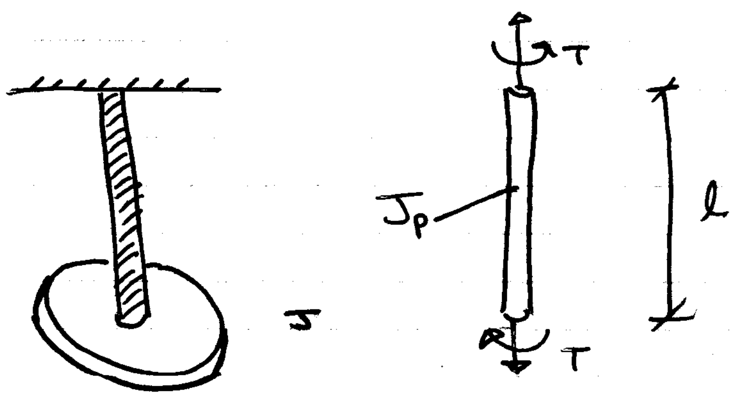


The change in length

$$\Delta u = \frac{Pl}{AE}$$

$$P = \frac{AE}{l}\Delta u$$

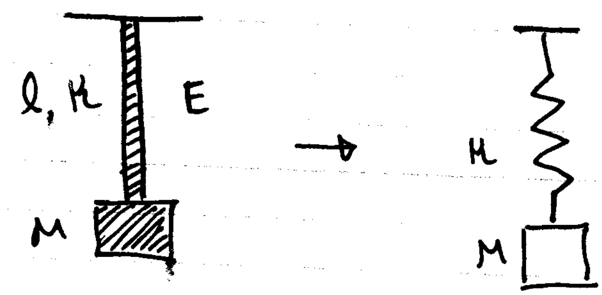
$$\therefore k = \frac{AE}{l}$$



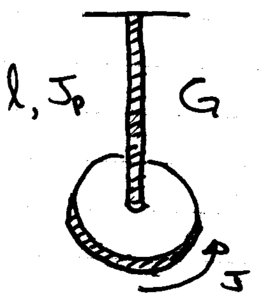
$$\theta = \frac{Tl}{GJ}$$

$$T = \frac{GJ_p}{l}\theta$$

$$k = \frac{GJ_p}{l}$$



$$k = \frac{AE}{l}$$



$$k = \frac{GJ_p}{l}$$

$$\omega_n = \sqrt{\frac{k}{J}}$$

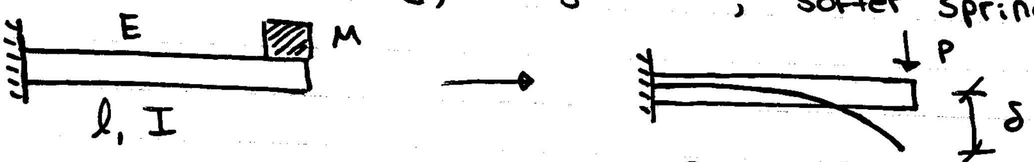
natural frequency

moment of inertia of disk

number of turns

→ shorter spring, smaller "n", stronger spring

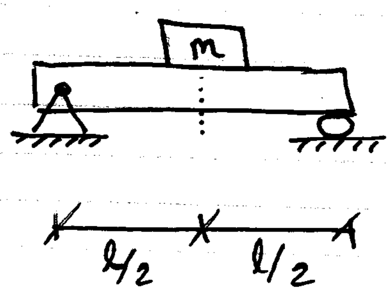
→ longer spring, larger "n", softer spring

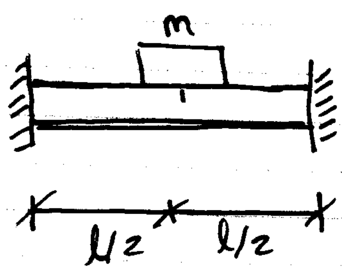


$$\delta = \frac{Pl^3}{3EI} \rightarrow P = \boxed{\frac{3EI}{l^3}}\,\delta$$

$$k = \frac{3EI}{l^3}$$

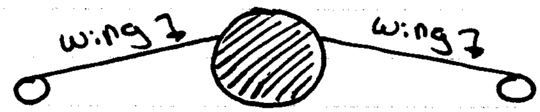
$$\omega_n = \sqrt{\frac{k}{m}}$$





→ Page 53, table 1.3

Example Consider front of airplane:



No Fuel, $m = 10$ kg
Full Fuel, $m = 1000$ kg
$I = 5.2 \times 10^{-5}$
$l = 2m$
$E = 6.9 \times 10^{9}$ Pa
→ Find the Freq. of the wing for the two cases

Solution:
Full Fuel, $\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{3EI}{ml^3}}$

$$= \sqrt{\frac{(3)\times(6.9\times10^{9})\times(5.2\times10^{-5})}{(1000)\times(2)^3}}$$

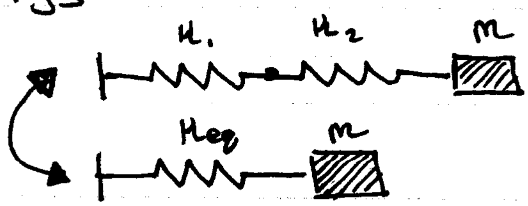
$\omega_n = 11.6$ rad/s

No fuel, $\omega_n = \sqrt{\frac{(3)(6.9\times10^{9})(5.2\times10^{-5})}{(10)(2)^3}}$

$\omega_n = 115$ rad/s
↳ this doesn't consider mass of wing

Combining Springs

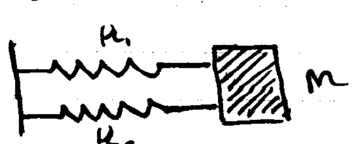
→ series



Each spring has the same Force.

$$k_{eq} = \frac{1}{1/k_1 + 1/k_2} = \frac{k_1 k_2}{k_1 + k_2}$$

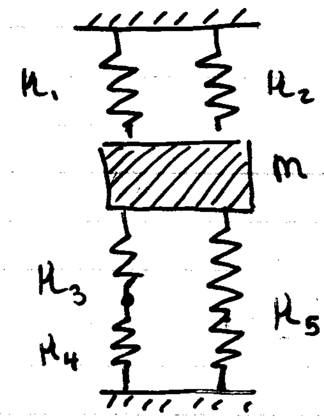
→ Parallel



Each spring has the same deformation
$k_{eq} = k_1 + k_2$

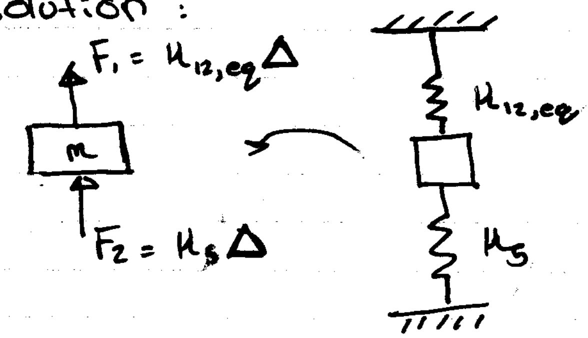
Example :

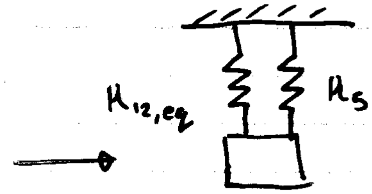
Find the equivalent stiffness (k_{eq}) :



Solution :

consider :



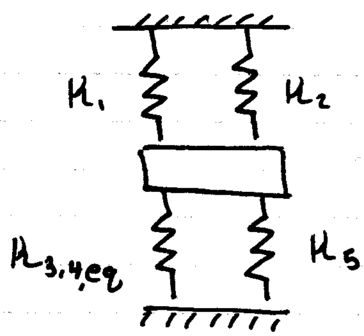


Thus, <u>Same deformation</u>

$$F = F_1 + F_2 = (k_{12,eq} + k_5)\Delta = k_{eq}$$

Springs k_3 and k_4 are in series.

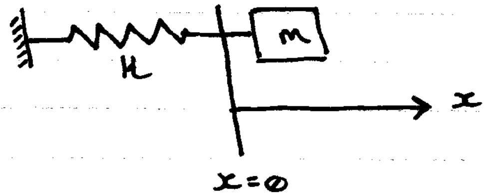
$$k_{34,eq} = \frac{k_3 k_4}{k_3 + k_4}$$



k_1 k_2 $k_{34,eq}$ and k_5 are in parallel.

$$k_{eq} = k_1 + k_2 + k_5 + k_{34,eq}$$
$$= k_1 + k_2 + k_5 + \frac{k_3 k_4}{k_3 + k_4}$$

Harmonic motion



$$m\ddot{x} + kx = 0$$

(where: $\omega_n = \sqrt{\frac{k}{m}}$

(Divide by m) $\ddot{x} + \frac{k}{m}x = 0$

(Replace) $\boxed{\ddot{x} + \omega_n^2 x = 0}$

Displacement : $x = A\sin(\omega_n t + \phi)$

Velocity : $\dot{x} = \omega_n A\cos(\omega_n t + \phi)$

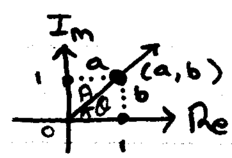
Acceleration : $\ddot{x} = -\omega_n^2 \underbrace{A\sin(\omega_n t + \phi)}_{x} = -\omega_n^2 x$

Max displacement : $x_{max} = A$

Max velocity : $v_{max} = \omega_n A$ (or when $\cos(\omega_n t + \phi) = 1$)

Max acceleration : $a_{max} = \omega_n^2 A$ (or when $x = A$)

Complex number



$c = a + ib$

$i = \sqrt{-1}$

where $a = A\cos\theta$

$b = A\sin\theta$

then $c = A\cos\theta + iA\sin\theta = A(\cos\theta + i\sin\theta)$

$C = Ae^{i\theta}$ $\quad \boxed{e^{i\theta} = \cos\theta + i\sin\theta}$

Differential equation

$m\ddot{x} + kx = 0$

Solution of the ODE

$x(t) \triangleq ae^{\lambda t}$ (*)

$\rightarrow \dot{x} = \lambda a e^{\lambda t} = \lambda x(t)$

$\ddot{x} = \lambda\dot{x} \triangleq \lambda^2 x(t)$ (*)

Substitution: $m\lambda^2 x(t) + kx(t) = 0$

$(m\lambda^2 + k)x(t) = 0$

Since $x(t) \neq 0$

$$m\lambda^2 + k = 0$$

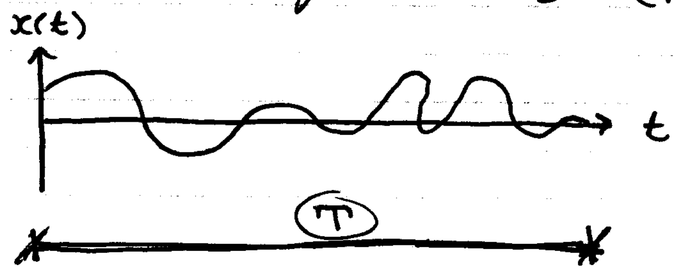
$$\rightarrow \lambda = \pm\sqrt{-k/m} = \pm i\sqrt{k/m}$$

$$\lambda = \pm i\omega_n$$

$x(t) = a_1 e^{i\omega_n t} + a_2 e^{-i\omega_n t}$; a_1 and a_2 are constant

$$x(t) = A\sin(\omega_n t + \phi)$$

Root mean square values (RMS):



$$\lim_{T\to\infty} \frac{1}{T}\int_0^T x(t)\,dt = \bar{x}$$

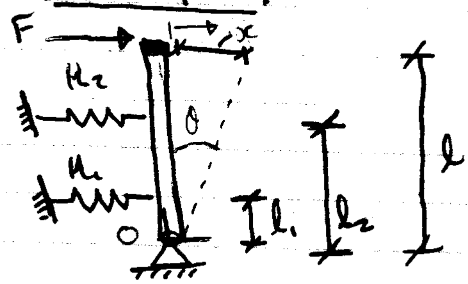
average value

$$\overline{x^2} = \lim_{T\to\infty} \frac{1}{T}\int_0^T x^2(t)\,dt$$

mean square value

$$x_{rms} = \sqrt{\overline{x^2}}$$

Example



Find the equivalent stiffness of the system that relates the applied force to the resulting displacement x

$$F = k_{eq} x$$

Solution potential energy in the real system equals to the energy stored in the equivalent spring:

For small disp. x, angle θ is small as well.

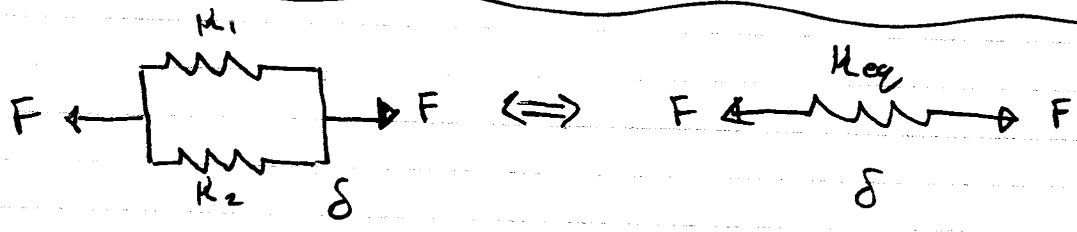
$x_1 = l_1\theta$; $x_2 = l_2\theta$ (only when ↻)

$$(1/2)k_1x_1^2 + (1/2)k_2x_2^2 = (1/2)k_{eq}x^2$$

Since $x = l\theta$:

$$(1/2)k(l_1\theta)^2 + (1/2)k_2(l_2\theta)^2 = (1/2)(l\theta)^2$$

$$\boxed{k_{eq} = k_1(l_1/l)^2 + k_2(l_2/l)^2}$$



$$(1/2)k_1\delta^2 + (1/2)k_2\delta^2 = (1/2)k_{eq}\delta^2$$

The decibel dB scale

measure the vibration relative to some reference value:

$$10 \log_{10}\left(\frac{x}{x_0}\right)^2 \leftarrow \text{dB}$$

Reference power P_0, the sound produces twice as much as the reference.

If $P = 2P_0$:

$$10 \log_{10}(P/P_0) = 10\log_{10} 2 = 3\,\text{dB}$$

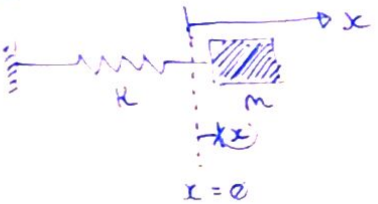
If $P = 10P_0$:

$$10 \log_{10} 10 = 10\,\text{dB}$$

If $P = 10^6 P_0$:

$$10 \log_{10}(10^6) = 60\,\text{dB}$$

Modeling and Energy Methods



Potential energy
$\rightarrow U = (1/2)kx^2$
Kinetic energy
$\rightarrow T = (1/2)m\dot{x}^2$

For rotating about a fixed axis:
$\rightarrow T = (1/2)J\dot{\theta}^2$

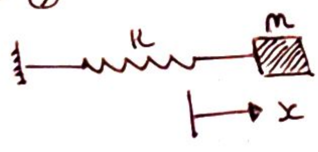
Conservation of energy:
$\rightarrow T + U = \text{const.}$

$T_1 + U_1 = T_2 + U_2$

$T_{max} = U_{max}$

$d/dt(T+U) = 0$

Spring-mass:



$T = (1/2)m\dot{x}^2$

$U = (1/2)kx^2$

$d/dt(T+U) = 0$

$\rightarrow d/dt((1/2)m\dot{x}^2 + (1/2)kx^2) = 0$

$1/2\, m \cdot 2\dot{x}\, d\dot{x}/dt + 1/2\, k \cdot 2x\, dx/dt = 0$

$m\dot{x}\ddot{x} + kx\dot{x} = 0$

$\dot{x}(m\ddot{x} + kx) = 0$

Since $\dot{x}$ cannot be zero all the time,

$\rightarrow m\ddot{x} + kx = 0$

$\boxed{\text{Example}}$ Find the natural frequency from the energy:

The displacement : $x = A\sin(\omega_n t + \phi)$

$x_{max} = A$

Velocity : $\dot{x} = A\omega_n \cos(\omega_n t + \phi)$

$\dot{x}_{max} = A\omega_n$

T_{max} : $(1/2)m(\dot{x}_{max})^2 = (1/2)m(A\omega_n)^2$

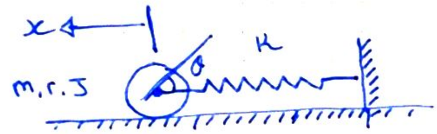
U_{max} : $(1/2)k(x_{max})^2 = (1/2)kA^2$

Since $T_{max} = U_{max}$;

$(1/2)m(A\omega_n)^2 = (1/2)kA^2$

$\omega_n = \sqrt{k/m}$

$\boxed{\text{Example}}$



Assume it is a conservative system and rolls without slipping
Find the natural frequency of the disk.

Solution : rolling w/o slipping

$x = r\theta$

$(x = A\sin(\omega_n t + \phi))$

$\dot{\theta} = \dot{x}/r$

The kinetic energy

$T = (1/2)J\dot{\theta}^2 + (1/2)m\dot{x}^2$

$= (1/2)J(\dot{x}/r)^2 + (1/2)m\dot{x}^2$

$= (1/2)\underbrace{[(J/r^2) + m]}_{\text{equivalent mass}}\dot{x}^2$

$\therefore T_{max} = (1/2)[(J/r^2) + m](A\omega_n)$

Potential energy

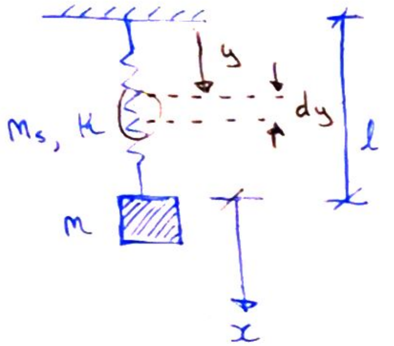
$U = (1/2) Kx^2$

$U_{max} = (1/2) KA^2$

$\Rightarrow (1/2)\left[(J/r^2) + m\right](A\omega_n)^2 = (1/2) KA^2$

$$\Rightarrow \omega_n = \sqrt{\frac{K}{(J/r^2) + m}}$$

$\boxed{\text{Example}}$ The effect of including the mass of the spring on the value of the frequency



Solution : The mass per unit length of the spring

m_s/ℓ

Assumptions
$$\begin{cases} \text{The mass element } dy: \\ \dfrac{m_s}{\ell}\, dy \\ \hookrightarrow \text{the velocity :} \\ \dfrac{y}{\ell}\,\dot{x} \end{cases}$$

$$T_s = \int_0^\ell (1/2)\left[(m_s/\ell)dy\right]\left[(y/\ell)\,\dot{x}\right]^2$$

$$= \left(\frac{1}{2}\right)\frac{m_s}{\ell}\left(\frac{\dot{x}}{\ell}\right)^2 \int_0^\ell y^2\, dy$$

$$= (1/2)(m_s/3)\,\dot{x}^2$$

The total kinetic energy :

$T = (1/2)(m_s/3)\,\dot{x}^2 + (1/2)m\dot{x}^2$

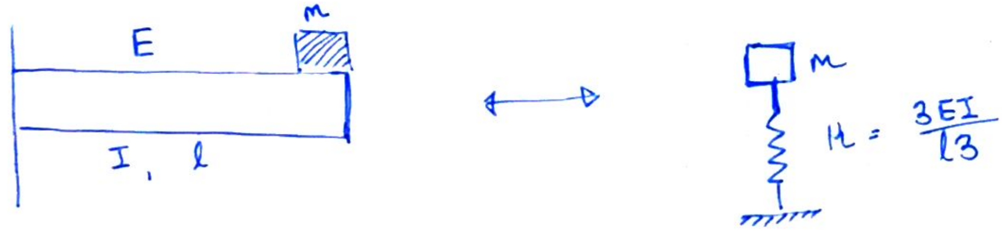
$T = (1/2)\left[(m_s/3) + m\right]\dot{x}^2$

$\longrightarrow T_{max} = (1/2)(m_s/3 + m)(A\omega_n)^2$

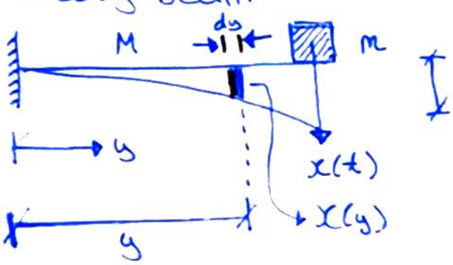
$U_{max} = (1/2) KA^2$

$\longrightarrow T_{max} = U_{max}$

$$\omega_n = \sqrt{\frac{K}{\left(\frac{m_s}{3} + m\right)}}$$



"Heavy beam"



The deflection at position y is:

$$x(y) = \frac{Py^2}{6EI}(3l-y)$$

The maximum deflection occurs at $y = l$

$$x_{max} = \frac{Pl^3}{3EI} \quad ; \quad P = \left(\frac{3EI}{l^3}\right) x_{max}$$

$$\rightarrow x(y) = \frac{3EI}{l^3} x_{max} \cdot \frac{y^2}{6EI}(3l-y)$$

$$= \frac{y^2}{2l^3}(3l-y) \cdot x_{max}$$

$$\dot{x}(y) = \frac{y^2(3l-y)}{2l^3} \dot{x}_{max}$$

For a small beam segment dy,

$$T_{beam} = \int_0^l \left(\frac{1}{2}\right)\left(\frac{M}{l}\,dy\right)\left(\dot{x}(y)\right)^2$$

$$\Rightarrow \int_0^l \left(\frac{1}{2}\right)\left(\frac{M}{l}\,dy\right)\left(\frac{y^2(3l-y)}{2l^3}\dot{x}_{max}\right)^2$$

$$= \left(\frac{1}{2}\right)\left(\frac{33}{140}M\right)\dot{x}^2_{max}$$

The total kinetic energy:

$$\rightarrow T = (1/2)\left[(33/140)M + m\right]\dot{x}^2_{max}$$

The equivalent mass of the system is:

$$\rightarrow m_{eq} = (33/140)M + m$$

$$\therefore \omega_n = \sqrt{\frac{\frac{3EI}{l^3}}{\frac{33}{140}M + m}}$$

Sept. 17/19

Legrange's Method for deriving equations of motion.

For a conservative system:

Kinetic energy $T = (1/2) m\dot{x}^2$

Potential energy $V = (1/2) k x^2$

Define the Legrangian L:

$$L = T - V = (1/2) m\dot{x}^2 - (1/2) k x^2$$

velocity — displacement

$L = L(x, \dot{x}, t)$ $(\theta, \dot{\theta}, t)$

The equation of motion:

$$\boxed{\frac{d}{dx}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0}$$

Since $L = T - V$:

$$\frac{\partial L}{\partial \dot{x}} = \frac{\partial T}{\partial \dot{x}} \quad \left(\text{since } \frac{\partial V}{\partial \dot{x}} = 0\right)$$

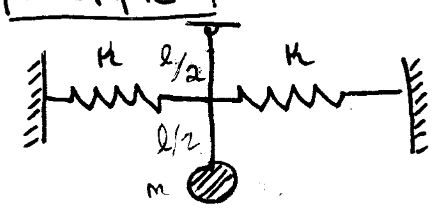
$$\frac{\partial L}{\partial x} = \frac{\partial T}{\partial x} - \frac{\partial V}{\partial x}$$

$$\Rightarrow \boxed{\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}}\right) - \frac{\partial T}{\partial x} + \frac{\partial V}{\partial x} = 0}$$

Let q be the generalized coordinate,

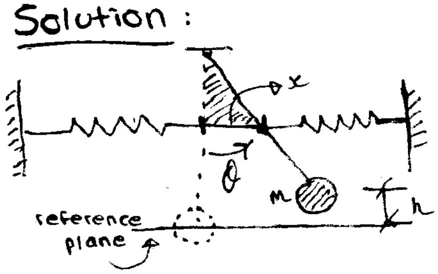
$$\Rightarrow \boxed{\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}}\right) - \frac{\partial T}{\partial q} + \frac{\partial V}{\partial q} = 0}$$

Example



Derive the equation of motion.

Solution:



θ: the generalized coordinate

Kinetic Energy:

$$T = (1/2) J\omega^2 = (1/2)(ml^2)\dot{\theta}^2 \quad \left(\frac{1}{2} m (l\dot{\theta})^2\right)$$

Potential Energy:

$$V = (1/2)Kx^2 + (1/2)K(-x)^2 + mgh$$

(should be expressed in terms of θ)

where $x = (l/2)\sin\theta$

$h = l - l\cos\theta = l(1-\cos\theta)$

$$\therefore \quad V = (1/2)K[(l/2)\sin\theta]^2 + (1/2)K[(l/2)\sin\theta]^2 + mgl(1-\cos\theta)$$

$$V = (1/4)Kl^2\sin^2\theta + mgl(1-\cos\theta)$$

$$\frac{\partial T}{\partial\dot\theta} = \left(\frac{1}{2}\right)ml^2 \cdot 2\dot\theta = ml^2\dot\theta$$

$$\frac{\partial T}{\partial\theta} = 0$$

$$\frac{\partial V}{\partial\theta} = \left(\frac{1}{4}\right)Kl^2 \cdot 2\sin\theta\cos\theta + mgl(0-(-\sin\theta))$$

$$\frac{\partial V}{\partial\theta} = \left(\frac{1}{2}\right)Kl^2\sin\theta\cos\theta + mgl\sin\theta$$

$$\Rightarrow \quad \frac{d}{dt}\left(ml^2\dot\theta\right) - 0 + \left(\frac{1}{2}\right)Kl^2\sin\theta\cos\theta + mgl\sin\theta = 0$$

$$ml^2\ddot\theta + (1/2)Kl^2\sin\theta\cos\theta + mgl\sin\theta = 0$$

Linearization: θ small, then $\sin\theta \approx \theta$, $\cos\theta \approx 1$

$$\Rightarrow \quad ml^2\ddot\theta + (1/2)Kl^2\theta + mgl\theta = 0$$

$$ml^2\ddot\theta + [(1/2)Kl^2 + mgl]\theta = 0$$

$$\Rightarrow \quad \ddot\theta + \left[\frac{(1/2)Kl^2 + mgl}{ml^2}\right]\theta = 0$$

The natural frequency:

$$\omega_n = \sqrt{\frac{(1/2)Kl^2 + mgl}{ml^2}}$$

$$\omega_n = \sqrt{\frac{Kl + 2mg}{2ml}}$$

Taylor series exp.

$$\left\{\begin{array}{l} \sin\theta = \theta - \dfrac{\theta^3}{3!} + \ldots = \theta \\ \cos\theta = 1 - \dfrac{\theta^2}{2!} + \dfrac{\theta^4}{4!} + \ldots = 1 - \dfrac{\theta^2}{2} \end{array}\right.$$

→ to linearize earlier ...

$$\Rightarrow \quad V = (1/4)Kl^2\theta^2 + mgl(\theta^2/2)$$

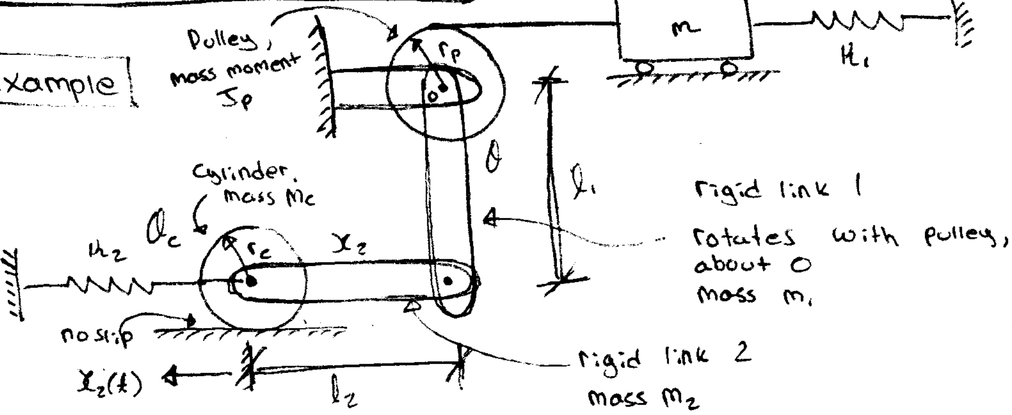
cont'd From prev...

$$\Rightarrow \quad V = (1/4)kl^2\theta^2 + mgl(\theta^2/2)$$

$$V = \left[(1/4)kl^2 + (1/2)mgl\right]\theta^2$$

$$V = (1/2)\left[(1/2)kl^2 + mgl\right]\theta^2$$

Example



Kinetic Energy :

$$T = (1/2)m\dot{x}^2 + (1/2)J_p\dot{\theta}^2 + (1/2)\left[(1/3)m_1 l_1^2\right]\dot{\theta}^2 + (1/2)m_2\dot{x}_2^2 \cdots$$

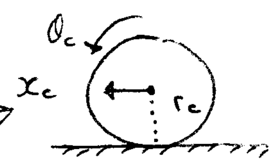
$$\cdots + (1/2)m_c\dot{x}_2^2 + (1/2)\left[(1/2)m_c r_c^2\right]\dot{\theta}_c^2$$

Pulley : $\quad x = r_p\theta \rightarrow \theta = x/r_p$

Bars : $\quad x_2 = l_1\theta \rightarrow x_2 = \dfrac{l_1 x}{r_p} = \dfrac{l_1}{r_p}x$

Cylinders :

→ From kinematics



$$\begin{cases} x_c = r_c\theta_c \\ x_c = x_2 \end{cases}$$

$$\theta_c = \frac{x_2}{r_c} = \frac{l_1}{r_c r_p}x$$

$$\Rightarrow \dot{\theta} = \frac{\dot{x}}{r_p} \; ; \quad \dot{x}_2 = \frac{l_1}{r_p}\dot{x} \; ; \quad \dot{\theta}_c = \frac{l_1}{r_c r_p}\dot{x}$$

$$\therefore \quad T = (1/2)m\dot{x}^2 + (1/2)J_p(\dot{x}/J_p)^2 + (1/2)(1/3\, m_1 l_1^2)(\dot{x}/r_p)^2 \cdots$$

$$\cdots + (1/2)m_2\left[(l_1/r_p)\dot{x}\right]^2 + (1/2)m_c\left[(l_1/r_p)\dot{x}\right]^2 \cdots$$

$$\cdots + (1/2)(1/2\, m_c r_c^2)\left(l_1/r_c r_p\, \dot{x}\right)$$

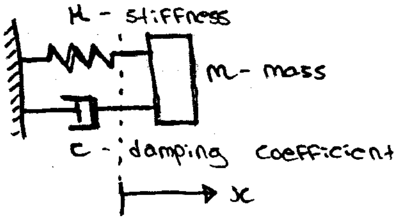
$$= (1/2)m_{eq}\dot{x}^2$$

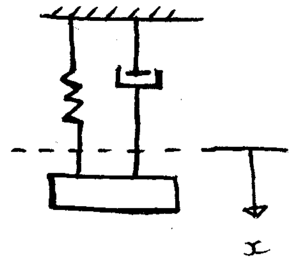
Here

$$m_{eq} = m + J_p/r_p^2 + 1/3\, m_1\, l_1^2/r_p^2 + m_2\, l_1^2/r_p^2 + m_c(l_1/r_p)^2 + (1/2)m_c(l_1/r_p)^2$$

$$= m + J_p/r_p^2 + (1/3)m_1(l_1^2/r_p^2) + m_2(l_1^2/r_p^2) + (3/2)m_c(l_1^2/r_p^2)$$

Free response with viscous damping





damping Force

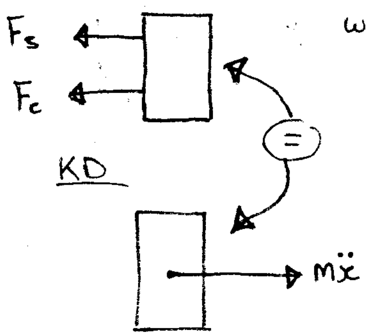
$$\underset{N}{f_c} = -c\underset{m/s}{v} = -c\dot{x}$$

unit of the damping constants $[c]$

$$[c] = \frac{[f]}{[m]} = \frac{N}{m/s} = \frac{N\cdot s}{m}$$

$$[c] = \frac{kg\cdot m}{s^2}\cdot\frac{s}{m} = kg/s$$

FBD



where $F_s = kx$
$F_c = c\dot{x}$

$\Rightarrow \quad -kx - c\dot{x} = m\ddot{x}$

$\Rightarrow \quad \boxed{m\ddot{x} + c\dot{x} + kx = 0}$

Equation of Motion (ODE)

Assume $x = \underset{\text{const.}}{a}e^{\overset{\text{const.}}{\lambda} t}$

Then $\dot{x} = a\lambda e^{\lambda t} = \lambda x$ (*)

$\ddot{x} = \lambda\dot{x} = \lambda^2 x$ (*)

$\Rightarrow \quad m\lambda^2 \underline{x} + c\lambda \underline{x} + k\underline{x} = 0$

$\Rightarrow \quad \boxed{(m\lambda^2 + c\lambda + k)x = 0}$

Since $x \neq 0$

$$\boxed{m\lambda^2 + c\lambda + k = 0}$$

$$\lambda^2 + \frac{c}{m}\lambda + \frac{k}{m} = 0$$

The roots:

$$\lambda_{1,2} = \frac{1}{2}\left(-\frac{c}{m} \pm \sqrt{\left(\frac{c}{m}\right)^2 - \frac{4k}{m}}\right)$$

Define the critical damping constant C_{CR} :

$$\left(\frac{C_{CR}}{m}\right)^2 - \frac{4K}{m} = \emptyset$$

$$\Rightarrow \quad C_{CR} = 2\sqrt{Km}$$

Define the damping ratio:

$$\boxed{\zeta = \frac{c}{C_{CR}}}$$

$$c = \zeta \cdot C_{CR} = \zeta \cdot 2\sqrt{Km}$$

$$\Rightarrow \lambda^2 + \frac{\zeta \cdot 2\sqrt{Km}}{m}\lambda + \frac{K}{m} = \emptyset$$

$$\lambda^2 + 2\zeta\sqrt{K/m}\,\lambda + K/m = \emptyset$$

$\omega_n = \sqrt{K/m}$ ← natural frequency

$$\Rightarrow \boxed{\lambda^2 + 2\zeta\omega_n\lambda + \omega_n^2 = \emptyset}$$

$$\Rightarrow \lambda_{1,2} = -\zeta\omega_n \pm \sqrt{\zeta^2 - 1}\,\omega_n$$

Case #1 Critically Damped Motion : $\zeta = 1$

$$\lambda_1 = \lambda_2 = -\zeta\omega_n$$

The solution of the system:

$$x = a_1 e^{-\omega_n t} + a_2 t e^{-\omega_n t} = (a_1 + a_2 t)e^{-\omega_n t}$$

The initial conditions:

$$\underbrace{x(\emptyset) = x_o}_{\text{given}} \quad ; \quad \underbrace{\dot{x}(\emptyset) = v_o}_{\text{given}}$$

Since $x(\emptyset) = (a_1 + a_2 t)e^{-\omega_n t}\Big|_{t=\emptyset} = a_1 = x_o$ (*)

$$\dot{x}(\emptyset) = a_2 e^{-\omega_n t} + (a_1 + a_2 t)(-\omega_n e^{-\omega_n t})$$

$$= (a_2 - [a_1 + a_2 t]\omega_n)e^{-\omega_n t}$$

$$\dot{x}(\emptyset) = a_2 - a_1\omega_n = v_o$$

(*) $\Rightarrow a_2 = v_o + x_o\omega_n$

$\therefore \; x = [x_o + (v_o + x_o\omega_n)t]e^{-\omega_n t}$ ← response of system w/ critical damping

$$t \to +\infty, \quad x \to \emptyset$$

Example $m = 100\ kg$
$k = 225\ N/m$
$\zeta = 1$

Find the disp of the system for different initial conditions.

1°: $x_0 = 0.4\,mm$; $v_0 = 1\ mm/s$
2°: $x_0 = 0.4\,mm$; $v_0 = 0$
3°: $x_0 = 0.4\,mm$; $v_0 = -1\ mm/s$

Solution: $\omega_n = \sqrt{k/m}$
$\omega_n = \sqrt{225/100} = 1.5\ rad/s$

$$\left.\begin{aligned} 1°:\ & x(t) = (0.4 + 1.6t)e^{-1.5t} \\ 2°:\ & x(t) = (0.4 + 0.6t)e^{-1.5t} \\ 3°:\ & x(t) = (0.4 - 0.4t)e^{-1.5t} \end{aligned}\right\}$$ (See picture for graph)

Case #2: Overdamped motion $(\zeta > 1)$

$$\lambda_{1,2} = -\zeta\omega_n \pm \sqrt{\zeta^2-1}\,\omega_n$$

The disp.:

$$x = a_1 e^{\lambda_1 t} + a_2 e^{\lambda_2 t}$$
$$= a_1 e^{-\zeta\omega_n t + \sqrt{\zeta^2-1}\,\omega_n} + a_2 e^{-\zeta\omega_n t - \sqrt{\zeta^2-1}\,\omega_n}$$

(*) $\longrightarrow$ $$x = e^{-\zeta\omega_n t}\left(a_1 e^{\sqrt{\zeta^2-1}\,\omega_n t} + a_2 e^{\sqrt{\zeta^2-1}\,\omega_n t}\right)$$

When $x(0) = x_0$
$\dot{x}(0) = v_0$

$$a_2 = \frac{-v_0 + (-\zeta + \sqrt{\zeta^2-1})\,\omega_n x_0}{2\omega_n\sqrt{\zeta^2-1}}$$

$$a_1 = \frac{v_0 + (\zeta + \sqrt{\zeta^2-1})\,\omega_n x_0}{2\omega_n\sqrt{\zeta^2-1}}$$

Case #3: Underdamped motion $(\zeta < 1)$ $\rightarrow (0 < \zeta < 1)$

$\zeta^2 - 1 < 0$

$$\lambda_{1,2} = -\zeta\omega_n \pm \sqrt{\zeta^2-1}\,\omega_n$$
$$= -\zeta\omega_n \pm j\sqrt{1-\zeta^2}\,\omega_n \qquad (j = \sqrt{-1})$$

$\lambda_2 = \bar{\lambda}_1^{*}$ (conjugate)

$$x = a_1 e^{\lambda_1 t} + a_2 e^{\lambda_2 t}$$
$$= a_1 e^{-\zeta\omega_n t + j\sqrt{1-\zeta^2}\,\omega_n t} + a_2 e^{-\zeta\omega_n t - j\sqrt{1-\zeta^2}\,\omega_n t}$$

$$= e^{-\zeta\omega_n t}\left(a_1 e^{j\sqrt{1-\zeta^2}\,\omega_n t} + a_2 e^{-j\sqrt{1-\zeta^2}\,\omega_n t}\right)$$

where $e^{j\alpha} = \cos\alpha + j\sin\alpha$

Define ω_d :

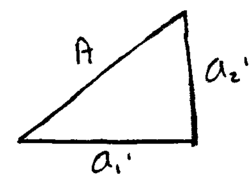
$$\boxed{\omega_d = \sqrt{1-\zeta^2}\,\omega_n}$$

damped natural freq.

$$x = e^{-\zeta\omega_n t}\left(a_1 e^{j\omega_d t} + a_2 e^{-j\omega_d t}\right)$$

$$= e^{-\zeta\omega_n t}\left(a_1\cos(\omega_d t) + j a_1 \sin(\omega_d t) + a_2\cos(\omega_d t) + j a_2 \sin(-\omega_d t)\right)$$

$$= e^{-\zeta\omega_n t}\left(\underbrace{(a_1+a_2)}_{a_1'}\cos\omega_d t + j\underbrace{(a_1-a_2)}_{a_2'}\sin\omega_d t\right)$$



$$\boxed{x = e^{-\zeta\omega_n t}\left(a_1'\cos(\omega_d t) + a_2'\sin(\omega_d t)\right)}$$

$$\boxed{x = Ae^{-\zeta\omega_n t}\sin(\omega_d t + \phi)}$$

Phase angle

$x(0) = x_0$; $\dot{x}(0) = v_0$

Since :

$$x(0) = A\sin(\phi) = x_0$$

$$\dot{x}(0) = (-\zeta\omega_n A e^{-\zeta\omega_n t})\sin(\omega_d t+\phi) + Ae^{-\zeta\omega_n t}\cos(\omega_d t+\phi)(\omega_d)\Big|_{t=0}$$

$$= -\zeta\omega_n A\sin(\phi) + A\omega_d\cos(\phi) = v_0$$

$$= -\zeta\omega_n x_0 + A\omega_d\cos(\phi) = v_0$$

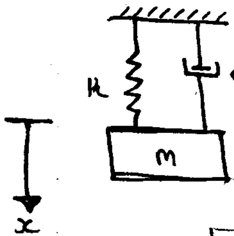
$$\Rightarrow A\cos(\phi) = \frac{v_0 + \zeta\omega_n x_0}{\omega_d}$$

$$\Rightarrow A\sin(\phi) = x_0$$

$$\Rightarrow A = \sqrt{x_0^2 + \left(\frac{v_0 + \zeta\omega_n x_0}{\omega_d}\right)^2}$$

$$\phi = \tan^{-1}\left(\frac{x_0\omega_d}{v_0 + \zeta\omega_n x_n}\right)$$

Sept. 24/19



$m\ddot{x} + c\dot{x} + kx = 0$ (w/ no damping)

$\omega_n = \sqrt{\frac{k}{m}}$

damping ratio

$\zeta = c/c_r$; $c_{cr} = 2\sqrt{mk}$

$\Rightarrow \boxed{\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = 0}$ (w/ damping)

Initial conditions:

$x(0) = x_0$

$\dot{x}(0) = v_0$

$\zeta > 1$ overdamped

$\zeta = 1$ critical

$\zeta < 1$ underdamped

Example: $m = 49.2 \times 10^{-3}$ kg

$k = 857.8$ N/m

$c = 0.11$ kg/s

→ Determine the damping ratio..

Solution: $c_{cr} = 2\sqrt{mk}$

$= 2\sqrt{(49.2\times10^{-3})(857.8)} = 12.99$ kg/s

Damping ratio $= \zeta = c/c_{cr} = \frac{0.11}{12.99}$

$\zeta = 0.0085$ ($\ll 1$)

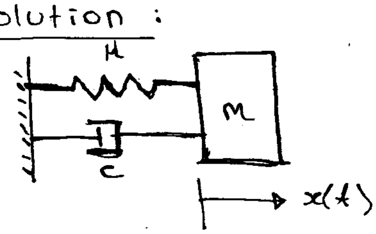
∴ underdamped.

Example: $\omega_n = 20$ Hz

$\zeta = 0.224$

→ Find the response of the tip if the initial velocity is $v_0 = 0.6$ m/s and initial displacement $x_0 = 0$. What is the maximum acceleration experienced by the leg? (Assuming no damping)

Solution:



cont'd :

Response :

$$x(t) = Ae^{-\zeta\omega_n t}\sin(\omega_d t + \phi)$$

Here,

damped natural Frequency

$$\omega_d = \sqrt{1-\zeta^2}\,\omega_n \quad (\text{always} < \omega_n)$$

$$A = \frac{1}{\omega_d}\sqrt{(v_0 + \zeta\omega_n x_0)^2 + (x_0\omega_d)^2}$$

$$\phi = \tan^{-1}\left(\frac{x_0\omega_d}{v_0 + \zeta\omega_n x_0}\right)$$

$$\omega_n = 20\ Hz = 20\ \tfrac{1}{s}$$

$$\omega_n = 20(2\pi)\ rad/s$$
$$= 40\pi\ rad/s$$

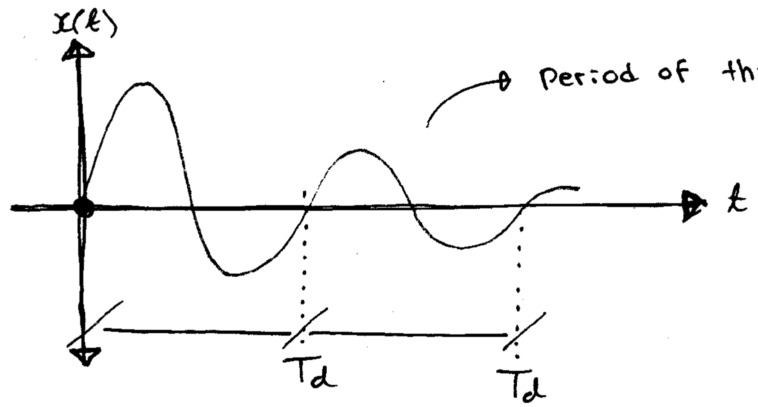
$$\omega_d = (1-\zeta^2)^{1/2}\omega_n = (1-0.224^2)^{1/2}(125.66)$$
$$= 122.467\ rad/s$$

$$\Rightarrow \begin{cases} A = 0.005 \\ \phi = 0 \end{cases}$$

both terms +ve, less than 90°

both -ve, greater than 180°

$$\Rightarrow x(t) = 0.005e^{-2\times 8.148\,t}\sin(122.46)t$$



Period of this system : $\boxed{T = 2\pi/\omega_d}$

Maximum acceleration (by assuming no damping)

$$a_{max} = A\omega_n^2$$

no damping :

$$x = A\sin(\omega_n t + \phi)$$
$$\dot{x} = A\omega_n\cos(\omega_n t)$$
$$\ddot{x} = -A\omega_n^2\sin(\omega_n t)$$

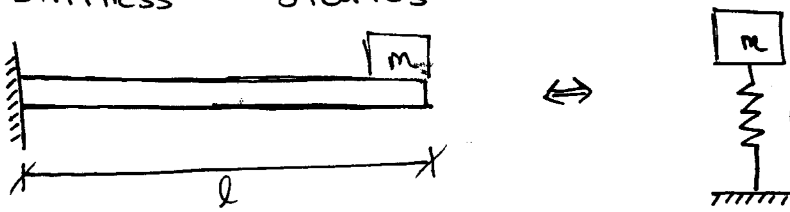
Then the max is just the coefficients (when sin/cos = 1)

$$a_{max} = 0.005(125.68)^2 = 75.396\ m/s^2$$

Measurement

Mass :

Stiffness : Statics



$$k = \frac{3EI}{l^3}$$

Measure the period T.

$T = 2\pi/\omega_n \qquad ; \qquad \omega_n = \sqrt{k/m}$

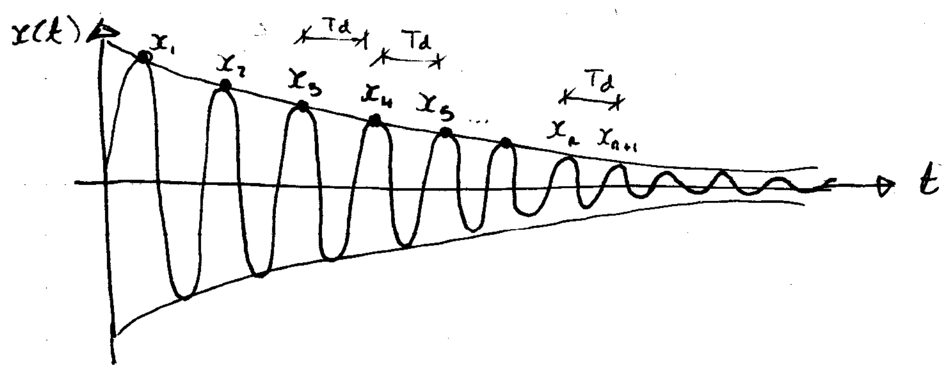
$\omega_n = 2\pi/T = \sqrt{k/m}$

$$\Rightarrow \frac{k}{m} = \frac{4\pi^2}{T^2}$$

$$\Rightarrow \frac{3EI}{l^3 m} = \frac{4\pi^2}{T^2}$$

$$\Rightarrow E = \left(\frac{4\pi^2}{T^2}\right)\left(\frac{m l^3}{3I}\right)$$

Damping (underdamped) : . . . . . $x(t) = Ae^{-\zeta\omega_n t}\sin(\omega_d t + \phi)$



At time $t + T_d$:

$x(t+T_d) = Ae^{-\zeta\omega_n(t+T_d)}\sin(\omega_d(t+T_d)+\phi)$

Ratio :

$$\frac{x(t)}{x(t+T_d)} = \frac{Ae^{-\zeta\omega_n t}\sin(\omega_d t+\phi)}{Ae^{-\zeta\omega_n(t-T_d)}\sin(\omega_d t+\omega_d T_d+\phi)}$$

$$= \left(\frac{1}{e^{-\zeta\omega_n T_d}}\right)\cdot\left(\frac{\cancel{\sin(\omega_d t+\phi)}}{\cancel{\sin(\omega_d t+\phi)}}\right) \qquad \cdots \quad \sin(2\pi + x) = \sin(x)$$

$$= e^{\zeta\omega_n T_d}$$

$\left(\begin{array}{l}\omega_d = 2\pi/T_d \\ T_d = 2\pi/\omega_d\end{array}\right)$

Define the logarithmic decrement:

$$\delta = \ln\left[\frac{x(t)}{x(t+T_d)}\right]$$

$$\Rightarrow \delta = \ln e^{\zeta\omega_n T_d}$$

$$\delta = \zeta\omega_n T_d$$

Since,

$$T_d = \frac{2\pi}{\omega_d} = \frac{2\pi}{\omega_n\sqrt{1-\zeta^2}}$$

$$\Rightarrow \boxed{\delta = 2\pi\left(\frac{\zeta}{\sqrt{1-\zeta^2}}\right)}$$

If $\zeta \ll 1$; $\sqrt{1-\zeta^2} \approx 1$

$$\delta = 2\pi\zeta$$

$$\boxed{\zeta = \frac{\delta}{2\pi}}$$

If ζ is <u>not</u> small

$$\boxed{\zeta = \frac{\delta}{\sqrt{4\pi^2+\delta^2}}}$$

(From diagram...)

$$\frac{x_1}{x_2} = e^{\zeta\omega_n T_d}$$

$$\frac{x_2}{x_3} = e^{\zeta\omega_n T_d} \quad \Rightarrow \quad \frac{x_1}{x_2}\cdot\frac{x_2}{x_3}\cdot\frac{x_3}{x_4}\cdots\frac{x_n}{x_{n+1}} = e^{(\zeta\omega_n T_d)^n}$$

$$\vdots$$

$$\frac{x_n}{x_{n+1}} = e^{\zeta\omega_n T_d}$$

$$\Rightarrow \frac{x_1}{x_{n+1}} = \left(e^{\zeta\omega_n T_d}\right)^n$$

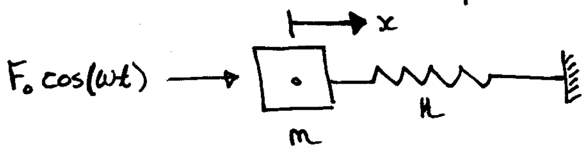
$$\Rightarrow \boxed{\ln\left(\frac{x_1}{x_{n+1}}\right) = n\zeta\omega_n T_d}$$

$$\delta = \ln(x_1/x_2) = \ln(x_2/x_3) = \cdots = \ln(x_n/x_{n+1})$$

$$\boxed{\delta = \left(\frac{1}{n}\right)\ln(x_1/x_{n+1})}$$

Chapter 2 : Response to Harmonic Excitation

(2.1) Undamped System



ω : driving frequency

F_o : magnitude of force

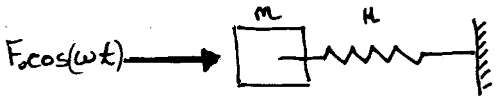
$$m\ddot{x} + kx = F_o \cos(\omega t)$$

$$\Rightarrow \ddot{x} + \omega_n^2 x = \frac{F_o}{m} \cos(\omega t) = f_o \cos(\omega t)$$

$$f_o = \frac{F_o}{m} \quad ; \quad \omega_n = \sqrt{\frac{k}{m}}$$

Ch. 2 - Response to Harmonic Motion Excitation

2.1 underdamped system



ω : driving force
F_o : magnitude

$$m\ddot{x} + kx = F_o\cos(\omega t)$$
$$\ddot{x} + (k/m)x = (F_o/m)\cos(\omega t)$$

where $f_o = \dfrac{F_o}{m}$; $\omega_n = \sqrt{\dfrac{k}{m}}$

Particular solution:

$$x_p(t) = X\cos(\omega t)$$

unknown const.

Since $\dot{x}_p(t) = -\omega X\sin(\omega t)$

$\ddot{x}_p(t) = -\omega^2 X\cos(\omega t)$

$$-\omega^2 X\cos(\omega t) + \omega_n^2 X\cos(\omega t) = f_o\cos(\omega t)$$
$$-\omega^2 X = \omega_n^2 X = f_o$$
$$\omega \neq \omega_n \longrightarrow X = \frac{f_o}{(\omega_n^2-\omega^2)}$$
$$x_p(t) = \frac{f_o}{(\omega_n^2-\omega^2)}\cos(\omega t) \quad (\text{where } \omega \neq \omega_n)$$

The general solution of the forced vibration:

$$x(t) = A_1\sin(\omega_n t) + A_2\cos(\omega_n t) + \frac{f_o}{(\omega_n^2-\omega^2)}\cos(\omega t)$$

Initial conditions:

$x(0) = x_o$; $\dot{x}(0) = v_o$

Since $x(0) = 0 + A_2 + \dfrac{f_o}{(\omega_n^2-\omega^2)} = x_o$

$$A_2 = x_o - [f_o/[\omega_n^2-\omega^2]]$$
$$\dot{x}(t) = \omega_n A_1\cos(\omega_n t) - \omega_n A_2\sin(\omega_n t) - [f_o/(\omega_n^2-\omega^2)]\sin(\omega t)$$
$$\dot{x}(0) = \omega_n A_1 = v_o$$
$$A_1 = v_o/\omega_n$$

$$\therefore x(t) = \left(\frac{v_0}{\omega_n}\right)\sin(\omega_n t) + \left(x_0 - \frac{f_0}{\omega_n^2-\omega^2}\right)\cos(\omega_n t) + \ldots$$

$$\ldots\left(\frac{f_0}{\omega_n^2-\omega^2}\right)\cos(\omega t)$$

Example:

$\omega_n = 1$ rad/s
$\omega = 2$ rad/s
$x_0 = 0.01$
$v_0 = 0.01$
$f_0 = 0.1$

$$x(t) = (0.01)\sin(t) + 0.0433\cos(t) + (-0.0333\cos(2t))$$

$\longrightarrow$ see text- becomes periodic, but no longer harmonic

Example:

$m = 10$ kg
$k = 1000$ N/m
$x_0 = 0$
$v_0 = 0.2$ m/s
$F = 23$ N

$\omega = 2\omega_n$
↑ excitation frequency

Find the response

Solution: $\omega_n = \sqrt{k/m}$; $\omega_n = 10$ rad/s

$\omega = 2\omega_n \Rightarrow \omega = 20$ rad/s

$f_0 = F/m = 23/10 = 2.3$

$$\therefore x(t) = \left(\frac{v_0}{\omega_n}\right)\sin(\omega_n t) + \left(x_0 - \frac{f_0}{\omega_n^2-\omega}\right)\cos(\omega_n t) + \left(\frac{f_0}{\omega_n^2-\omega^2}\right)\cos(\omega t)$$

$$= \left(\frac{0.2}{10}\right)\sin(10t) + \left(0 - \frac{2.3}{10^2-20^2}\right)\cos(10t) + \left(\frac{2.3}{10^2-20^2}\right)\cos(20t)$$

$$= 0.02\sin(10t) + (7.9667\times10^{-3})\cos(10t) - 7.9667\times10^{-3}\cos(20t)$$

$\longrightarrow$ When ω is near ω_n, what will happen?

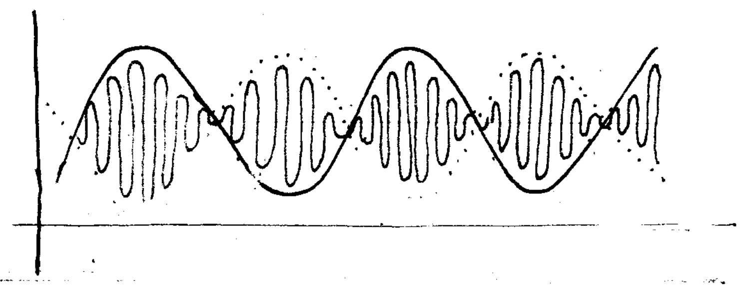
Consider: $f_0 = 1$, $\omega_n = 2\pi$ rad/s, $x_0 = v_0 = 0$

$$x(t) = \left(\frac{f_0}{\omega_n^2-\omega^2}\right)\left[\cos(\omega t) - \cos(\omega_n t)\right]$$

$$\left(\omega_{slow} = \frac{\omega_n - \omega}{2} \quad ; \quad \omega_{fast} = \frac{\omega_n + \omega}{2} \right)$$

when $\omega \to \omega_n$, becomes a beat (or a beating freq. occurs)

Beat: $\omega_{beat} = |\omega_n - \omega|$



(fix from textbook.)

What if $\omega = \omega_n$?

Particular solution

$x_p(t) = X \cdot t \cdot \sin \omega t$

$\Rightarrow \dot{x}_p(t) = X \cdot \sin \omega t + X \omega t \cos \omega t$

$\Rightarrow \ddot{x}_p(t) = X \omega \cos \omega t + X \omega \cos \omega t + (-X t \omega^2 \sin \omega t)$

$\therefore \ddot{x}_p + \omega_n^2 x_p = 2X\omega \cos \omega t$

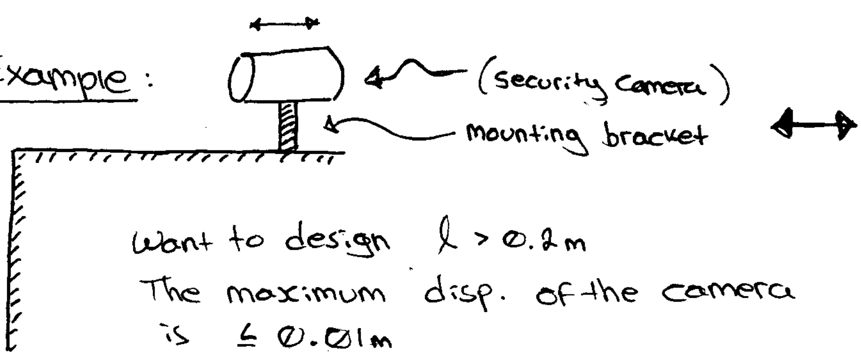
$\ddot{x}_p + \omega_n^2 x_p = \boxed{f_0 \cos \omega t}$

$\rightarrow 2X\omega = f_0 \quad ; \quad X = \dfrac{f_0}{2\omega}$

$\rightarrow x(t) = A_1 \sin(\omega t) + A_2 \cos(\omega t) + (f_0 / 2\omega)\, t \sin(\omega t)$

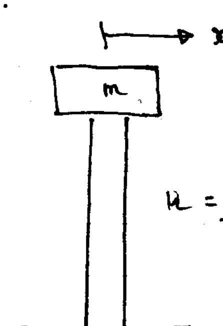
$\rightarrow$ the amplitude of the vibration grows without bounds, this is known as a <u>resonance</u> condition.

Example:



(security camera)

mounting bracket



$k = \dfrac{3EI}{l^3}$

Want to design $l > 0.2\,m$
The maximum disp. of the camera is $\leq 0.01\,m$

With load: $F = 15\,N$, $\omega = 10\,Hz$

Camera: $m = 3\,kg$

beam: $0.02 \times 0.02\,m$

Find the length.

(Refer to camera example)

Solution : The forced response of the undamped mass-spring is:

$$x(t) = \frac{v_0}{\omega_n}\sin(\omega_n t) + \left(x_0 - \frac{f_0}{\omega_n^2-\omega^2}\right)\cos(\omega_n t) + \frac{f_0}{\omega_n^2-\omega^2}\cos(\omega t)$$

For zero initial conditions

$v_0 = 0$, $x_0 = 0$

$$\boxed{x(t) = \frac{f_0}{\omega_n^2-\omega^2}\left(\cos(\omega t) - \cos(\omega_n t)\right)}$$

$$|x(t)| = \left|\frac{f_0}{\omega_n^2-\omega^2}\right| \cdot \left|\cos(\omega t) - \cos(\omega_n t)\right|$$

$(|a+b| \leq |a| + |b|)$

$$|x(t)| \leq \left|\frac{f_0}{\omega_n^2-\omega^2}\right| \left(|\cos(\omega t)| + |\cos(\omega_n t)|\right)$$

$$\leq \frac{2f_0}{|\omega_n^2-\omega^2|}$$

$\therefore$ maximum displacement is $\dfrac{2f_0}{|\omega_n^2-\omega^2|}$

$$\boxed{\therefore \frac{2f_0}{|\omega_n^2-\omega^2|} \leq 0.01}$$

Case 1: $\omega_n < \omega = 10\ \text{Hz} = 2\pi(10)\ \text{rad/s} = 62.832\ \text{rad/s}$

$$\frac{2f_0}{\omega^2-\omega_n^2} \leq 0.01$$

Since $f_0 = \dfrac{F_0}{m} = \dfrac{15\ \text{N}}{3\ \text{kg}} = 5\ \text{N/kg}$

$\rightarrow \omega^2 - \omega_n^2 \geq 2f_0/0.01$

$\rightarrow \omega_n^2 \leq \omega^2 - \dfrac{2f_0}{0.01}$

$$= (62.832)^2 - \frac{2(5)}{0.01}$$

$\omega_n^2 \leq 2947.86$

$\omega_n \leq 54.294$

Since $k = \dfrac{3EI}{l^3}$ (Cross-section of beam changed to 0.01×0.01) *

$$\omega_n^2 = \frac{k}{m}$$

$E = 71$ GPa,

$I = (1/12)(10 \times 10^{-8})\ m^4$

$\longrightarrow \omega_n^2 = k/m = \dfrac{3EI}{3l^3}$

$$\boxed{\omega_n^2 = \frac{59.1667}{l^3}}$$

$\Rightarrow \dfrac{59.1667}{l^3} \leq 2947.86$

$\Rightarrow \boxed{l \geq 0.272\ m}$

Case 2 $\quad \omega_n > \omega = 62.832$

$\dfrac{2f_0}{\omega_n^2 - \omega^2} \leq 0.01$

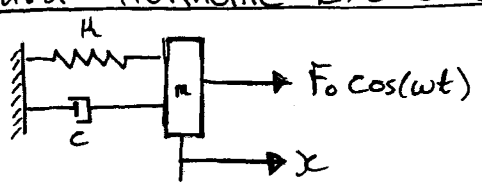
$\Rightarrow \omega_n^2 \geq \omega^2 + \dfrac{2f_0}{0.01} = 4947.86$

$\Rightarrow \dfrac{59.1667}{l^3} \geq 4947.86$

$\Rightarrow \boxed{l \leq 0.229\ m}$

Choose $l = 0.22m$ (requirement of $l > 0.2m$)

2.2 Harmonic Excitation of Damped Systems



Equation of motion:

$m\ddot{x} = F_0 \cos(\omega t) - kx - c\dot{x}$

$\longrightarrow m\ddot{x} + c\dot{x} + kx = F_0 \cos(\omega t)$

$(k/m) = \omega_n^2 \Rightarrow k = m\omega_n^2$

$\longrightarrow c/c_{cr} = \zeta \quad ; \quad c_{cr} = 2\sqrt{mk} = 2m\omega_n$

$c = 2\zeta m\omega_n$

$\Rightarrow m\ddot{x} + 2\zeta m\omega_n\dot{x} + m\omega_n^2 x = F_0 \cos(\omega t)$

$\Rightarrow \boxed{\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = f_0 \cos(\omega t)}$

$f_0 = F_0/m$

The general solution of the homogeneous eq. is the free vibration of the damped system.

$$x_H(t) = Ae^{-\zeta\omega_n t}\sin(\omega_d + \phi)$$
$$\omega_d = \omega_n\sqrt{1-\zeta^2}$$

The particular solution

$$x_p(t) = A_s\cos(\omega t) + B_s\sin(\omega t)$$
$$\dot{x}_p(t) = -\omega A_s\sin(\omega t) + \omega B_s\cos(\omega t)$$
$$\ddot{x}_p(t) = -\omega^2 A_s\cos(\omega t) - \omega^2 B_s\sin(\omega t)$$
$$\left(-\omega^2(A_s\cos(\omega t) + B_s\sin(\omega t)) = -\omega^2 x_p\right)$$

Sub into the eq. of motion:

$$-\omega^2(A_s\cos\omega t + B_s\sin\omega t) + 2\zeta\omega_n(-\omega A_s\sin(\omega t) + \omega B_s\cos(\omega t)) + \cdots$$
$$\cdots\ \omega_n^2(A_s\cos\omega t + B_s\sin\omega t) = f_0\cos\omega t$$

$$(-\omega^2 A_s + 2\zeta\omega_n\omega B_s + A_s\omega_n^2)\cos(\omega t) + (-\omega^2 B_s - 2\zeta\omega_n\omega A_s + B_s\omega_n^2)\sin(\omega t)$$
$$= f_0\cos(\omega t)$$

$$\Rightarrow \left.\begin{aligned}(\omega_n^2-\omega^2)A_s + 2\zeta\omega_n\omega B_s &= f_0\\ -2\zeta\omega_n\omega A_s + (\omega_n^2-\omega^2)B_s &= 0\end{aligned}\right\}$$

$$\Rightarrow A_s = \frac{\omega_n^2-\omega^2}{(\omega_n^2-\omega^2)^2 + (2\zeta\omega_n\omega)^2} f_0$$
$$B_s = \frac{2\zeta\omega_n\omega}{(\omega_n^2-\omega^2)^2 + (2\zeta\omega_n\omega)^2} f_0$$

$$\therefore x_p(t) = A_s\cos(\omega t) + B_s\sin(\omega t)$$
$$= X\cos(\omega t - \theta)$$
$$= X\cos(\omega t)\cos\theta + X\sin(\omega t)\sin\theta$$

$$A_s = X\cos\theta\ ;\quad B_s = X\sin\theta$$
$$\Rightarrow X = \sqrt{A_s^2 + B_s^2},\quad \tan\theta = \frac{B_s}{A_s}$$

Here,

$$\left.\begin{aligned}X &= \frac{f_0}{\sqrt{(\omega_n^2-\omega^2)^2 + (2\zeta\omega_n\omega)^2}}\\ \theta &= \tan^{-1}\left[\frac{2\zeta\omega_n\omega}{\omega_n^2-\omega^2}\right]\end{aligned}\right\}$$

$\therefore$ the response :

$$\boxed{\begin{aligned}x(t) &= x_h(t) + x_p(t)\\ &= \underbrace{Ae^{-\zeta\omega_n t}\sin(\omega_d t + \phi)}_{\text{free vibration only}} + \underbrace{X\cos(\omega t - \phi)}_{\text{forced vibration added}}\end{aligned}}$$

Example: Find the response of the system.

$\omega_n = 10$ rad/s

$\omega = 5$ rad/s

$\zeta = 0.01$

$F_o = 1000$ N

$m = 100$ kg

$x_o = 0.05$ m

$v_o = 0$

Solution: $f_o = \dfrac{F_o}{m} = \dfrac{1000}{100} = 10$ N/kg

$$X = \frac{f_o}{\sqrt{(\omega_n^2-\omega^2)^2+(2\zeta\omega_n\omega)^2}} \quad 0.13332$$

$$\theta = \tan^{-1}\left[\frac{2\zeta\omega_n\omega}{\omega_n^2-\omega^2}\right] = 0.013333 \text{ rad}$$

$$\omega_d = \omega_n\sqrt{1-\zeta^2} = 9.9995 \text{ rad/s}$$

$$\therefore\ x(t) = Ae^{-(0.01)(10)t}\sin(9.9995t+\phi) + 0.13332\cos(5t-0.0133333)$$

Velocity

$$\dot{x} = -0.1\,Ae^{-0.1t}\sin(9.9995t+\phi) + (9.9995)Ae^{-0.1t}\cos(9.9995t+\phi) - \ldots$$
$$\ldots\ (0.13332)(5)\sin(5t-0.0133333)$$

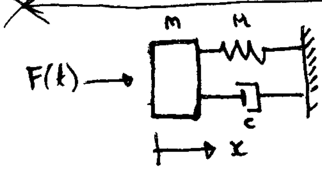
At $t=0$:

$$x(0) = A\sin\phi + 0.13332\cos(-0.013333) = 0.05$$
$$v(0) = -(0.1)A\sin\phi + (9.9995)A\cos\phi - (0.13332)(5)\sin(-0.013333) = 0$$

$$\Rightarrow \left.\begin{aligned} A &= 0.083327 \\ \phi &= 1.5501 \text{ (rad)} \end{aligned}\right\}$$

$$\therefore\ x(t) = 0.083327\,e^{-0.1t}\sin(9.9995t+1.5501) + \ldots$$
$$\ldots\ (0.13332)\cos(5t-0.013333) \qquad (\text{in m})$$

→ Midterm location to be emailed
Sections: 1.1 → 1.6
3 questions
Formula Sheet provided



$$m\ddot{x} + c\dot{x} + kx = F_0\cos(\omega t)$$

$$x = x_h(t) + x_p(t)$$
$$x_h(t) = Ae^{-\zeta\omega_n t}\sin(\omega_d t + \phi) \quad \leftarrow \text{transient response}$$
$$x_p(t) = X\cos(\omega t - \theta) \quad \leftarrow \text{steady-state response}$$

when $t \to \infty$, $x_h(t) \to 0$

$$X = \frac{f_0}{\sqrt{(\omega_n^2-\omega^2)^2 + (2\zeta\omega_n\omega)^2}}$$
$$\theta = \arctan\left(\frac{2\zeta\omega_n\omega}{\omega_n^2-\omega^2}\right)$$

$$f_0 = \frac{F}{m}$$

Define $r = \frac{\omega}{\omega_n}$ ← frequency ratio, $\omega = r\omega_n$

$$\therefore X = \frac{f_0}{\sqrt{(\omega_n^2-\omega^2)^2 + (2\zeta\omega_n\omega)^2}}$$
$$= \frac{f_0}{\omega_n^2}\frac{1}{\sqrt{(1-r^2)^2+(2\zeta r)^2}}$$

$$\frac{f_0}{\omega_n^2} = \frac{F_0/m}{k/m} = \frac{F_0}{k} = \delta_{st.}$$

$$\Rightarrow \boxed{\frac{X}{\delta_{st.}} = \frac{1}{\sqrt{(1-r^2)^2+(2\zeta r)^2}}}$$

amplitude factor (ratio between dynamic response and static response)

$$\boxed{\theta = \arctan\left(\frac{2\zeta\omega_n\omega}{\omega_n^2-\omega^2}\right) = \tan^{-1}\left(\frac{2\zeta r}{1-r^2}\right)}$$

Amplitude Factor :

(amplitude)

→ 1^{ST} $\zeta = 0$, $r = 1$ (resonance)

→ 2^{ND} Any amount of damping reduces the magnification Factor For all the Forcing Frequency

→ 3^{RD} Resonance Frequency : $\omega = \omega_n$

→ 4^{TH} $\frac{d}{dr}\left(\frac{X}{\delta_{st}}\right) = 0$

$0 < \zeta < (1/\sqrt{2}) \Rightarrow 0.707$

at $r = \sqrt{1-2\zeta^2}$, $X \longrightarrow max$

$\zeta > 1/\sqrt{2}$ $(= 0.707)$; X_{max} occurs at $r = 0$

→ 5^{TH} $0 < \zeta < 1/\sqrt{2}$

at $r = \sqrt{1-2\zeta^2}$; $\left(\frac{X}{\delta_{st}}\right)_{max} = \frac{1}{2\zeta\sqrt{1-\zeta^2}}$

at $r = 1$; $\frac{X}{\delta_{st}} = \frac{1}{2\zeta}$

Phase $\theta = \tan^{-1}\left[\frac{2\zeta r}{1-r^2}\right]$

Section 2.3

$m\ddot{x} + c\dot{x} + kx = F_0\cos(\omega t)$

$e^{i\omega t} = \underbrace{\cos(\omega t)}_{\text{real part}} + \underbrace{i\sin(\omega t)}_{\text{imaginary part}}$

→ $m\ddot{x} + c\dot{x} + kx = F_0 e^{i\omega t}$

Assume: $x(t) = Xe^{i\omega t}$

$\dot{x} = i\omega X e^{i\omega t}$

$\ddot{x} = (i\omega)^2 X e^{i\omega t} = -\omega^2 X e^{i\omega t}$

$\Rightarrow (-m\omega^2 + i\omega c + k) X e^{i\omega t} = F_0 e^{i\omega t}$

$\Rightarrow X = \frac{F_0}{-m\omega^2 + i\omega c + k}$

Define: $\frac{X}{F_0} = H(\omega) = \frac{1}{k - m\omega^2 + i\omega c}$

$H(i\omega)$: Frequency response Function.

$$m\ddot{x} + c\dot{x} + kx = F_o \cos(\omega t)$$

Laplace Transform

$$X(s) = \int_0^\infty x(t)\, e^{-st}\, dt$$

$$\Rightarrow \int_0^\infty (m\ddot{x} + c\dot{x} + kx)\, e^{-st}\, dt$$

$$\Rightarrow \int_0^\infty F_o \cos(\omega t) e^{-st}\, dt$$

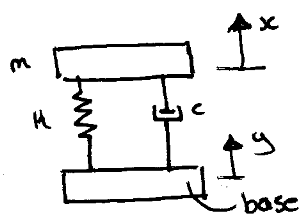
$$\Rightarrow (ms^2 + cs + k)\, X(s) = \boxed{\frac{F_o \cdot s}{s^2 + \omega^2}}$$

$\Rightarrow$ Transfer Function

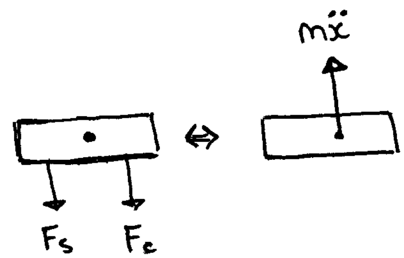
$$\boxed{H(s) = \frac{1}{ms^2 + cs + k}}$$

$\left(\begin{array}{l} s \to i\omega \\ \text{gives Freq. Function} \end{array} \right)$

2.4 Base Extraction



FBD



where
$$F_s = k(x-y)$$
$$F_c = c(\dot{x} - \dot{y})$$

Newton's 2nd Law:

$$m\ddot{x} = -k(x-y) - c(\dot{x} - \dot{y})$$

$$\Rightarrow m\ddot{x} + c\dot{x} + kx = ky + c\dot{y}$$

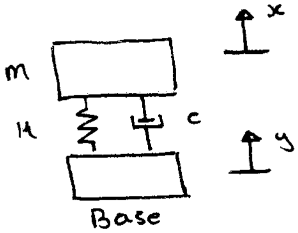
Assume: $y(t) = Y \sin(\omega t)$

$$m\ddot{x} + c\dot{x} + kx = kY\sin(\omega t) + cY\cos(\omega t)\omega$$

$$\Rightarrow \ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = (k/m)\, Y \sin(\omega t) + (c/m)\,\omega Y \cos(\omega t)$$

$$\begin{cases} f_{os} = (k/m)Y = \omega_n^2 Y \\ f_{oc} = (c/m)\omega Y = 2\zeta\omega_n\omega Y \end{cases}$$

Oct. 8/19



$$m\ddot{x} + c\dot{x} + kx = c\dot{y} + ky$$

→ Equation of motion

$$y(t) = Y\sin(\omega_b t)$$

$$m\ddot{x} + c\dot{x} + kx = c\omega_b Y\cos(\omega_b t) + kY\sin(\omega_b t)$$

$$\sqrt{k/m} = \omega_n, \quad \zeta = c/2\sqrt{mk}$$

$$\Rightarrow \ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = 2\zeta\omega_n\omega_b Y\cos(\omega_b t) + \omega_n^2 Y\sin(\omega_b t)$$

$$\Rightarrow \omega_n Y\left(2\zeta\omega_b\cos(\omega_b t) + \omega_n\sin(\omega_b t)\right)$$

$\Rightarrow$ Since

$$2\zeta\omega_b\cos(\omega_b t) + \omega_n\sin(\omega_b t)$$
$$= P_0\cos(\omega_b t - \theta_2)$$
$$= P_0\cos(\omega_b t)\cos(\theta_2) + P_0\sin(\omega_b t)\cdot\sin(\theta_2)$$

$$\Rightarrow \begin{cases} P_0\cos(\theta_2) = 2\zeta\omega_b \\ P_0\sin(\theta_2) = \omega_n \end{cases}$$

$$P_0 = \sqrt{2\zeta\omega_b^2 + \omega_n^2}$$

$$\theta_2 = \arctan\left(\frac{\omega_n}{2\zeta\omega_b}\right)$$

∴ Equation of motion:

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = \omega_n Y P_0\cos(\omega_b t - \theta_2)$$

The particular solution (Forced response)

$$x(t) = X\cos(\omega_b t - \theta_2 - \theta_1)$$

and:

$$X = \frac{f_0}{\sqrt{(\omega_n^2 - \omega_b^2)^2 + (2\zeta\omega_n\omega_b)^2}}$$

$$\theta_1 = \arctan\left(\frac{2\zeta\omega_n\omega_b}{\omega_n^2 - \omega_b^2}\right)$$

→ The magnitude of the response:

$$X = \frac{\omega_n Y P_0}{\sqrt{(\omega_n^2 - \omega_b^2)^2 + (2\zeta\omega_n\omega_b)^2}}$$

$$X = \frac{\omega_n Y\sqrt{(2\zeta\omega_b)^2 + \omega_n^2}}{\sqrt{(\omega_n^2 - \omega_b^2)^2 + (2\zeta\omega_n\omega_b)^2}}$$

$$= \omega_n Y\sqrt{\frac{\omega_n^2 + (2\zeta\omega_b)^2}{(\omega_n^2 - \omega_b^2)^2 + (2\zeta\omega_b\omega_n)^2}}$$

NOTE: ; Let $r = \omega_b / \omega_n$ → $\omega_b = r\omega_n$

Frequency ratio (r), excitation Frequency (ω_b), natural Frequency (ω_n)

$$\Rightarrow \boxed{\frac{X}{Y} = \sqrt{\frac{1 + (2\zeta r)^2}{(1 - r^2)^2 + (2\zeta r)^2}}}$$

~~transmission~~ transmissibility (or Displacement Ratio)

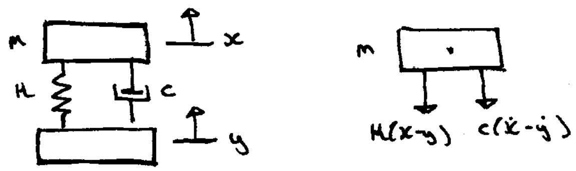
Resonance Freq. @ $r=1$ (not necessarily maximum)

To find max displacement ratio:

$$\frac{d}{dr}\left(\frac{X}{Y}\right) = 0$$

$$r = \frac{1}{2\zeta}\left[\sqrt{1+8\zeta^2} - 1\right]^{1/2}$$

Force Transmitted



$$F = k(x-y) + c(\dot{x}-\dot{y}) = -m\ddot{x}$$

$$x(t) = X\cos(\omega_b t - \theta_1 - \theta_2)$$

$$\ddot{x} = -\omega_b^2 X\cos(\omega_b t - \theta_1 - \theta_2)$$

$$\therefore\ F = m\omega_b^2 X\cos(\omega_b t - \theta_1 - \theta_2)$$

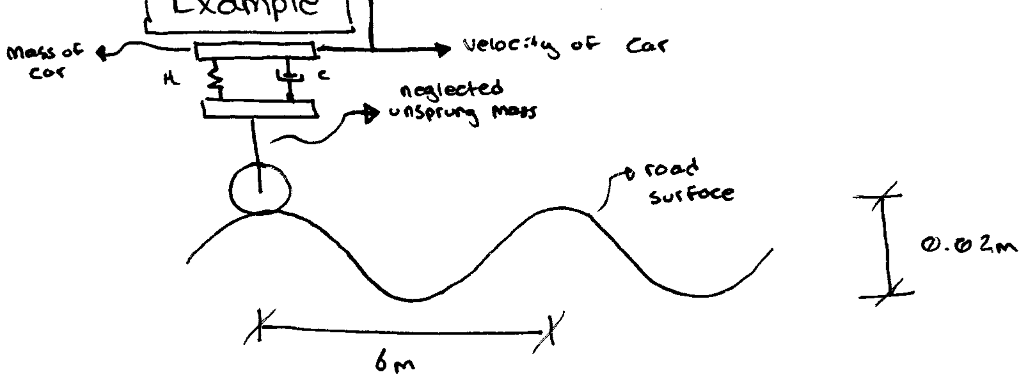
max transmitted Force

$$|F_T| = m\omega_b^2 X = mr^2\omega_n^2 X = r^2 kX$$

$$= r^2 kY\sqrt{\frac{1+(2\zeta r)^2}{(1-r^2)^2+(2\zeta r)^2}}$$

$$\Rightarrow \frac{|F_T|}{kY} = r^2\sqrt{\frac{1+(2\zeta r)^2}{(1-r^2)^2+(2\zeta r)^2}}$$

Example



$m = 1007$ kg

$k = 40000$ N/m

$c = 2000$ Ns/m

$v = 20$ km/h

The base excitation: $Y\sin(\omega_b t)$

$$Y = \frac{0.02}{2} = 0.01$$

Period: $T = \frac{d}{v} = \frac{6\text{m}}{20\text{ km/h}}$

Frequency of the base excitation:

$$\omega_b = \frac{2\pi}{T} = \frac{2\pi}{6m/v} = \frac{2\pi v}{6}$$

If v is km/h:

$$\omega_b = \frac{2\pi v}{6} \times \frac{1000}{3600} \longrightarrow \text{rad/s}$$

$$\omega_b = 0.2909\,v \quad \text{rad/s} \quad (v \text{ is km/h})$$

When $v = 20$ km/h:

$$\omega_b = 5.818 \text{ rad/s}$$

$$r = \frac{\omega_b}{\omega_n}$$

$$r = \frac{5.818}{6.303}$$

$$r = 0.9231$$

Since $\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{40000}{1007}} = 6.303$

$$\zeta = \frac{c}{2\sqrt{mk}} = \frac{2000}{2\sqrt{1007 \times 40000}} = 0.158 < 1$$

$$\therefore \frac{X}{Y} = \sqrt{\frac{1 + (2\zeta r)^2}{(1-r^2)^2 + (2\zeta r)^2}} = 3.19$$

$$X = 3.19\,Y \qquad (\text{Since } Y = 0.01\text{m})$$

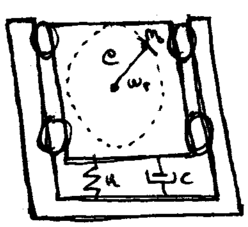
$\longrightarrow X = 0.0319$ m

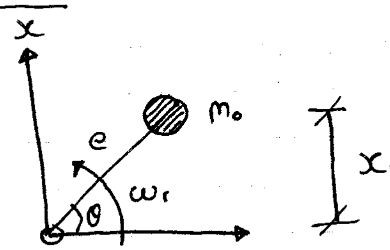
As $v \uparrow$, response $\downarrow$

Heavier objects feel less vibration (from displacement point of view)

$\hookrightarrow$ what about forces?

2.5 Rotating Unbalance





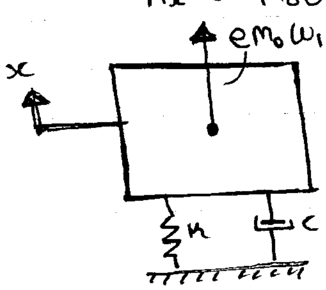
$$\theta = \omega_r t$$

$$x_r = e\sin(\omega_r t)$$

$$a_x = \ddot{x}_r = -e\omega_r^2 \sin(\omega_r t)$$

$\rightarrow$ The force along the x-axis:

$$R_x = m_0 a_x = -e m_0 \omega_r^2 \sin(\omega_r t)$$



$$m\ddot{x} + c\dot{x} + kx = F(t)$$

$$m\ddot{x} + c\dot{x} + kx = e m_0 \omega_r^2 \sin(\omega_r t)$$

$$\ddot{x} + 2\zeta\omega_n \dot{x} + \omega_n^2 x = \left(\frac{m_0}{m}\right) e\omega_r^2 \sin(\omega_r t)$$

The Forced response :

$$x(t) = X \sin(\omega_1 t - \theta)$$

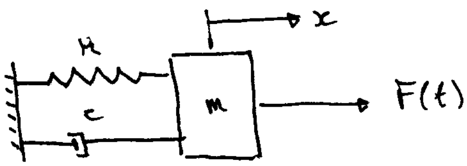
Here :

$$X = \left(\frac{m_o}{m}\right) e \cdot \frac{r^2}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}}$$

$\nearrow$ (increasing mass reduces response, but can unbalance system)

$$\theta = \arctan\left(\frac{2\zeta r}{1-r^2}\right)$$

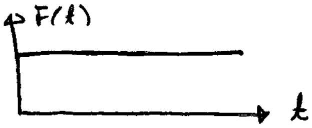
$$\left(\frac{m}{m_o}\right)\left(\frac{X}{e}\right) = \frac{r^2}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}}$$



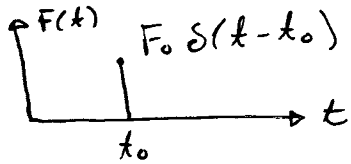
$$\begin{cases} m\ddot{x} + c\dot{x} + kx = F(t) \\ t = 0 : \; x(0) = x_0 , \quad \dot{x}(0) = v_0 \end{cases}$$

$F(t) = F_0 \cos(\omega t)$

(Constant Force)



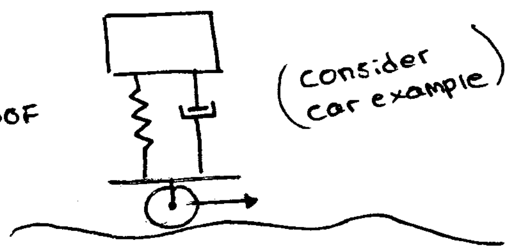
(Impulsive Force)



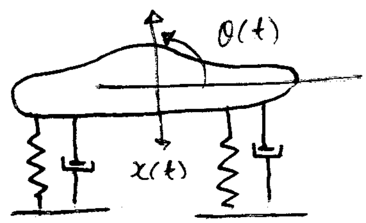
(Variable Force)

Chapter 4 : Multiple Degree of Freedom Systems

1 DOF



(consider car example)



2 DOF

DOF's :

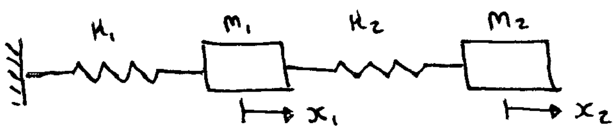
$1 \rightarrow 2 \rightarrow 3 \rightarrow \cdots \rightarrow n \rightarrow \cdots \rightarrow \infty$

Introduce concepts $(1 \rightarrow 2)$

Same concepts $(2 \rightarrow n)$

Introduce concepts $(n \rightarrow \infty)$

4.1 : 2 DOF Model



$$k_1 x_1 \longleftarrow \boxed{m_1} \longrightarrow k(x_2 - x_1)$$

$$(\xrightarrow{+ve}) \quad m_1 \ddot{x}_1 = -k_1 x_1 + k(x_2 - x_1)$$

$$\Rightarrow \quad m_1 \ddot{x}_1 + (k_1 + k_2) x_1 - k_2 x_2 = 0$$

$$k_2(x_2 - x_1) \longleftarrow \boxed{m_2}$$

$$(\xrightarrow{+ve}) \quad m_2 \ddot{x}_2 = -k_2(x_2 - x_1)$$

$$\Rightarrow \quad m_2 \ddot{x}_2 - k_2 x_1 + k_2 x_2 = 0$$

Initial conditions :

$$x_1(0) = \underbrace{x_{10}}, \qquad \dot{x}_1(0) = \underbrace{\dot{x}_{10}}$$

given numbers

$$x_2(0) = \overbrace{x_{20}}, \qquad \dot{x}_2(0) = \overbrace{\dot{x}_{20}}$$

Define :

$$\vec{x}(t) = \begin{Bmatrix} x_1(t) \\ x_2(t) \end{Bmatrix} = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

$$\dot{\vec{x}}(t) = \begin{Bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{Bmatrix} \qquad \ddot{\vec{x}}(t) = \begin{Bmatrix} \ddot{x}_1(t) \\ \ddot{x}_2(t) \end{Bmatrix}$$

$$\begin{cases} m_1 \ddot{x}_1 + (k_1 + k_2) x_1 - k_2 x_2 = 0 \\ m_2 x_2 - k_2 x_1 + k_2 x_2 = 0 \end{cases}$$

$$[M] = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \qquad [K] = \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix}$$

$$\boxed{[M]\ddot{\vec{x}} + [K]\vec{x} = 0}$$

Initial conditions :

$$\vec{x}(0) = \begin{Bmatrix} x_{10} \\ x_{20} \end{Bmatrix} \qquad \dot{\vec{x}}(0) = \begin{Bmatrix} \dot{x}_{10} \\ \dot{x}_{20} \end{Bmatrix}$$

1 DOF : $x(t) = Ae^{i\omega t} \Rightarrow A\sin(\omega t + \phi)$

2 DOF : $x_1(t) = u_1 e^{i\omega t}$

$x_2(t) = u_2 e^{i\omega t}$

$\rightarrow \vec{x}(t) = \vec{u}\, e^{i\omega t}, \qquad \vec{u} = \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$

unknown

$$\dot{\vec{x}}(t) = i\omega \vec{u}\, e^{i\omega t}$$

$$\ddot{\vec{x}}(t) = (i\omega)(i\omega)\vec{u}\, e^{i\omega t}$$

$$= -\omega^2 \vec{u}\, e^{i\omega t}$$

$$\rightarrow -\omega^2[M]\vec{u}\, e^{i\omega t} + [K]\vec{u}\, e^{i\omega t} = 0$$

$$(-\omega^2[M] + [K])\vec{u}\, e^{i\omega t} = 0$$

$$\rightarrow (-\omega^2[M] + [K])\vec{u} = 0$$

$\vec{u}$ is non-zero vector

$\rightarrow \det(-\omega^2[M] + [K]) = 0$ (condition of this equation)

the characteristic equation

1 DOF : $[M] = m \quad ; \quad [K] = K$

$$\det(-\omega^2 m + K) = 0$$

$$\Rightarrow -\omega^2 m + K = 0 \rightarrow \omega = \sqrt{K/m}$$

$$\det(-\omega^2[M] + [K])$$

$$= \det\left(-\omega^2 \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} + \begin{bmatrix} K_1K_2 & -K_2 \\ -K_2 & K_2 \end{bmatrix}\right)$$

$$= \det\begin{pmatrix} K_1+K_2-m_1\omega^2 & -K_2 \\ -K_2 & K_2-m_2\omega^2 \end{pmatrix}$$

$$\rightarrow (K_1+K_2-m_1\omega^2)(K_2-m_2\omega^2) - (-K_2)(-K_2) = 0$$

$$\rightarrow m_1m_2\,\omega^4 - (m_1K_2 + m_2K_1 + m_2K_2)\omega^2 + K_1K_2 = 0$$

Two roots : $\omega_1^2,\ \omega_2^2$

Four roots : $\omega_1,\ -\omega_1,\ \omega_2,\ -\omega_2$

For $\omega = \omega_1$:

$$(-\omega_1^2[M] + [K])\vec{u} = 0$$

For $\omega = \omega_2$:

$$(-\omega_2^2[M] + [K])\vec{u} = 0$$

$$\rightarrow \quad \vec{x}(t) = c_1 \vec{u}_1 e^{i\omega_1 t} + c_2 \vec{u}_1 e^{-i\omega t} + c_3 \vec{u}_2 e^{i\omega t} + c_4 \vec{u}_2 e^{-i\omega t}$$

$$= A_1 \sin(\omega_1 t + \phi_1)\vec{u}_1 + A_2 \sin(\omega_2 t + \phi_2)\vec{u}_2$$

$\boxed{\text{Example:}}$ $m_1 = 9$ kg, $m_2 = 1$ kg ; $k_1 = 24$ N/m, $k_2 = 3$ N/m

Find ω and $\vec{u}$

ω: natural freq. ; $\vec{u}$: mode shape

Solution:

$$[M] = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \rightarrow \begin{bmatrix} 9 & 0 \\ 0 & 1 \end{bmatrix}$$

$$[K] = \begin{bmatrix} K_1+K_2 & -K_2 \\ -K_2 & K_2 \end{bmatrix} \rightarrow \begin{bmatrix} 27 & -3 \\ -3 & 3 \end{bmatrix}$$

$$\det\left[-\omega^2[M] + [K]\right] = 0$$

$$\rightarrow \begin{vmatrix} 27-9\omega^2 & -3 \\ -3 & 3-\omega^2 \end{vmatrix} = 0$$

$$\rightarrow (27-9\omega^2)(3-\omega^2) - 9 = 0$$

$$\rightarrow (81 - 27\omega^2 - 27\omega^2 + 9\omega^4 - 9) = 0$$

$$\rightarrow \omega^4 - 6\omega^2 + 8 = 0$$

$$\hookrightarrow (\omega^2-2)(\omega^2-4) = 0$$

$$\omega_1 = \sqrt{2} \; ; \; \omega_2 = 2$$

For $\omega = \omega_1 = \sqrt{2}$

$$(-\omega^2[M] + [K])\vec{u}_1 = 0$$

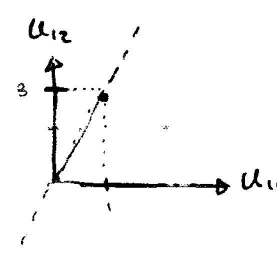
$$\rightarrow \left(-2\begin{bmatrix} 9 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 27 & -3 \\ -3 & 0 \end{bmatrix}\right)\begin{Bmatrix} u_{11} \\ u_{12} \end{Bmatrix} = 0$$

$$\rightarrow \begin{bmatrix} 9 & -3 \\ -3 & 1 \end{bmatrix}\begin{Bmatrix} u_{11} \\ u_{12} \end{Bmatrix} = 0$$

$$\rightarrow \begin{matrix} 9u_{11} - 3u_{12} = 0 \\ -3u_{11} \quad u_{12} = 0 \end{matrix} \rightarrow \frac{u_{11}}{u_{12}} = \frac{1}{3}$$

$\rightarrow u_{11} = 1, \; u_{12} = 3$

$\rightarrow u_{11} = 1/3, \; u_{12} = 1$



Choose $u_{11} = 1/3$; $u_{12} = 1$

$$\vec{u}_1 = \begin{pmatrix} 1/3 \\ 1 \end{pmatrix}$$

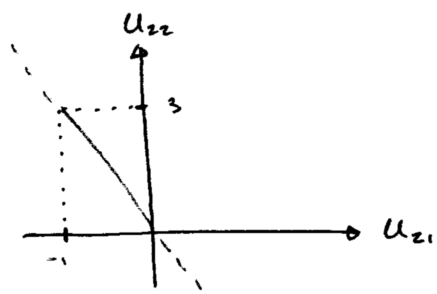
For $\omega = \omega_2 = 2$

$$\left(-\omega_2^2 [M] + [K]\right) u_2 = \emptyset$$

$$\Rightarrow \left(-4 \begin{bmatrix} 9 & \emptyset \\ \emptyset & 1 \end{bmatrix} + \begin{bmatrix} 27 & -3 \\ -3 & 3 \end{bmatrix} \right) \begin{Bmatrix} u_{21} \\ u_{22} \end{Bmatrix} = \emptyset$$

$$\Rightarrow \begin{bmatrix} -9 & -3 \\ -3 & -1 \end{bmatrix} \begin{Bmatrix} u_{21} \\ u_{22} \end{Bmatrix} = \emptyset$$

$$\Rightarrow \begin{array}{l} -9u_{21} - 3u_{22} = \emptyset \\ -3u_{21} - u_{22} = \emptyset \end{array} \Longrightarrow \frac{u_{21}}{u_{22}} = \frac{-1}{3}$$



$u_{22} = 1$; $u_{21} = -1/3$

$$\vec{u}_2 = \begin{pmatrix} -1/3 \\ 1 \end{pmatrix}$$

$$[M]\ddot{\vec{x}} + [K]\vec{x} = \emptyset$$

$$\vec{x} = \vec{u}e^{i\omega t}$$

$$\rightarrow \left([K] - \omega^2[M]\right)\vec{u} = \emptyset \quad ; \text{ where } \quad \vec{u} \neq \emptyset$$

$$\det\left([K] - \omega^2[M]\right) = \emptyset$$

$\hookrightarrow$ Solving quadratic eqn yields ω_1, ω_2

natural Freq. $\left\{\begin{array}{l} \omega_1 \longrightarrow \vec{u}_1 \\ \omega_2 \longrightarrow \vec{u}_2 \end{array}\right\}$ mode shape

The solution :

$$\vec{x} = \left(ae^{i\omega_1 t} + be^{-i\omega_1 t}\right)\vec{u}_1 + \left(ce^{i\omega_2 t} + de^{-i\omega_2 t}\right)\vec{u}_2$$

$$= \underline{A_1}\sin(\omega_1 t + \underline{\phi_1})\vec{u}_1 + \underline{A_2}\sin(\omega_2 t + \underline{\phi_2})\vec{u}_2$$

Initial conditions such that

$$A_2 = \phi_1 = \phi_2 = \emptyset$$

Then :

$$\vec{x} = A_1\sin(\omega_1 t)\,\vec{u}_1$$

Let :

$$\vec{u}_1 = \left\{\begin{array}{c} u_{11} \\ u_{21} \end{array}\right\}$$

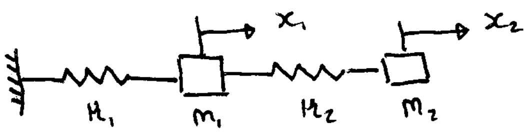
Then :

$$x_1 = A_1\sin(\omega_1 t)\cdot u_{11}$$

$$x_2 = A_2\sin(\omega_1 t)\,u_{21}$$

$$\rightarrow \frac{x_1}{x_2} = \frac{u_{11}}{u_{21}}$$

Example:



Initial conditions:

$$x_1(0) = 1\ \text{mm}, \qquad x_2(0) = \dot{x}_1(0) = \dot{x}_2(0) = 0$$

Given: $m_1 = 9$ kg / $m_2 = 1$ kg

$k_1 = 24$ N/m / $k_2 = 2$ N/m

Solution: $\omega_1 = \sqrt{2}$ rad/s

$\omega_2 = 2$ rad/s

$$\vec{u}_1 = \begin{Bmatrix} 1/3 \\ 1 \end{Bmatrix} \quad / \quad \vec{u}_2 = \begin{Bmatrix} -1/3 \\ 1 \end{Bmatrix}$$

The free vibration:

$$\vec{x}(t) = A_1 \sin(\omega_1 t + \phi_1)\vec{u}_1 + A_2 \sin(\omega_2 t + \phi_2)\vec{u}_2$$
$$= A_1 \sin(\omega_1 t + \phi_1)\begin{pmatrix} 1/3 \\ 1 \end{pmatrix} + A_2 \sin(\omega_2 t + \phi_2)\begin{pmatrix} -1/3 \\ 1 \end{pmatrix}$$

$\rightarrow$ $x_1(t) = (1/3)A_1 \sin(\omega_1 t + \phi_1) - (1/3) A_2 \sin(\omega_2 t + \phi_2)$

$x_2(t) = A_1 \sin(\omega_1 t + \phi_1) + A_2 \sin(\omega_2 t + \phi_2)$

$\rightarrow$ $\dot{x}_1(t) = (1/3) A_1 \omega_1 \cos(\omega_1 t + \phi_1) - (1/3) A_2 \omega_2 \cos(\omega_2 t + \phi_2)$

$\dot{x}_2(t) = A_1 \omega_1 \cos(\omega_1 t + \phi_1) + A_2 \omega_2 \cos(\omega_2 t + \phi)$

At $t = 0$:

$x_1(0) = (1/3) A_1 \sin(\phi_1) - (1/3) A_2 \sin(\phi_2) = 1$

$x_2(0) = A_1 \sin(\phi_1) + A_2 \sin(\phi_2) = 0$

$\rightarrow$ $A_1 \sin\phi_1 = 1.5$ / $\rightarrow$ $A_2 \sin\phi_2 = -1.5$

$\dot{x}_1(0) = (1/3) A_1 \omega_1 \cos(\phi_1) - (1/3) A_2 \omega_2 \cos(\phi_2) = 0$

$\dot{x}_2(0) = A_1 \omega_1 \cos(\phi_1) + A_2 \omega_2 \cos(\omega_2) = 0$

$\rightarrow$ $A_1 \cos(\phi_1) = 0$ / $\rightarrow$ $A_2 \cos(\phi_2) = 0$

$\cos(\phi_1) = 0$ when $\phi_1 = 90°$

$\cos(\phi_2) = 0$ when $\phi_2 = 90°$

thus, $A_1 = 1.5$

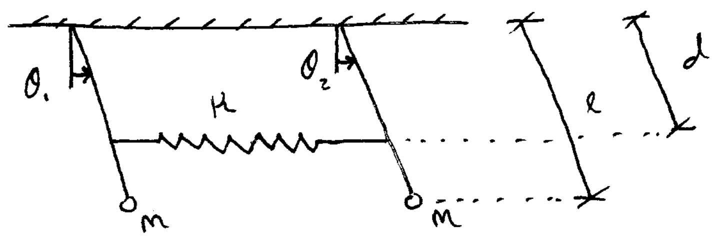
$A_2 = -1.5$

$$x_1(t) = (0.5)\cos(\sqrt{2}\,t) + 0.5\cos(2t)$$
$$x_2(t) = (1.5)\cos(\sqrt{2}\,t) - 1.5\cos(2t)$$

since $\phi_{1,2} = 90°$

$\sin(\omega t + 90°) = \cos(\omega t)$

$\boxed{\text{Example:}}$ Two connected pendulums



Mass matrix: $[M] = \begin{bmatrix} m\ell^2 & 0 \\ 0 & m\ell^2 \end{bmatrix}$

Stiffness matrix: $[K] = \begin{bmatrix} mg\ell + kd^2 & -kd^2 \\ -kd^2 & mg\ell + kd^2 \end{bmatrix}$

Natural Freq: $\left| -\omega^2 [M] + [K] \right| = 0$

$$(*) \quad \begin{vmatrix} mg\ell + kd^2 - \omega^2 m\ell^2 & -kd^2 \\ -kd^2 & mg\ell + kd^2 - \omega^2 m\ell^2 \end{vmatrix} = 0$$

$$\rightarrow (mg\ell + kd^2 - \omega^2 m\ell^2)^2 - (kd^2)^2 = 0$$
$$mg\ell + kd^2 - \omega^2 \cdot m\ell^2 = \pm kd^2$$
$$\omega^2 m\ell^2 = mg\ell + kd^2 \pm kd^2$$
$$\rightarrow \omega^2 m\ell^2 = mg\ell \longrightarrow \omega_1^2 = g/\ell$$
$$\rightarrow \omega^2 m\ell^2 = mg\ell + 2kd^2 \longrightarrow \omega_2^2 = g/\ell + \left(2kd^2 / mg\ell^2\right)$$

For $\omega = \omega_1 = \sqrt{g/\ell}$

$$\begin{pmatrix} mg\ell + kd^2 - (g/\ell)m\ell^2 & -kd^2 \\ -kd^2 & mg\ell + kd^2 - (g/\ell)m\ell^2 \end{pmatrix} \vec{u}_1 = 0$$

ω_1^2

$$\begin{pmatrix} kd^2 & -kd^2 \\ -kd^2 & kd^2 \end{pmatrix} \begin{pmatrix} u_{11} \\ u_{21} \end{pmatrix} = 0$$

$$\rightarrow kd^2 \cdot u_{11} - kd^2 \cdot u_{21} = 0 \quad ; \quad u_{11} = u_{21}$$

$$\therefore \vec{u}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

For $\omega = \omega_2 = \sqrt{g/\ell + 2Kd^2/m\ell^2}$

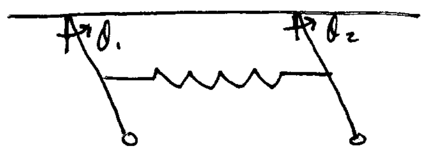
$$\begin{pmatrix} mg\ell + Kd^2 - (g/\ell + 2Kd^2/m\ell^2)\,m\ell^2 & -Kd^2 \\ -Kd^2 & mg\ell + Kd^2 - (g/\ell + 2Kd^2/m\ell^2)\,m\ell^2 \end{pmatrix} \vec{u}_2 = \emptyset$$

$$\begin{pmatrix} -Kd^2 & -Kd^2 \\ -Kd^2 & -Kd^2 \end{pmatrix} \begin{pmatrix} u_{12} \\ u_{22} \end{pmatrix} = \emptyset$$

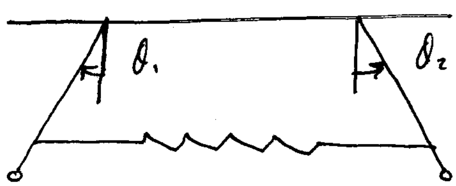
$$-Kd^2 u_{12} - Kd^2 u_{22} = \emptyset \longrightarrow u_{12} = u_{22}$$

$$\vec{u}_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$\longrightarrow \quad \omega_1 = \sqrt{g/\ell}$



$\longrightarrow \quad \omega_2 = \sqrt{\dfrac{g}{\ell} + \dfrac{2Kd^2}{m\ell^2}}$



Take $\ell = 1\text{m}$; $d = \emptyset.3\text{m}$; $K = 4$ N/m ; $m = 1$ kg

$\omega_1 = \sqrt{g/\ell}$; $\omega_2 = 3.245$ rad/s

Choose $\theta_1(\emptyset) = 1$; $\theta_2(\emptyset) = \emptyset$

$\dot{\theta}_1(\emptyset) = \emptyset$; $\dot{\theta}_2(\emptyset) = \emptyset$

4.2 Eigenvalues and Natural Frequencies

Symmetric Matrix :

$$[M]^T = [M]$$

A Symmetric matrix M is <u>positive definite</u> :

row vector $\vec{x}^T [M] \vec{x} > 0$ column vector

For all non-zero vector $\vec{x}$

A Symmetric positive definite matrix M can be Factored :

$$[M] = [L][L]^T$$

Here $[L]$ is upper triangular.

Cholesky matrix

If $[L]$ is diagonal, $[L]$ is the matrix $[M]$ square root.

<u>2 DOF</u> :

$$[M] = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \quad ; \quad [M]^{1/2} = \begin{bmatrix} \sqrt{m_1} & 0 \\ 0 & \sqrt{m_2} \end{bmatrix}$$

$$[M]^{-1/2} = \begin{bmatrix} 1/\sqrt{m_1} & 0 \\ 0 & 1/\sqrt{m_2} \end{bmatrix}$$

Equation of motion:

$$[M]\ddot{\vec{x}} + [K]\vec{x} = 0$$

Define

$$\vec{x} = [M]^{-1/2}\vec{q}(t)$$

$$\underbrace{[M]^{-1/2}[M][M]^{-1/2}}\ddot{\vec{q}} + \underbrace{[M]^{-1/2}[K][M]^{-1/2}}\vec{q} = 0$$

$$\ddot{\vec{q}} + [\tilde{K}]\vec{q} = 0 \qquad (A)$$

Here, $[\bar{K}] = [M]^{-1/2}[K][M]^{-1/2}$

$[\bar{K}]^T = [\bar{K}]$

mass normalized stiffness

1 DOF :

$$m\ddot{x} + kx = \emptyset$$

$$\ddot{x} + (k/m)\,x = \emptyset$$

B

(compare A → B
thus, ω_n is contained within.)

For the free vibration :

$$\ddot{\vec{q}} + [\bar{K}]\vec{q} = \emptyset$$

Take $\vec{q} = \vec{v}e^{i\omega t}$

$$(-\omega^2\vec{v} + [\bar{K}]\vec{v})e^{i\omega t} = \emptyset$$

$$\longrightarrow \boxed{[\bar{K}]\vec{v} = \omega^2\vec{v}}, \quad \vec{v} \neq \emptyset$$

(would mean no motion at all)

$$[M]\ddot{\vec{x}} + [K]\vec{x} = \emptyset$$

$\vec{x}$: the physical coordinates

Transformation :

$$\vec{x} = [M]^{-1/2}\vec{q}$$ not physical coordinate (general coordinate)

$$\longrightarrow \ddot{\vec{q}} + [\bar{K}]\vec{q} = \emptyset$$

Here $[\bar{K}] = [M]^{-1/2}[K][M]^{-1/2}$

Free vibration $\vec{q}(t) = \vec{v}e^{i\omega t}$

$$\longrightarrow ([\bar{K}] - \omega^2[I])\vec{v} = \emptyset$$

$$[I] = diag(1)$$

$$[\bar{K}]\vec{v} = \omega^2\vec{v}$$ eigenvector

eigenvalue

* Real eigenvalue & real eigenvector
* Eigenvalues are positive if and only if $[\bar{K}]$ is positive definite
* The set of eigenvectors can be chosen to be orthogonal

$$\vec{x} = \begin{Bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{Bmatrix}, \quad \vec{y} = \begin{Bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{Bmatrix}$$

Inner product

$$\vec{x}^T \cdot \vec{y} = \underbrace{(x_1, x_2, \dots, x_n)}_{1\times n}\underbrace{\begin{Bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{Bmatrix}}_{n\times 1} = x_1y_1 + x_2y_2 + \dots + x_ny_n$$

same

results in 1×1

Norm :

$$\|\vec{x}\| = \sqrt{\vec{x}^T \cdot \vec{x}} = \sqrt{x_1^2 + x_2^2 + \dots + x_n^2}$$

Vector :

$$\frac{1}{\|x\|}\vec{x}$$ normal vector

$\vec{x}$ and $\vec{y}$ orthogonal $\vec{x}^T\vec{y} = \emptyset$

$\boxed{\text{Example :}}$ Normalize $\vec{x} = \begin{Bmatrix} 1 \\ 2 \\ 2 \end{Bmatrix}$

$$\vec{x}^T \cdot \vec{x} = 1^2 + 2^2 + 2^2 = 9$$

$$\therefore \|\vec{x}\| = \sqrt{\vec{x}^T \cdot \vec{x}} = \sqrt{9} = 3$$

$\therefore$ the normalized vector of $\vec{x}$ is

$$\frac{1}{\|\vec{x}\|}\vec{x} = \left(\frac{1}{3}\right)\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 1/3 \\ 2/3 \\ 2/3 \end{pmatrix}$$

$\boxed{\text{Example :}}$ $[M] = \begin{bmatrix} 9 & \emptyset \\ \emptyset & 1 \end{bmatrix}$; $[K] = \begin{bmatrix} 27 & -3 \\ -3 & 3 \end{bmatrix}$

Since $[M]^{1/2} = \begin{bmatrix} 3 & \emptyset \\ \emptyset & 1 \end{bmatrix} \longrightarrow [M]^{-1/2} = \begin{bmatrix} 1/3 & \emptyset \\ \emptyset & 1 \end{bmatrix}$

$$\therefore [\bar{K}] = [M]^{-1/2}[K][M]^{-1/2}$$

$$= \begin{bmatrix} 1/3 & \emptyset \\ \emptyset & 1 \end{bmatrix}\begin{bmatrix} 27 & -3 \\ -3 & 3 \end{bmatrix}\begin{bmatrix} 1/3 & \emptyset \\ \emptyset & 1 \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix}$$

Eigenvalues and eigenvectors

$$([\bar{K}] - \lambda[I])\vec{v} = \emptyset$$

$$\longrightarrow \begin{pmatrix} 3-\lambda & -1 \\ -1 & 3-\lambda \end{pmatrix}\vec{v} = \emptyset$$

$$\det\begin{pmatrix} 3-\lambda & -1 \\ -1 & 3-\lambda \end{pmatrix} = \emptyset$$

$$(3-\lambda)^2 - (-1)^2 = \emptyset$$

$(3-\lambda) = (-1)$ or $(3-\lambda) = 1$

$\lambda_1 = 2$ $\qquad\qquad \lambda_2 = 4$

For $\lambda_1 = 2$

$$\begin{pmatrix} 3-2 & -1 \\ -1 & 3-2 \end{pmatrix}\begin{Bmatrix} v_{11} \\ v_{21} \end{Bmatrix} = \emptyset$$

$$v_{11} - v_{21} = \emptyset$$

$\longrightarrow \vec{v}_1 = \begin{Bmatrix} v_{11} \\ v_{21} \end{Bmatrix} = \alpha\begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$ $\qquad \alpha \neq \emptyset$

$$\vec{v_1}^T \cdot \vec{v_1} = (\alpha \ \ \alpha)\begin{pmatrix}\alpha \\ \alpha\end{pmatrix} = 2\alpha^2 = 1$$

$$\alpha = \frac{1}{\sqrt{2}} \quad \text{or} \quad \alpha = \frac{-1}{\sqrt{2}}$$

$$\therefore \quad \vec{v_1} = \frac{1}{\sqrt{2}}\begin{pmatrix}1 \\ 1\end{pmatrix} = \begin{pmatrix}1/\sqrt{2} \\ 1/\sqrt{2}\end{pmatrix}$$

For $\lambda_2 = 4$

$$\vec{v_2} = \frac{1}{\sqrt{2}}\begin{pmatrix}1 \\ -1\end{pmatrix} = \begin{pmatrix}1/\sqrt{2} \\ -1/\sqrt{2}\end{pmatrix}$$

Since $\vec{v_1}^T \vec{v_2} = (1/\sqrt{2})^2 - (1/\sqrt{2})^2 = 0$

$\hookrightarrow$ means they're perpendicular to each other

Mode shape :

$$\begin{cases} \vec{u_1} = [M]^{-1/2}\,\vec{v_1} \\ \vec{u_2} = [M]^{-1/2}\,\vec{v_2} \end{cases}$$

Define :

$$[P] = [\vec{v_1} \ \ \vec{v_2}]$$

Since :

$$[P^T][P] = \begin{bmatrix}\vec{v_1}^T \\ \vec{v_2}^T\end{bmatrix}\begin{bmatrix}\vec{v_1} & \vec{v_2}\end{bmatrix}$$

$$= \begin{bmatrix}\vec{v_1}^T\vec{v_1} & v_1^T v_2 \\ \vec{v_2}^T\vec{v_1} & v_2^T v_2\end{bmatrix} = \begin{bmatrix}1 & 0 \\ 0 & 1\end{bmatrix} = [I]$$

Then $[P^T][P] = [I]$

$[P]$: orthogonal matrix

$[P]^{-1} = [P]^T$

$$[P]^T[\bar{K}][P] = [P]^T[\bar{K}][\vec{v}_1\ \vec{v}_2]$$
$$= [P]^T[\bar{K}\vec{v}_1\ \ \bar{K}\vec{v}_2]$$

Since $[\bar{K}]\vec{v}_1 = \lambda_1\vec{v}_1$

$[\bar{K}]\vec{v}_2 = \lambda_2\vec{v}_2$

$$\rightarrow [P]^T[\bar{K}][P] = \begin{bmatrix} \vec{v}_1^{\,T} \\ \vec{v}_2^{\,T} \end{bmatrix}[\lambda_1\vec{v}_1\ \ \lambda_2\vec{v}_2]$$
$$= \begin{bmatrix} \lambda_1\vec{v}_1^{\,T}\vec{v}_1 & \lambda_2\vec{v}_1^{\,T}\vec{v}_2 \\ \lambda_1\vec{v}_2^{\,T}\vec{v}_1 & \lambda_2\vec{v}_2^{\,T}\vec{v}_2 \end{bmatrix}$$
$$= \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = [\Lambda]$$

Mode shape :

$$[M]\ddot{\vec{x}} + [K]\vec{x} = 0$$
$$\vec{x} = \vec{u}e^{i\omega t}$$
$$\rightarrow (-[M]\omega^2 + [K])\vec{u} = 0$$
$$\omega_m \ \& \ \vec{u}_m$$

$$(-[M]\omega_m^2 + [K])\vec{u}_m = 0$$
$$\rightarrow \vec{u}_m^{\,T}(-[M]\omega_m^2 + [K])\vec{u}_m = 0$$
$$\rightarrow -\vec{u}_m^{\,T}[M]\vec{u}_m \cdot \omega_m^2 + \vec{u}_m^{\,T}[K]\vec{u}_m = 0$$
$$\omega_m^2 = \frac{\vec{u}_m^{\,T}[K]\vec{u}_m}{\vec{u}_m^{\,T}[M]\vec{u}_m}$$

(For single degree of Freedom :

$$\omega^2 = \frac{K}{m} = \frac{(1/2)KA^2}{(1/2)mA^2} = \frac{(1/2)\cdot A\cdot K\cdot A}{(1/2)\cdot A\cdot m\cdot A}$$
)

Mass normalization of the mode shape :

$$\vec{u}_m^{\,T}[M]\vec{u}_m = 1$$

Then :

$$\omega_m^2 = \vec{u}_m^{\,T}[K]\vec{u}_m$$

4.3 Modal Analysis

$$\begin{cases} [M]\ddot{\vec{x}} + [K]\vec{x} = \emptyset \\ \vec{x}(0) = \vec{x}_0 \, , \quad \dot{\vec{x}}(0) = \dot{\vec{x}}_0 \end{cases}$$

$\longrightarrow$ Transformation #1:

$$\vec{x} = [M]^{-1/2}\vec{q}$$

$$\longrightarrow \quad \ddot{\vec{q}} + [\tilde{K}]\vec{q} = \emptyset$$

The eigenvector matrix $[P]$

$\longrightarrow$ Transformation #2:

$$\vec{q} = [P]\vec{r}$$

$$\longrightarrow \quad [P]\ddot{\vec{r}} + [\tilde{K}][P]\vec{r} = \emptyset$$

$$\longrightarrow \quad [P]^T[P]\ddot{\vec{r}} + [P]^T[\tilde{K}][P]\vec{r} = \emptyset$$

$$\longrightarrow \quad \ddot{\vec{r}} + [\Lambda]\vec{r} = \emptyset$$

Here $[\Lambda] = \begin{bmatrix} \lambda_1 & \emptyset \\ \emptyset & \lambda_2 \end{bmatrix} = \begin{bmatrix} \omega_1^2 & \emptyset \\ \emptyset & \omega_2^2 \end{bmatrix}$

$\hookrightarrow$ for 2 DOF

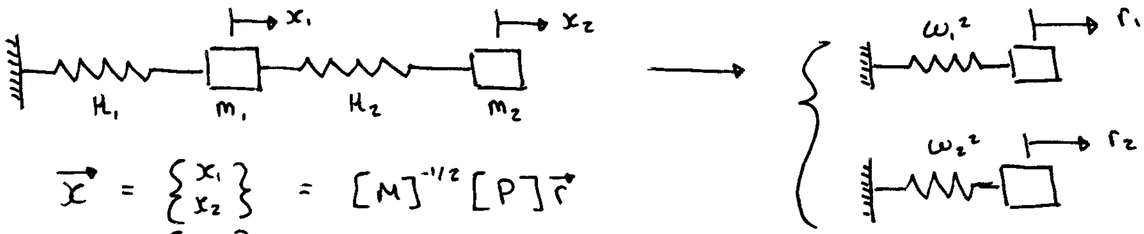
$\hookrightarrow$ (for 3 DOF, matrix would be 3×3 w/ λ_3)

$$\vec{r} = \begin{Bmatrix} r_1(t) \\ r_2(t) \end{Bmatrix}$$

$$\longrightarrow \quad \begin{Bmatrix} \ddot{r}_1 \\ \ddot{r}_2 \end{Bmatrix} + \begin{bmatrix} \omega_1^2 & \emptyset \\ \emptyset & \omega_2^2 \end{bmatrix} \begin{Bmatrix} r_1 \\ r_2 \end{Bmatrix} = \emptyset$$

$$\longrightarrow \quad \begin{cases} \ddot{r}_1 + \omega_1^2 r_1 = \emptyset \\ \ddot{r}_2 + \omega_2^2 r_2 = \emptyset \end{cases}$$

$\vec{r} = \begin{Bmatrix} r_1 \\ r_2 \end{Bmatrix}$ modal coordinates



$$\vec{x} = \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = [M]^{-1/2}[P]\vec{r}$$

$$\vec{r} = \begin{Bmatrix} r_1 \\ r_2 \end{Bmatrix}$$

$$[M]\ddot{\vec{x}} + [K]\vec{x} = 0$$

$$\vec{x} = (x_1, x_2, \dots, x_n)^T \quad \text{coupled}$$

$$\boxed{\vec{x} = [M]^{-1/2}\vec{q}}$$

$$\ddot{\vec{q}} + [\tilde{K}]q = 0 \qquad \& \quad [\tilde{K}] = [M]^{-1/2}[K][M]^{-1/2}$$

$$([\tilde{K}] - \lambda[I])\vec{v} = 0$$

$$\lambda_m = \omega_m^2, \quad \vec{v}_m, \quad m = 1, 2, \dots, n$$

$$[P] = [\vec{v}_1, \vec{v}_2, \dots, \vec{v}_m]$$

$$[P]^T[P] = [I]$$

$$\boxed{\vec{q} = [P]\vec{r}}$$

$$\ddot{\vec{r}} + [\Lambda]\vec{r} = 0 \quad \text{decoupled}$$

$$[\Lambda] = \mathrm{diag}(\omega_m^2) = [P]^T[\tilde{K}][P]$$

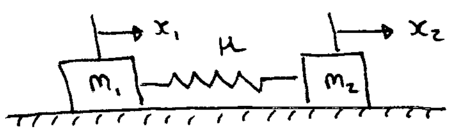
$$\ddot{r}_1 + \omega_1^2 r_1 = 0$$

$$\ddot{r}_2 + \omega_2^2 r_2 = 0$$

$$\vdots$$

$$\ddot{r}_n + \omega_n^2 r_n = 0$$

Example



$m_1 = 1\ kg$

$m_2 = 4\ kg$

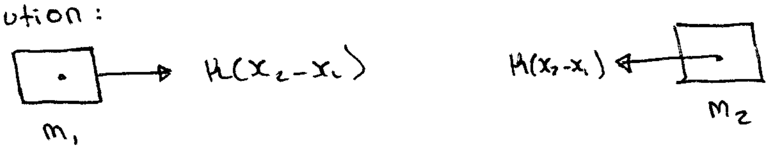
$k = 400\ N/m$

$$\vec{x}(0) = \begin{bmatrix} 0.01 \\ 0 \end{bmatrix} m$$

$$\dot{\vec{x}}(0) = 0$$

Find the response of the system.

Solution:



$m\ddot{x} + kx \quad Ix = 0$

m_1: $\quad m_1\ddot{x}_1 = k(x_2 - x_1)$

m_2: $\quad m_2\ddot{x}_2 = -k(x_2 - x_1)$

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} k & -k \\ -k & k \end{bmatrix}\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = 0$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} 400 & -400 \\ -400 & 400 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = 0$$

$$[M]^{1/2} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} ; \quad [M]^{-1/2} = \begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix} \begin{bmatrix} 400 & -400 \\ -400 & 400 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix}$$

$$= \begin{bmatrix} 400 & -200 \\ -200 & 100 \end{bmatrix}$$

$$\det([\bar{K}] - \lambda[I]) = \begin{vmatrix} 400-\lambda & -200 \\ -200 & 100-\lambda \end{vmatrix} = 0$$

$$\lambda_1 = 0 \qquad \lambda_2 = 500$$

$$\omega_1 = \sqrt{\lambda_1} = 0 \qquad \omega_2 = \sqrt{500} = 22.36 \text{ rad/s} \quad \text{(natural freq.)}$$

$$\lambda_1 = 0 : \begin{bmatrix} 400-\lambda_1 & -200 \\ -200 & 100-\lambda_1 \end{bmatrix} \begin{Bmatrix} v_{11} \\ v_{12} \end{Bmatrix} = 0$$

$$\vec{v}_1 = \begin{Bmatrix} v_{11} \\ v_{21} \end{Bmatrix} = \begin{Bmatrix} 1 \\ 2 \end{Bmatrix}$$

$$\|\vec{v}\| = \sqrt{\vec{v}_1^T \vec{v}_1} = \sqrt{1^2 + 2^2} = \sqrt{5}$$

Normalized: $\vec{v}_1 = (1/\sqrt{5}) \begin{Bmatrix} 1 \\ 2 \end{Bmatrix} = \begin{Bmatrix} 0.4472 \\ 0.8944 \end{Bmatrix}$

$$\lambda_2 = 500 : \begin{bmatrix} 400-\lambda_2 & -200 \\ -200 & 100-\lambda_2 \end{bmatrix} \begin{Bmatrix} v_{12} \\ v_{22} \end{Bmatrix} = 0$$

$$\vec{v}_2 = \begin{Bmatrix} -2 \\ 1 \end{Bmatrix}$$

Normalized: $\vec{v}_2 = (1/\sqrt{5}) \begin{Bmatrix} -2 \\ 1 \end{Bmatrix} = \begin{Bmatrix} -0.8944 \\ 0.4472 \end{Bmatrix}$

$$[P] = [\vec{v}_1 \ \vec{v}_2]$$

$$= \begin{bmatrix} 0.4472 & -0.8944 \\ 0.8944 & 0.4472 \end{bmatrix}$$

Verify: $[P]^T[P] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$; $[P]^T[\bar{K}][P] = \begin{bmatrix} 0 & 0 \\ 0 & 500 \end{bmatrix}$

$$\underbrace{\vec{x} = [M]^{-1/2}\vec{q}} = [M]^{-1/2}[P]\vec{r}$$

$$[S] = [M]^{-1/2}[P] = \begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix}\begin{bmatrix} 0.4472 & -0.8944 \\ 0.8944 & 0.4472 \end{bmatrix}$$

$$= \begin{bmatrix} 0.4472 & -0.8944 \\ 0.4472 & 0.2236 \end{bmatrix}$$

$$(\ddot{\vec{r}} + [\Lambda]\vec{r} = 0)$$

$$r(0) = [S]^{-1}\vec{x}(0) = \begin{bmatrix} 0.4472 & 1.7889 \\ -0.8944 & 0.8944 \end{bmatrix}\begin{Bmatrix} 0.01 \\ 0 \end{Bmatrix}$$

$$= \begin{Bmatrix} 0.004472 \\ -0.008944 \end{Bmatrix}$$

$$r_1(0) = 0.004472 \quad \& \quad r_2(0) = -0.008944$$

$$\dot{\vec{r}}(0) = [S]^{-1}\dot{\vec{x}}(0) = 0$$

$$\dot{\vec{r}}_1(0) = 0, \quad \dot{\vec{r}}_2(0) = 0$$

$$\ddot{r}_1 + \omega_1^2 r_1 = 0 \qquad (\omega_1 = 0)$$

$$\ddot{r}_1 = 0$$

$$\Rightarrow r_1(t) = a + bt$$

$$\Rightarrow r_1(t) = 0.004472$$

$$\ddot{r}_2 + \omega_2^2 r_2 = 0 \qquad (\omega_2 = 500)$$

$$\Rightarrow r_2(t) = -0.0089\cos(22.36t)$$

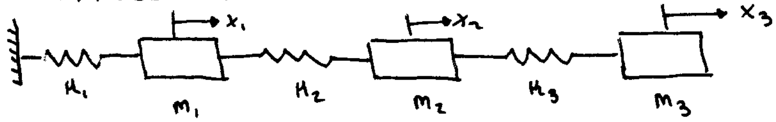
$$\therefore \vec{x} = [S]\vec{r}$$

$$= \begin{bmatrix} 0.4472 & -0.8944 \\ 0.4472 & 0.2236 \end{bmatrix}\begin{Bmatrix} 0.004472 \\ -0.0089\cos(22.36t) \end{Bmatrix}$$

$$\Rightarrow \vec{x} = \begin{Bmatrix} 2.012 + 7.60\cos(22.36t) \\ 2.012 - 1.990\cos(22.36t) \end{Bmatrix} \times 10^{-3}$$

Influence coefficients

Stiffness influence coefficients



k_{ij} : the force at point i due to a unit displacement at point j when all the other points other than j are fixed.

k_{11}:

If point j has displacement x_j ($j = 1, 2, 3$), then the force at point i :

$$F_i = k_{i1} x_1 + k_{i2} x_2 + k_{i3} x_3 = \sum_{j=1}^{3} k_{ij} x_j$$

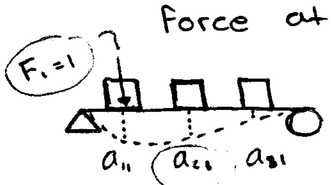
$i = 1, 2, 3$

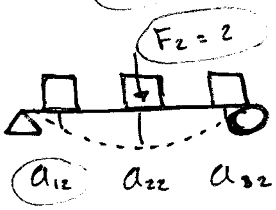
$$\rightarrow \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix} = \begin{bmatrix} k_{11} & \boxed{k_{12}} & k_{13} \\ \boxed{k_{21}} & k_{22} & \underline{k_{23}} \\ k_{31} & \underline{k_{32}} & k_{33} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}$$

symmetric matrix

Flexibility Influence Coefficients

a_{ij} : the displacement at point i due to a unit force at point j





Flexibility Influence Matrix

$$[A] = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

The relationship: $[K][A] = [I]$

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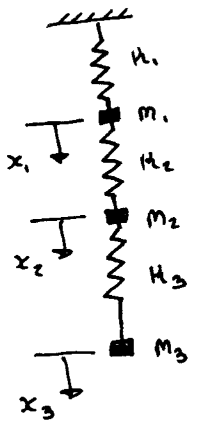
k_{ij} : the force at point i due to a unit displacement at point j, when all the other points have zero displacement.

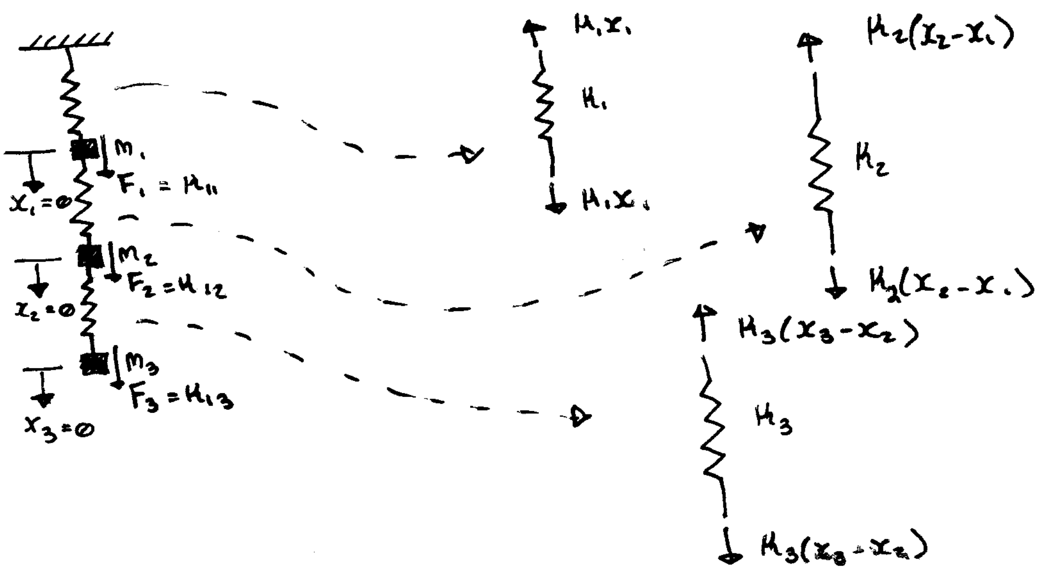
Example:

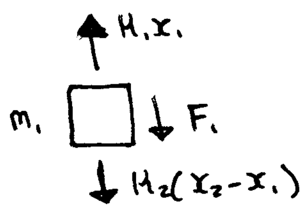


- Find the stiffness influence matrix.

$$K = \begin{bmatrix} k_{11} & k_{12} & k_{13} \\ k_{21} & k_{22} & k_{23} \\ k_{31} & k_{32} & k_{33} \end{bmatrix}$$

Solution: $x_1 = 0$; $x_2 = 0$; $x_3 = 0$

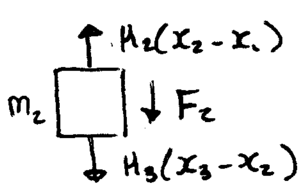




i.e. $\Sigma F = 0$

$$F_1 + k_2(x_2 - x_1) - k_1 x_1 = 0$$

$$\boxed{F_1 = (k_1 + k_2) x_1 - k_2 x_2}$$



$$\ldots \boxed{F_2 = -k_2 x_1 + (k_2 + k_3) x_2 - k_3 x_3}$$

$$\ldots \boxed{F_3 = -k_3 x_2 + k_3 x_3}$$

Set $x_1 = 1$; $x_2 = x_3 = 0$

$F_1 = (k_1 + k_2)$ $\quad F_2 = -k_2$ $\quad F_3 = 0$

$k_{11} = k_1 + k_2$ $\quad k_{21} = -k_2$ $\quad k_{31} = 0$

Set $x_1 = x_3 = 0$; $x_2 = 1$

$F_1 = -k_2$ $\quad F_2 = k_2 + k_3$ $\quad F_3 = -k_2$

$k_{21} = -k_2$ $\quad k_{22} = k_2 + k_3$ $\quad k_{32} = -k_3$

Set $x_1 = x_2 = 0$; $x_3 = 1$

$F_1 = 0$ $\quad F_2 = -k_{31}$ $\quad F_3 = k_3$

$k_{13} = 0$ $\quad k_{23} = -k_3$ $\quad k_{33} = k_3$

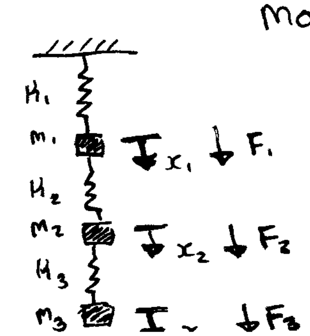
$$\rightarrow [K] = \begin{bmatrix} k_1 + k_2 & -k_2 & 0 \\ -k_2 & k_2 + k_3 & -k_3 \\ 0 & -k_3 & k_3 \end{bmatrix}$$ (stiffness matrix by statics)

$\underline{\underline{a_{ij}}}$: the deflection at point i due to a unit force at point j.

$$[A] = [a_{ij}] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix}$$

$$[A][K] = [I]$$

$\boxed{\text{Example}}$ Find the flexibility influence coefficient matrix



Solution:

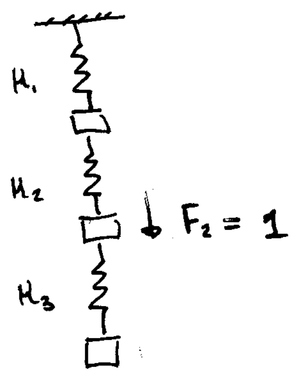
$F_1 = 1$; $F_2 = F_3 = 0$

disp: $\quad x_1 = a_{11} \quad x_2 = a_{21} \quad x_3 = a_{31}$

Spring 1 : $F_1 = K_1 x_1 = 1$ (*)

$x_1 = 1/K_1 = x_2 = x_3$

Let $F_1 = F_3 = \emptyset$, $F_2 = 1$



$$K_{eq} = \frac{K_1 K_2}{K_1 + K_2} \quad ; \quad x_2 = \frac{F_2}{K_{eq}}$$

$$\therefore \; x_2 = \frac{1}{K_1} + \frac{1}{K_2} = x_3$$

$$a_{22} = x_2 = \frac{1}{K_1} + \frac{1}{K_2}$$

$$a_{32} = x_3 = \frac{1}{K_1} + \frac{1}{K_2}$$

$$\boxed{a_{12} = a_{21} = \frac{1}{K_1}}$$

$$\frac{1}{K_{eq}} = \frac{1}{K_1} + \frac{1}{K_2} + \frac{1}{K_3}$$

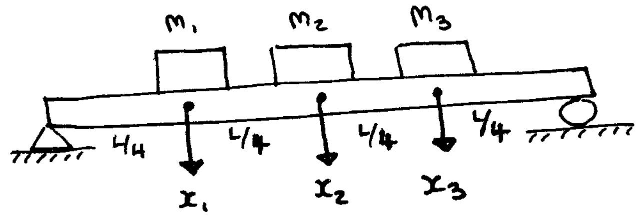
$$x_3 = \frac{F_3}{K_{eq}} = \frac{1}{K_{eq}} = \frac{1}{K_1} + \frac{1}{K_2} + \frac{1}{K_3}$$

$$\therefore \; a_{33} = x_3 = \frac{1}{K_1} + \frac{1}{K_2} + \frac{1}{K_3}$$

$$a_{13} = a_{31} = \frac{1}{K_1} \quad ; \quad a_{23} = a_{32} = \frac{1}{K_1} + \frac{1}{K_2}$$

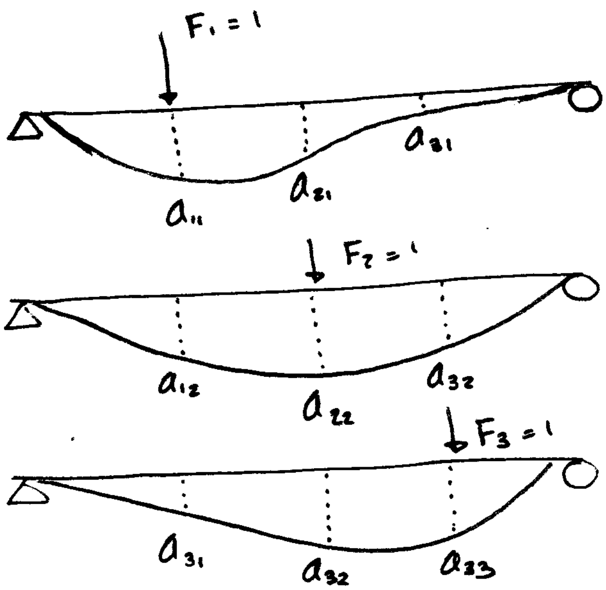
$$\therefore \; [A] = \begin{bmatrix} 1/K_1 & 1/K_1 & 1/K_1 \\ 1/K_1 & 1/K_1 + 1/K_2 & 1/K_1 + 1/K_2 \\ 1/K_1 & 1/K_1 + 1/K_2 & 1/K_1 + 1/K_2 + 1/K_3 \end{bmatrix}$$

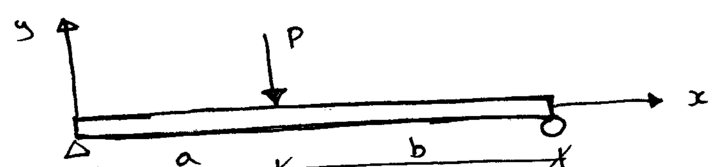
$\boxed{\text{Example}}$ Find the Flexibility matrix of the weightless beam shown:



EI = const.

Solution:





$L = a + b$

$$y = \begin{cases} \dfrac{Pbx}{6EIL}\left(L^2 - b^2 - x^2\right) & ; \quad 0 \le x \le a \\[2ex] -\dfrac{Pa(L-x)}{6EIL}\left(a^2 + x^2 - 2Lx\right) & ; \quad a \le x \le L \end{cases}$$

a_{11} : $a = L/4$; $b = 3/4\,L$

At $x = L/4$

$$a_{11} = \frac{P(3/4\,L)(1/4\,L)}{6EIL}\left(L^2 - (3L/4)^2 - (L/4)^2\right)$$

$$a_{11} = \left(\frac{9}{[illegible]}\right)\left(\frac{L^3}{EI}\right)$$

At $x = L/2$

$$a_{21} = \left(\frac{11}{768}\right)\left(\frac{L^3}{EI}\right)$$

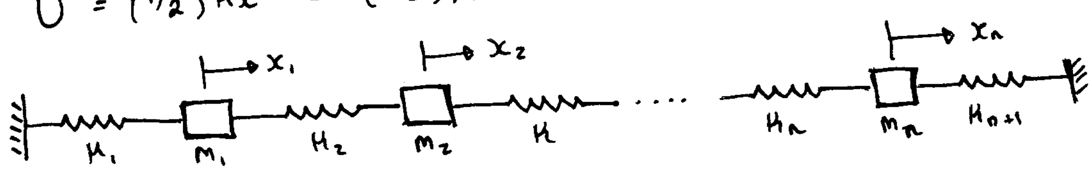
At $x = 3L/4$

$$a_{31} = \left(\frac{7}{768}\right)\left(\frac{L^3}{EI}\right)$$

$$[A] = \frac{L^3}{768EI}\begin{bmatrix} 9 & 11 & 7 \\ 11 & 16 & 11 \\ 7 & 11 & 9 \end{bmatrix}$$

Potential and Kinetic Energy:

$$U = (1/2)Kx^2 = (1/2)Fx$$



$$\{\vec{F}\} = [K]\{\vec{x}\}$$

Here: $\{\vec{F}\} = (F_1, F_2, \dots, F_n)^T$

$\{\vec{x}\} = (x_1, x_2, \dots, x_n)^T$

The potential energy:

$$U = (1/2)F_1x_1 + (1/2)F_2x_2 + \dots + (1/2)F_nx_n$$

$$= (1/2)(F_1 \ F_2 \ \dots \ F_n)\begin{Bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{Bmatrix}$$

$$= (1/2)\{\vec{F}\}^T\{x\}$$

$$U = (1/2)\left([K]\{\vec{x}\}\right)^T\{\vec{x}\}$$

$$= (1/2)\{\vec{x}\}^T[K]\{\vec{x}\}$$

Stiffness matrix

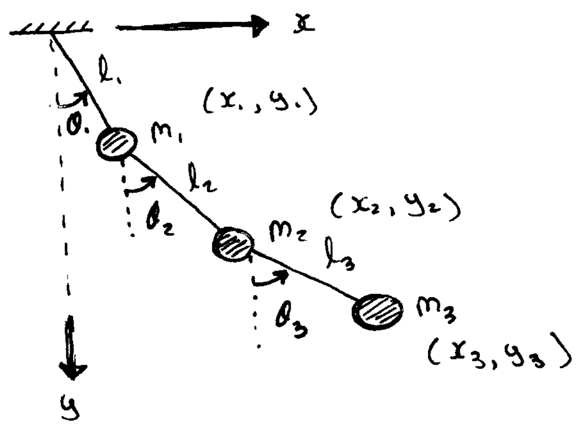
The kinetic energy :

$$T = (1/2) m_1 \dot{x}_1^2 + (1/2) m_2 \dot{x}_2^2 + \cdots + (1/2) m_n \dot{x}_n^2$$

$$= (1/2) \{\dot{\vec{x}}\}^T [M] \{\dot{\vec{x}}\}$$

Here $[M] = \begin{bmatrix} m_1 & & \emptyset \\ & m_2 \cdots & \\ \emptyset & & m_n \end{bmatrix}$

Generalized coordinates :



$$x_1^2 + y_1^2 = l_1^2$$

$$(x_2 - x_1)^2 + (y_2 - y_1)^2 = l_2^2$$

$$(x_3 - x_2)^2 + (y_3 - y_2)^2 = l_3^2$$

6 : $x_1 \ y_1 \ x_2 \ y_2 \ x_3 \ y_3$

3 constrained eq'ns

only 6 - 3 = 3 are independent.

(*) $\theta_1 \ \theta_2 \ \theta_3$ are three independent generalized coordinates.

Generalized coordinates :

$q_1 = \theta_1$; $q_2 = \theta_2$; $q_3 = \theta_3$

$$\begin{cases} x_1 = l_1 \sin(\theta_1) & ; \quad y_1 = l_1 \cos(\theta_1) \\ x_2 = x_1 + l_2 \sin(\theta_2) & ; \quad y_2 = y_1 + l_2 \cos(\theta_2) \\ x_3 = x_2 + l_3 \sin(\theta_3) & ; \quad y_3 + y_2 + l_3 \cos(\theta_3) \end{cases}$$

$$\begin{cases} x_i = x_i(q_1, q_2, q_3) \\ y_i = y_i(q_1, q_2, q_3) \end{cases} \qquad i = 1, 2, 3$$

Virtual displacement :

$q_1, q_2, \ldots, q_n \longrightarrow \delta q_1, \delta q_2, \ldots, \delta q_n$

The work done : $\delta \omega_1, \delta \omega_2, \ldots, \delta \omega_n$

The generalized Force :

$$Q_1 = \frac{\delta \omega_1}{\delta q_1}, \quad Q_2 = \frac{\delta \omega_2}{\delta q_2}, \quad \ldots, \quad Q_n = \frac{\delta \omega_n}{\delta q_n}$$

Define the Lagrangian

$L = T - V$

Then the equations of motion

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} + \frac{\partial V}{\partial q_i} = Q_i \quad ; \quad i = 1, 2, \ldots, n$$

suddenly changing from i to j, but same thing

If F_{xk}, F_{yk}, F_{zk} are the external forces acting on the k^{th} mass in the x, y, z-directions,

$$Q_j = \sum_k \left[F_{xk}\left(\frac{\partial x_k}{\partial \dot{q}_j}\right) + F_{yk}\left(\frac{\partial y_k}{\partial \dot{q}_j}\right) + F_{zk}\left(\frac{\partial z_k}{\partial \dot{q}_j}\right) \right] \quad ; \quad j = 1, 2, \ldots, n$$

Viscously damped Systems

Rayleigh's dissipation function :

$$R = (1/2)\, \dot{x}^T [c]\, \dot{x}$$

here $[c]$ is the damping matrix

The equations of motion

$$[M]\{\ddot{\vec{x}}\} + [C]\{\dot{\vec{x}}\} + [K]\{\vec{x}\} = \{\vec{F}\}$$

Proportional damping matrix

$$[C] = \alpha[M] + \beta[K]$$

The generalized force of the viscously damping

$$Q_i = \frac{-\partial R}{\partial \dot{x}_i}$$

In the generalized coordinates,

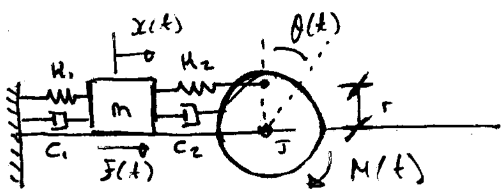
$$Q_j = \frac{-\partial R}{\partial \dot{q}_i}$$

The final equations of motion :

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} + \frac{\partial V}{\partial q_j} = -\frac{\partial R}{\partial \dot{q}_j} + Q_j$$

$$\boxed{\Rightarrow \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} + \frac{\partial R}{\partial \dot{q}_i} + \frac{\partial V}{\partial q_i} = Q_j}$$

Example:



Derive the equations of motion.

Solution : $q_1 = x(t)$; $q_2 = \theta(t)$

→ generalized coordinates

Kinetic :

$$T = (1/2)m\dot{x}^2 + (1/2)J\dot{\theta}^2$$

$$T = (1/2)m\dot{q}_1^2 + (1/2)J\dot{q}_2^2$$

Potential :

$$V = (1/2)K_1 x^2 + (1/2)K_2(r\theta - x)^2$$

$$V = (1/2)K_1 q_1^2 + (1/2)K_2(rq_2 - q_1)^2$$

Rayleigh's dissipation Function

$$R = (1/2)c_1\dot{x}^2 + (1/2)c_2(r\dot{\theta} - \dot{x})^2$$

$$= (1/2)c_1\dot{q}_1^2 + (1/2)c_2(r\dot{q}_1 - \dot{q}_2)^2$$

For q_1 :

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_1}\right) - \frac{\partial T}{\partial q_1} + \frac{\partial R}{\partial \dot{q}_1} + \frac{\partial V}{\partial q_1} = Q_1$$

$\rightarrow \frac{\partial T}{\partial \dot{q}_1} = m\dot{q}_1$; $\rightarrow \frac{\partial T}{\partial q_1} = \emptyset$

$\rightarrow \frac{\partial R}{\partial \dot{q}_1} = c\dot{q}_1 + C_2(r\dot{q}_2 - \dot{q}_1)\cdot(-1)$

$\rightarrow \frac{\partial V}{\partial q_1} = H_1 q_1 + H_2(rq_2 - q_1)(-1)$

$\rightarrow Q_1 = \mathfrak{F}(t)$

(1) $$m\ddot{q}_1 + C_1\dot{q}_1 + C_2(\dot{q}_1 - r\dot{q}_2) + H_1 q_1 + H_2(q_1 - rq_2) = \mathfrak{F}(t)$$

For q_2 :

$\rightarrow \frac{\partial T}{\partial \dot{q}_2} = J\dot{q}_2$; $\rightarrow \frac{\partial T}{\partial q_2} = \emptyset$

$\rightarrow \frac{\partial R}{\partial \dot{q}_2} = C_2(r\dot{q}_2 - \dot{q}_1)(r) = C_2 r(r\dot{q}_2 - \dot{q}_1)$

$\rightarrow \frac{\partial V}{\partial q_2} = H_2 r(rq_2 - q_1)$

$\rightarrow Q_2 = M(t)$

(2) $$J\ddot{q}_2 + C_2 r(r\dot{q}_2 - \dot{q}_1) + H_2 r(rq_2 - q_1) = M(t)$$

Putting (1) and (2) into matrix form :

$$\begin{bmatrix} m & 0 \\ 0 & J \end{bmatrix}\begin{Bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{Bmatrix} + \begin{bmatrix} C_1 + C_2 & -C_2 r \\ -C_2 r & C_2 r^2 \end{bmatrix}\begin{Bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{Bmatrix} + \begin{bmatrix} H_1 + H_2 & -H_2 r \\ -H_2 r & H_2 r^2 \end{bmatrix}\begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix} = \begin{Bmatrix} \mathfrak{F}(t) \\ M(t) \end{Bmatrix}$$

$$T = (1/2)m\dot{q}_1^2 + (1/2)J\dot{q}_2^2 = (1/2)(\dot{q}_1\ \dot{q}_2)\begin{bmatrix} m & 0 \\ 0 & J \end{bmatrix}\begin{Bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{Bmatrix}$$

Define $\vec{q} = \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix}$

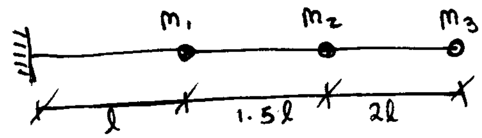
$$T = (1/2)\{\dot{\vec{q}}\}^T[M]\{\dot{\vec{q}}\}$$

$$V = (1/2)H_1 q_1^2 + (1/2)H_2(r^2 q_2^2 - 2rq_1q_2 + q_1^2)$$

$$= (1/2)\left[(H_1 + H_2)q_1^2 - 2H_2 r q_1 q_2 + H_2 r^2 q_2^2\right]$$

$$= (1/2)\{\vec{q}\}^T\begin{bmatrix} H_1 + H_2 & -H_2 r \\ -H_2 r & H_2 r^2 \end{bmatrix}\{\vec{q}\}$$

Example



$m_1 = 3m$; $m_2 = 2m$; $m_3 = m$; EI = const.

Find the natural frequencies and mode shapes :

Solution :

$$[A] = \frac{l^3}{24EI}\begin{bmatrix} 8 & 26 & 50 \\ 26 & 125 & 275 \\ 50 & 275 & 729 \end{bmatrix}$$

$$= \frac{l^3}{EI}\begin{bmatrix} 0.33333 & 1.083333 & 2.08333 \\ & 5.20833 & 11.4583 \\ \text{Sym.} & & 3.03750 \end{bmatrix}$$

The Stiffness matrix

$$[K] = [A]^{-1}$$

$$[K] = \frac{EI}{l^3}\begin{bmatrix} 11.0399 & -3.70655 & 0.641026 \\ & 2.37322 & -0.641026 \\ \text{Sym.} & & 0.230769 \end{bmatrix}$$

Mass matrix :

$$[M] = \begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix} = m\begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Natural Freq. and mode shape :

$$(-\omega^2 [M] + [K])\{\vec{u}\} = 0$$

$$\Rightarrow \left(-m\omega^2\begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \frac{EI}{l^3}\begin{bmatrix} x & x & x \\ x & x & x \\ x & x & x \end{bmatrix}\right)\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = 0$$

$$\Rightarrow \left(\begin{bmatrix} x & x & x \\ x & x & x \\ x & x & x \end{bmatrix} - \frac{ml^3}{EI}\omega^2\begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = 0$$

Define : $\lambda = \dfrac{m\ell^3}{EI}\omega^2$

$$\Rightarrow \left(\begin{bmatrix} \times & \times & \times \\ \times & \times & \times \\ \times & \times & \times \end{bmatrix} - \lambda \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = 0$$

$\Rightarrow \lambda_1 = 0.02502413$; $\omega_1 = \sqrt{EI/m\ell^3} \cdot \sqrt{\lambda_1}$

$\lambda_2 = 0.612216$

$\lambda_3 = 4.46008$

$\Rightarrow \omega_1 = 0.158224 \sqrt{EI/m\ell^3}$

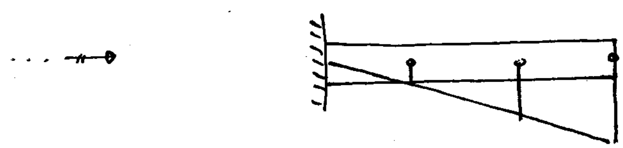
$\omega_2 = 0.782442 \sqrt{EI/m\ell^3}$

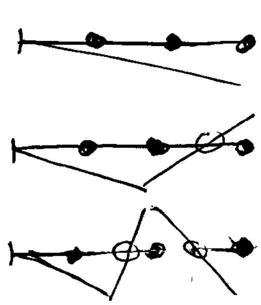
$\omega_3 = 2.11189 \sqrt{EI/m\ell^3}$

The modal shapes :

$$\{\vec{u}_i\} = \begin{Bmatrix} 0.0654970 \\ 0.343615 \\ 0.866597 \end{Bmatrix} ; \begin{Bmatrix} 0.25073 \\ 0.537842 \\ -0.483263 \end{Bmatrix} ; \begin{Bmatrix} 0.51646 \\ -0.304393 \\ 0.121367 \end{Bmatrix}$$

↳ 0 sign change ↳ 1 sign change ↳ 2 sign change





mode 1

mode 2

mode 3

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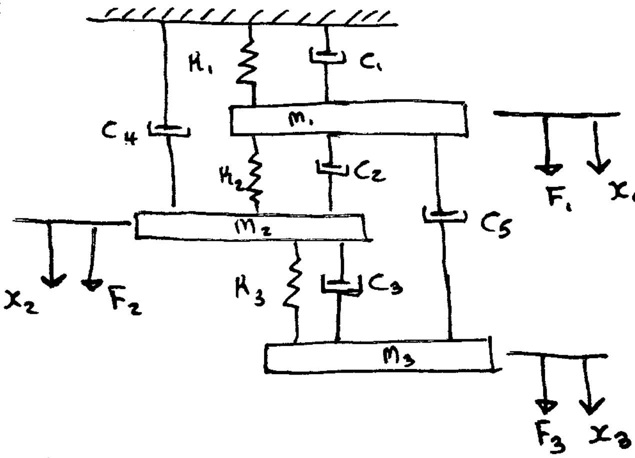
Modal summation

Example:



Find the eq'n's of motion and calculate the forced response.

Solution:

Kinetic energy: $T = (1/2) m_1 \dot{x}_1^2 + (1/2) m_2 \dot{x}_2^2 + (1/2) m_3 \dot{x}_3^2$

Potential energy: $V = (1/2) K_1 x_1^2 + (1/2) K_2 (x_2 - x_1)^2 + (1/2) K_3 (x_3 - x_2)^2$

Rayleigh's Formula: $R = (1/2) C_1 \dot{x}_1^2 + (1/2) C_2 (\dot{x}_2 - \dot{x}_1)^2 + (1/2) C_3 (\dot{x}_3 - \dot{x}_2)^2 + \ldots$

$\ldots (1/2) C_4 \dot{x}_2^2 + (1/2) C_5 (\dot{x}_3 - \dot{x}_1)^2$

The generalized force

$$\vec{Q} = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix}$$

Equations of Motion:

$$(d/dt)(\partial T/\partial \dot{q}_1) + (\partial V/\partial q_1) - (\partial T/\partial q_1) + (\partial R/\partial \dot{q}_1) = Q$$

where: $q_1 = x_1$

then $(\partial V/\partial q_1) = K_1 x_1 + K_2 (x_1 - x_2)$

$(\partial R/\partial \dot{q}_1) = C_1 \dot{x}_1 + C_2 (\dot{x}_1 - \dot{x}_2) + C_5 (\dot{x}_1 - \dot{x}_3)$

$$\therefore \quad m_1 \ddot{x}_1 + c_1 \dot{x}_1 + c_2(\dot{x}_1 - \dot{x}_2) + c_5(\dot{x}_1 - \dot{x}_3) + k_1 x_2 + k_2(x_1 - x_2) = F_1$$

. . . etc. For other equations

Matrix Form :

$$[M] = \begin{bmatrix} m_1 & \emptyset & \emptyset \\ \emptyset & m_2 & \emptyset \\ \emptyset & \emptyset & m_3 \end{bmatrix}$$

$$[C] = \begin{bmatrix} c_1 + c_2 + c_5 & -c_2 & -c_5 \\ -c_2 & c_2 + c_3 + c_4 & -c_3 \\ -c_5 & -c_3 & c_3 + c_5 \end{bmatrix}$$

$$[K] = \begin{bmatrix} k_1 + k_2 & -k_2 & \emptyset \\ -k_2 & k_2 + k_3 & -k_3 \\ \emptyset & -k_3 & k_3 \end{bmatrix}$$

Given:

$$m_1 = m_2 = m_3 = m$$
$$k_1 = k_2 = k_3 = k$$
$$c_4 = c_5 = \emptyset \quad ; \quad \boxed{\zeta_1 = \zeta_2 = \zeta_3 = \emptyset.\emptyset 1}$$
$$F_1 = F_2 = F_3 = F_o \cos(\omega t)$$
$$\hookrightarrow \omega = 1.75 \sqrt{k/m}$$

Step 1 : The natural frequencies and modal shape (w/o damping)

$$[M] = m \begin{bmatrix} 1 & \emptyset & \emptyset \\ \emptyset & 1 & \emptyset \\ \emptyset & \emptyset & 1 \end{bmatrix}$$

$$[K] = k \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ \emptyset & -1 & 1 \end{bmatrix}$$

$$\Rightarrow \left(-\omega^2 m \begin{bmatrix} 1 & \emptyset & \emptyset \\ \emptyset & 1 & \emptyset \\ \emptyset & \emptyset & 1 \end{bmatrix} + k \begin{bmatrix} 2 & -1 & \emptyset \\ -1 & 2 & -1 \\ \emptyset & -1 & 1 \end{bmatrix} \right) \vec{u} = \emptyset$$

After solving:

$\omega_1 = 0.44504 \times \sqrt{K/m}$

$\omega_2 = 1.2470 \times \sqrt{K/m}$

$\omega_3 = 1.8019 \times \sqrt{K/m}$

$$\vec{u}_1 = \frac{1}{\sqrt{m}} \begin{Bmatrix} 0.32799 \\ 0.59101 \\ 0.73698 \end{Bmatrix}$$

$$\vec{u}_2 = \frac{1}{\sqrt{m}} \begin{Bmatrix} 0.73698 \\ 0.32799 \\ -0.59101 \end{Bmatrix}$$

$$\vec{u}_3 = \frac{1}{\sqrt{m}} \begin{Bmatrix} 0.59101 \\ -0.73688 \\ 0.32799 \end{Bmatrix}$$

Step 2 : $\vec{U} = [\vec{u}_1 \ \vec{u}_2 \ \vec{u}_3]_{3\times3}$

Verify : $[U]^T[M][U] = [I]$

$[U]^T[K][U] = \mathrm{diag}(\omega_1^2, \omega_2^2, \omega_3^2)$

Define : $\vec{x} = [U]\vec{q}$

The equations of motion

$$[M]\ddot{\vec{x}} + [C]\dot{\vec{x}} + [K]\vec{x} = \vec{Q}$$

$$[M][U]\ddot{\vec{q}} + [C][U]\dot{\vec{q}} + [K][U]\vec{q} = \vec{Q}$$

$$[U]^T[M][U]\ddot{\vec{q}} + [U]^T[C][U]\dot{\vec{q}} + [U]^T[K][U]\vec{q} = [U]^T\vec{Q}$$

$$\ddot{\vec{q}} + [U]^T[C][U]\dot{\vec{q}} + \mathrm{diag}(\omega_1^2 \omega_2^2 \omega_3^3)\vec{q} = [U]^T\vec{Q}$$

Proportional damping :

$$[U]^T[C][U] = \mathrm{diag}(2\zeta_1\omega_1, \ 2\zeta_2\omega_2, \ 2\zeta_3\omega_3)$$

$$\begin{Bmatrix} Q_{10} \\ Q_{20} \\ Q_{30} \end{Bmatrix} = [U]^T\vec{Q} = [U]^T \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix}$$

$$= [U]^T F_0 \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix} \cos(\omega t) = \frac{F_0}{\sqrt{m}} \begin{Bmatrix} 1.6560 \\ 0.47395 \\ 0.18202 \end{Bmatrix} \cos(\omega t)$$

Equations in the Modal Coordinates

$$\begin{cases} \ddot{q}_1 + 2\zeta_1\omega_1\dot{q}_1 + \omega_1^2 q_1 = Q_{10} = (F_0/\sqrt{m})(1.6560)\cos(\omega t) \\ \ddot{q}_2 + 2\zeta_2\omega_2\dot{q}_2 + \omega_2^2 q_2 = Q_{20} = (F_0/\sqrt{m})(0.47395)\cos(\omega t) \\ \ddot{q}_3 + 2\zeta_3\omega_3\dot{q}_3 + \omega_3^2 q_3 = Q_{30} = (F_0/\sqrt{m})(0.18202)\cos(\omega t) \end{cases}$$

Step 3 : Steady-state response of each modal coordinate

$$\boxed{q_i(t) = q_{i0}\cos(\omega t - \phi_i)} \quad ; \quad i = 1, 2, 3$$

and

$$q_{i0} = \frac{Q_{i0}}{\omega_i^2} \cdot \frac{1}{\sqrt{(1-(\omega/\omega_i)^2)^2 + (2\zeta_i\,\omega/\omega_i)^2}}$$

$$\phi_i = \tan^{-1}\left(\frac{2\zeta(\omega/\omega_i)}{1-(\omega/\omega_i)^2}\right)$$

$i = 1$:

$$\frac{\omega}{\omega_1} = \frac{1.75\sqrt{K/m}}{0.44504\sqrt{K/m}} = 3.9322$$

$$q_{10} = 0.57811\,(F_0\sqrt{m}/K)$$

$$\phi_1 = 3.1367$$

$i = 2$:

$$\frac{\omega}{\omega_2} = \frac{1.75}{1.2470} = 1.4034$$

$$q_{20} = 1.0980$$

$$\phi_2 = 3.1187$$

$i = 3$:

$$\frac{\omega}{\omega_3} = \frac{1.75}{1.809} = 0.97118$$

$$q_{30} = 8.4938$$

$$\phi_3 = 0.32941$$

$$\vec{x} = [U]\,\vec{q}$$

$$= [U]\begin{Bmatrix} q_1(t) \\ q_2(t) \\ q_3(t) \end{Bmatrix}$$

$$= [U]\begin{Bmatrix} q_{10}\cos(\omega t - \phi_1) \\ q_{20}\cos(\omega t - \phi_2) \\ q_{30}\cos(\omega t - \phi_3) \end{Bmatrix}$$

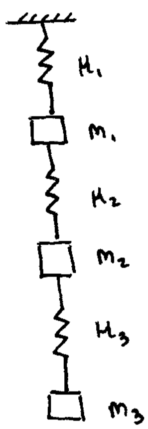
$$= \frac{1}{\sqrt{m}} \begin{bmatrix} 0.32799 & 0.73098 & 0.59101 \\ 0.59101 & 0.32799 & -0.73098 \\ 0.73098 & -0.59101 & 0.32799 \end{bmatrix} \begin{Bmatrix} 0.57811 \cos(\omega t - 3.1362) \\ 10.980 \cos(\omega t - 3.112) \\ 8.4930 \cos(\omega t - 0.32944) \end{Bmatrix} \cdot \frac{F_0 \sqrt{m}}{K}$$

$$= \frac{F_0}{K} \begin{Bmatrix} 3.7515 \cdot \cos \omega t + 1.6483 \sin(\omega t) \\ -6.6248 \cdot \cos \omega t - 2.0127 \sin(\omega t) \\ 2.8587 \cdot \cos \omega t + 0.88472 \sin(\omega t) \end{Bmatrix}$$

Determination of natural Frequencies

Dunkerley's Formula :

the Fundamental natural freq.



$$\left(-\omega^2 [M] + [K]\right) \vec{u} = 0$$

The Flexibility matrix

$$[A] = [K]^{-1}$$
$$[K] = [A]^{-1}$$

$$\longrightarrow [A]\left(-\omega^2 [M] + [K]\right) \vec{u} = 0$$
$$\left(-\omega^2 [A][M] + [A][K]\right) \vec{u} = 0$$

$$\longrightarrow \left(-[A][M] + (1/\omega^2)[I]\right) \vec{u} = 0$$

Eigenvalues :

$$\left| -[A][M] + (1/\omega^2)[I] \right| = 0$$

$$[A] = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \quad ; \quad [M] = \begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix}$$

$$[A][M] = \begin{bmatrix} a_{11}m_1 & a_{12}m_2 & a_{13}m_3 \\ a_{21}m_1 & a_{22}m_2 & a_{23}m_3 \\ a_{31}m_1 & a_{32}m_2 & a_{33}m_3 \end{bmatrix}$$

$$\Rightarrow \begin{vmatrix} a_{11}m_1 - (1/\omega^2) & a_{12}m_2 & a_{13}m_3 \\ a_{21}m_1 & a_{22}m_2 - (1/\omega^2) & a_{23}m_3 \\ a_{31}m_1 & a_{32}m_2 & a_{33}m_3 - (1/\omega^2) \end{vmatrix} = \varnothing$$

$$\Rightarrow \left[a_{11}m_1 - (1/\omega^2)\right]\left[a_{22}m_2 - (1/\omega^2)\right]\left[a_{33}m_3 - (1/\omega^2)\right] \cdots$$

$$\cdots + a_{12}m_2 a_{23}m_3 + a_{31}m_1 + a_{21}m_1 a_{32}m_2 a_{13}m_3 \cdots$$

$$\cdots - a_{13}m_3 a_{31}m_1 (a_{22}m_2 - 1/\omega^2) \cdots$$

$$\cdots - a_{12}m_2 a_{21}m_2 (a_{33}m_3 - 1/\omega^2) \cdots$$

$$\cdots - a_{22}m_3 a_{32}m_2 (a_{11}m_1 - 1/\omega^2) = \varnothing$$

Cubic eq. w.r.t. $1/\omega^2$

$$(1/\omega^2)^3 - (a_{11}m_1 + a_{22}m_2 + a_{33}m_3)(1/\omega^2)^2 + (\cdots)(1/\omega^2) + (\cdots) = \varnothing$$

The three roots : $1/\omega_1^2$; $1/\omega_2^2$; $1/\omega_3^2$

$$\boxed{1/\omega_1^2 + 1/\omega_2^2 + 1/\omega_3^2 = a_{11}m_1 + a_{22}m_2 + a_{33}m_3}$$

$$(1/\omega^2 - 1/\omega_1^2)(1/\omega^2 - 1/\omega_2^2)(1/\omega^2 - 1/\omega_3^2) = \varnothing$$

When $\omega_2 \gg \omega_1$, $\omega_3 \gg \omega_1$

$$1/\omega_1^2 \approx 1/\omega_1^2 + 1/\omega_2^2 + 1/\omega_3^2 = a_{11}m_1 + a_{22}m_2 + a_{33}m_3$$

$$\therefore \quad \omega_1 = \frac{1}{\sqrt{a_{11}m_1 + a_{22}m_2 + a_{33}m_3}}$$

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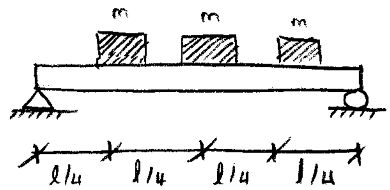
$$1/\omega_1^2 + 1/\omega_2^2 + \dots + 1/\omega_n^2 = a_{11}m_1 + a_{22}m_2 + \dots + a_{nn}m_n$$

$$\boxed{1/\omega_1^2 \approx a_{11}m_1 + \dots + a_{nn}m_n}$$

Example



$EI = \text{const.}$

Estimate the Fundamental Frequency.

Solution: $a_{11} = a_{33} = \left(\frac{3}{256}\right)\left(l^3/EI\right)$ — by symmetry

$a_{22} = (1/48)(l^3/EI)$

$\therefore \quad 1/\omega_1^2 = a_{11}m_1 + a_{22}m_2 + a_{33}m_3$

$= (3/256)(l^3/EI)m_1 + (1/48)(l^3/EI)m_2 + (3/256)(l^3/EI)m_3$

$1/\omega_1^2 = 0.04427\,(l^3/EI)m$

$\omega_1 = 4.754\sqrt{EI/ml^3}$

The exact solution:

$\omega_{1,exact} = 4.734\sqrt{EI/ml^3}$??? (should always be larger than ω_1)

$$1/\omega_{1,d}^2 = 1/\omega_{1,exact}^2 + 1/\omega_2^2 + \dots + 1/\omega_n^2 > 1/\omega_{1,exact}^2$$

Rayleigh's Method

$[M]\ddot{\vec{x}} + [K]\vec{x} = 0$

The motion:

$\vec{x} = e^{i\omega t}\vec{u}$

ω: the natural freq.

$\vec{u}$: the mode shape

$\Rightarrow \quad ([K] - \omega^2[M])\vec{u} = 0$

$\Rightarrow \quad \vec{u}^T([K] - \omega^2[M])\vec{u} = 0$

$\Rightarrow \quad \vec{u}^T[K]\vec{u} - \omega^2\vec{u}^T[M]\vec{u} = 0$

$\Rightarrow \quad \omega^2 = \dfrac{\vec{u}^T[K]\vec{u}}{\vec{u}^T[M]\vec{u}}$

Define: $R(\vec{x}) = \dfrac{\vec{x}^T[K]\vec{x}}{\vec{x}^T[M]\vec{x}}$

If x is close to modal shape, then it approximates the natural frequency

$T = (1/2)\,\dot{\vec{x}}^T [M]\, \dot{\vec{x}}$

$V = (1/2)\,\vec{x}^T [K]\, \vec{x}$

→ Harmonic motion:

$\vec{x} = \vec{X} e^{i\omega t}$

$\dot{\vec{x}} = \vec{X} \cdot i\omega e^{i\omega t}$

→ The max kinetic energy:

$T_{max} = (1/2)\,\omega^2 \vec{X}^T [M] \vec{X}$

→ The max potential energy:

$V_{max} = (1/2)\,\vec{X}^T [K] \vec{X}$

<u>Conservation:</u>

$T_{max} = V_{max}$

$(1/2)\,\omega^2 \vec{X}^T [M] \vec{X} = (1/2)\,\vec{X}^T [K] \vec{X}$

→ $$\omega^2 = \frac{\vec{X}^T [K] \vec{X}}{\vec{X}^T [M] \vec{X}}$$

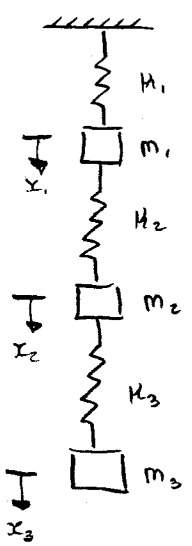
<u>Verify:</u>

$\omega_1^2 \le R(\vec{x}) \le \omega_n^2$

$\vec{X}$: the static deflection (disp.) of the system

Then $R(\vec{x})$: approximation of the Fundamental Freq.

$\boxed{\text{Example}}$



Let $m_1 = m_2 = m_3 = m$

$k_1 = k_2 = k_3 = k$

Estimate the Fundamental Freq. of the system.

<u>Solution</u> :

$$[M] = \begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{bmatrix} = m \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$[K] = k \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

Take $$\vec{x} = \begin{Bmatrix} 1 \\ 2 \\ 3 \end{Bmatrix} = \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}$$

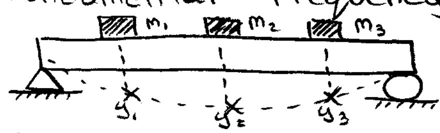
$$R(\vec{x}) = \frac{\vec{x}^T [K] \vec{x}}{\vec{x}^T [M] \vec{x}} = \frac{(1,2,3)\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}\begin{Bmatrix} 1 \\ 2 \\ 3 \end{Bmatrix}}{(1,2,3)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{Bmatrix} 1 \\ 2 \\ 3 \end{Bmatrix}} \frac{k}{m}$$

$$= 0.2143 \, (k/m)$$

$$\therefore \quad \omega_1^2 \simeq R(x) = 0.2143 \frac{k}{m}$$

$$\omega_1 = 0.4629 \sqrt{k/m} \qquad (\omega_{1,\,exact} = 0.4450 \sqrt{k/m})$$

Fundamental Frequency of beams and shafts



beam / shaft : weightless

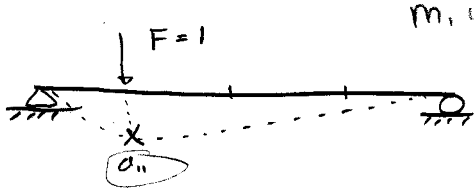
The max potential energy = the max strain energy
= the work done by all the forces

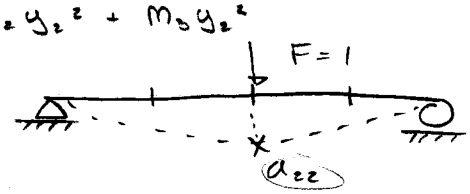
$$V_{max} = (1/2)(m_1 g y_1 + m_2 g y_2 + m_3 g y_3)$$

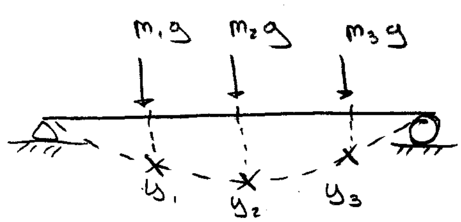
$$T_{max} = (1/2)\omega^2 (m_1 y_1^2 + m_2 y_2^2 + m_3 y_3^2)$$

$$\Rightarrow (1/2)(m_1 g y_1 + m_2 g y_2 + m_3 g y_3) = (1/2)\omega^2 (m_1 y_1^2 + m_2 y_2^2 + m_3 y_3^2)$$

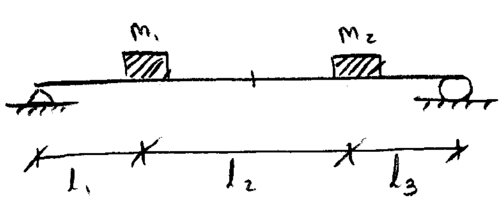
$$\rightarrow \omega^2 = \frac{m_1 g y_1 + m_2 g y_2 + m_3 g y_3}{m_1 y_1^2 + m_2 y_2^2 + m_3 y_3^2}$$







$\boxed{\text{Example:}}$ Estimate the fundamental freq. of the beam as shown.

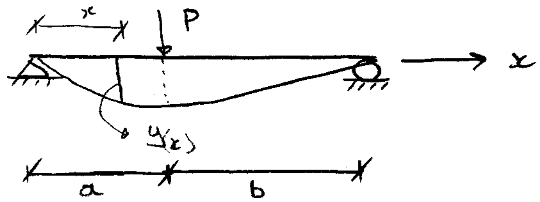


where $l_1 = 1\,m$; $l_2 = 3\,m$; $l_3 = 2\,m$

$m_1 = 20\,kg$; $m_2 = 50\,kg$

$EI = const.$

Solution :

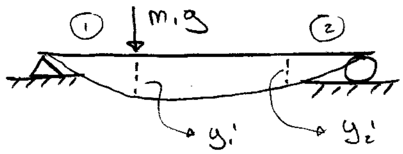


The deflection :

$$y(x) = \begin{cases} \dfrac{Pbx}{6EIl}\left(l^2 - b^2 - x^2\right) & ; \quad 0 \le x \le a \\ -\dfrac{Pa(l-x)}{6EIl}\left(a^2 + x^2 - 2lx\right) & ; \quad a \le x \le 6 \end{cases}$$

→ The deflection due to $P = m_1 g$

$l = l_1 + l_2 + l_3 = 6\,m$

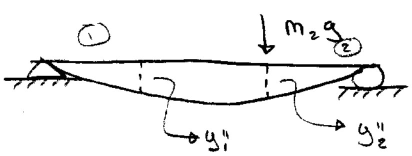


$$y_1' = \frac{Pbx}{6EIl}\left(l^2 - b^2 - x^2\right)\Big|_{x=1} = \frac{(20)(9.81)(5)(1)}{6EI(6)}\left(6^2 - 5^2 - 1^2\right)$$

$$= \frac{272.5}{EI}$$

$$y_2' = -\frac{(20)(9.81)(1)(6-4)}{6EI(6)}\left(1^2 + 4^2 - 2 \times 6 \times 4\right) = \frac{337.9}{EI}$$

→ The deflection due to $P = m_2 g$



$$y_1'' = \frac{844.75}{EI}$$

$$y_2'' = \frac{1744.0}{EI}$$

The total displacement :

$$y_1 = y_1' + y_1'' = (272.5/EI) + (844.75/EI) = (1117.25/EI)$$

$$y_2 = y_2' + y_2'' = (337.9/EI) + (1744.0/EI) = (2081.9/EI)$$

$$\therefore \quad \omega_1^2 = \frac{(m_1 y_1 + m_2 y_2) g}{m_1 y_1^2 + m_2 y_2^2}$$

$$\Rightarrow \quad \omega_1 = 0.07166 \sqrt{EI}$$

Dunkerley's Formula :

$$a_{11} = \frac{y_1'}{m_1 g} \quad ; \quad a_{22} = \frac{y_2''}{m_2 g}$$

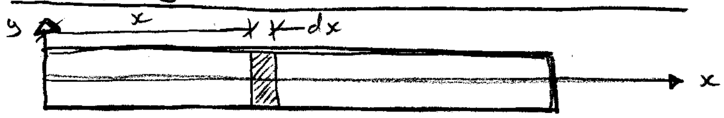
$$1/\omega_1^2 = a_{11} m_1 + a_{22} m_2$$

$$= \frac{y_1'}{m_1 g} m_1 + \frac{y_2''}{m_2 g} m_2$$

$$= \frac{1}{9.81} \left(\frac{272.5}{EI} + \frac{1744.0}{EI} \right)$$

$$\Rightarrow \quad \omega_1 = 0.06974 \sqrt{EI}$$

Rayleigh's Method For a beam



mass density $\rho(x)$

area of cross-section $A(x)$

bending stiffness $EI(x)$

$$y(x,t) = Y(x) e^{i\omega t}$$

Max Kinetic energy :

$$T_{max} = (1/2)\, \omega^2 \int_0^l \rho A \, dx \cdot Y(x)^2$$

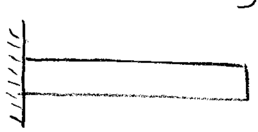
$$= (1/2)\, \omega^2 \int_0^l \rho A Y^2 \, dx$$

Max potential energy :

$$V_{max} = (1/2) \int_0^l EI (Y'')^2 \, dx$$

$$(Y'' = d^2 Y / dx^2)$$

$$\therefore \quad \omega^2 = \frac{\int_0^l EI (Y'')^2 \, dx}{\int_0^l \rho A Y^2 \, dx}$$



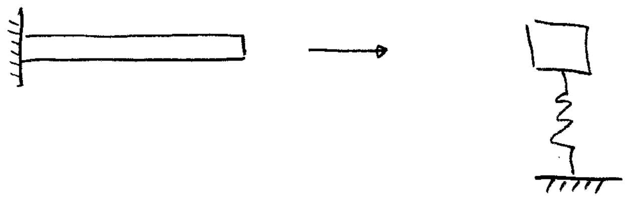
ρA , EI const.

$$y = \frac{-P x^2}{6EIl} (3l - x)$$

$$\omega^2 = (140/11) \ldots$$

$$\omega = 3.56753\sqrt{EI/\rho A l^4}$$

$$\omega_{1,exact} = 3.51602\sqrt{EI/\rho A l^4}$$



$$M = \frac{33}{140}\rho A l$$

$$k = \frac{3EI}{l^3}$$

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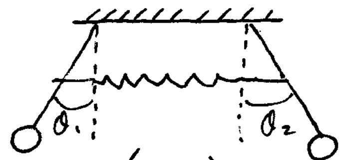
; where $\frac{\theta_1}{\theta_2} = 1$

(ω_1)



(ω_2)

beam bending

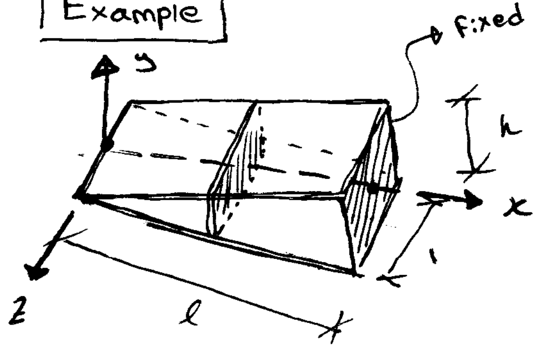
$$\omega^2 = \frac{\int_0^l EI(y'')^2\,dx}{\int_0^l \rho A y^2\,dx}$$

Here, $y = y(x)$: the assumed deflection

$y(x)$: the static deflection

ω^2 : an approximation of the fundamental freq. of the beam

$\boxed{\text{Example}}$



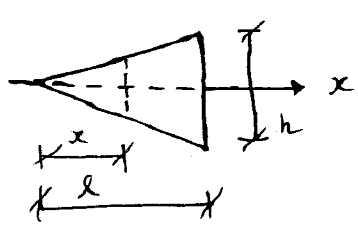
E is constant, estimate first natural frequency.

Solution : $y(x)$ trial function

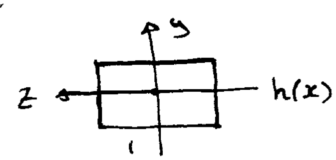
then $y(l) = 0$, $y'(l) = 0$

Let $y(x) = \left[1 - (x/l)\right]^2$

→ side view



$h(x) = \left(\frac{x}{l}\right)h$



$$I = (1/12)(1)(h(x)^3)$$
$$= (1/12)\left(\frac{x}{l}h\right)^3$$

Since $y'' = 2/\ell^2$

$$\therefore \quad \omega^2 = \frac{\int_0^\ell E\cdot(1/12)(x/\ell\, h)^3 \cdot (2/\ell^2)^2\, dx}{\int_0^\ell \rho \frac{x}{\ell} h \left[(1-\frac{x}{\ell})^2\right]^2 dx} = 2.5\,\frac{Eh^2}{\rho \ell^4}$$

$$\omega = 1.5811\sqrt{\frac{Eh^2}{\rho\ell^4}} \quad \& \quad \omega_{exact} = 1.5343\sqrt{\frac{Eh^2}{\rho\ell^4}}$$

— Matrix iteration method

$$[M]\ddot{\vec{x}} + [K]\vec{x} = 0$$

The natural Frequency and mode shape:

$$(-\omega^2[M] + [K])\vec{X} = 0$$

$$[K]^{-1}(-\omega^2[M] + [K])\vec{X} = 0$$

$$\Rightarrow (-\omega^2[K]^{-1}[M] + [I])\vec{X} = 0$$

Define: $[D] = [K]^{-1}[M]$; $\lambda = 1/\omega^2$

$$\Rightarrow ([D] - \lambda[I])\vec{X} = 0$$

then $[D]\vec{X} = \lambda\vec{X}$

1° : $\vec{X} = \vec{X}_1 \; (\neq 0)$

2° :

$$[D]\vec{X}_1 = \vec{X}_2$$

$$[D]\vec{X}_2 = \vec{X}_3$$

$$\vdots$$

$$[D]\vec{X}_r = \vec{X}_{r+1} \approx \lambda\vec{X}_r$$

$$\vec{X}_r = \begin{Bmatrix} X_{1r} \\ X_{2r} \\ \vdots \\ X_{nr} \end{Bmatrix} \qquad \vec{X}_{r+1} = \begin{Bmatrix} X_{1,r+1} \\ X_{2,r+1} \\ \vdots \\ X_{n,r+1} \end{Bmatrix}$$

$$\frac{X_{1,r+1}}{X_{1,r}} = \frac{X_{2,r+1}}{X_{2,r}} = \cdots = \frac{X_{n,r+1}}{X_{n,r}} \approx \frac{X_{n,r+1}}{X_{n,r}} \approx \lambda$$

Given $[D]$: $\lambda_1, \lambda_2, \ldots, \lambda_n$
$\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_n$

Then

$$\vec{X}_1 = c_1\vec{u}_1 + c_2\vec{u}_2 + \cdots + c_n\vec{u}_n \qquad (c_n \neq 0)$$

$$\vec{X}_2 = [D]\vec{X}_1 = c_1[D]\vec{u}_1 + c_2[D]\vec{u}_2 + \cdots + c_n[D]\vec{u}_n$$

$$\vec{X}_2 = c_1\lambda_1\vec{u}_1 + c_2\lambda_2\vec{u}_2 + \cdots + c_n\lambda_n\vec{u}_n$$

$$\vec{X}_3 = [D]\vec{X}_2 = c_1\lambda_1^2\vec{u}_1 + c_2\lambda_2^2\vec{u}_2 + \cdots + c_n\lambda_n^2\vec{u}_n$$

$\vdots$ (thus ...)

$$\vec{X}_r = c_1\lambda_1^{r-1}\vec{u}_1 + c_2\lambda_2^{r-1}\vec{u}_2 + \cdots + c_n\lambda_n^{r-n}\vec{u}_n$$

$$\vec{X}_{r+1} = c_1\lambda_1^{r}\vec{u}_1 + c_2\lambda_2^{r}\vec{u}_2 + \cdots + c_n\lambda_n^{r}\vec{u}_n$$

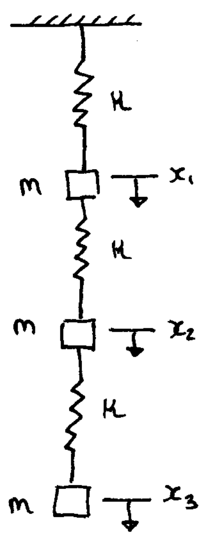
When $\vec{X}$ is large enough :

$$\vec{X}_r \approx c_n\lambda_n^{r-1}\vec{u}_n$$

$$\vec{X}_{r-1} \approx c_n\lambda_n^{r}\vec{u}_n$$

$$(\lambda_1 < \lambda_2 < \cdots < \lambda_n)$$

$\boxed{\text{Examples}}$ Find the natural freq. using iteration method.



Solution:

$$[M] = \begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{bmatrix} = m\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$[K] = k\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

$$[D] = [K]^{-1}[M] = \frac{m}{k}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$$

Consider : $\left([D_1] - \lambda[I]\right)x = 0$

Let : $\lambda = \frac{k}{m}\cdot\frac{1}{\omega^2}$ $\Rightarrow [D_1] = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$

(for this example only)

Take $\quad \vec{X_1} = \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix}$

Then $\quad \vec{X_2} = [D]\vec{X_1} = \begin{Bmatrix} 3 \\ 5 \\ 6 \end{Bmatrix} = 3 \begin{Bmatrix} 1 \\ 1.666\overline{7} \\ 2 \end{Bmatrix}$

$$\vec{X_3} = [D]\vec{X_2} = [D] \begin{Bmatrix} 1 \\ 1.666\overline{7} \\ 2 \end{Bmatrix} = \begin{Bmatrix} 4.6667 \\ 8.3333 \\ 10.3333 \end{Bmatrix}$$

$$= (4.6667) \begin{Bmatrix} 1 \\ 1.7857 \\ 2.2143 \end{Bmatrix}$$

$$\vec{X_4} = [D]\vec{X_3} = [D] \begin{Bmatrix} 1 \\ 1.7857 \\ 2.2143 \end{Bmatrix} = \begin{Bmatrix} 5.0000 \\ 9.0000 \\ 11.2143 \end{Bmatrix}$$

$$= (5.0000) \begin{Bmatrix} 1 \\ 1.80000 \\ 2.24286 \end{Bmatrix}$$

After 9 iterations :

$$= (5.04892) \begin{Bmatrix} 1.00000 \\ 1.80194 \\ 2.24698 \end{Bmatrix}$$

$$\therefore \lambda = 5.04892, \quad \vec{u} = \begin{Bmatrix} 1 \\ 1.80194 \\ 2.24698 \end{Bmatrix}$$

$$\therefore \omega = \sqrt{\frac{1}{\lambda}} \cdot \sqrt{\frac{k}{m}} = 0.44504 \sqrt{k/m}$$

The largest Freq:

$$(-\omega^2[M] + [K])\vec{X} = \emptyset$$

$$\Rightarrow (-\omega^2[I] + [M]^{-1}[K])\vec{X} = \emptyset$$

$$\Rightarrow [M]^{-1}[K]\vec{X} = \omega^2 \vec{X}$$

Define : $[E] = [M]^{-1}[K]$

$$[E]\vec{X} = \omega^2\vec{X} = \lambda\vec{X}$$

Iteration : $X_2 = [E]\vec{X_1}$

$$X_3 = [E]\vec{X_2}$$

$$\vdots$$

$$X_{r+1} = [E]\vec{X_r}$$

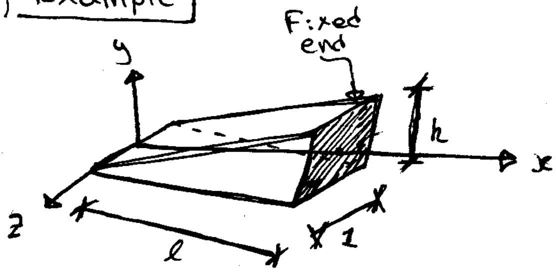
$$\downarrow$$

ω_n and $\vec{u_n}$

Rayleigh - Ritz Method :

$$\omega^2 = \frac{\int_0^l EI(y'')^2\,dx}{\int_0^l \rho A\, y^2\,dx}$$

$\boxed{\text{Example}}$



Solution :

$y_1 = (1 - x/l)^2$

$y_2 = (1 - x/l)^2 (x/l)$

Let $y(x) = c_1 y_1 + c_2 y_2$

Then $$\omega^2 = \frac{\int_0^l EI(c_1 y_1'' + c_2 y_2'')^2\,dx}{\int_0^l \rho A (c_1 y_1 + c_2 y_2)^2\,dx}$$

$$\rightarrow \quad \omega^2(c_1, c_2) = \frac{P(c_1, c_2)}{Q(c_1, c_2)}$$

where $\omega^2(c_1, c_2)$: stationary value

$$\frac{\partial \omega^2}{\partial c_1} = 0 \quad ; \quad \frac{\partial \omega^2}{\partial c_2} = 0$$

$$\frac{\partial \omega^2}{\partial c_1} = \frac{\partial}{\partial c_1}\left(\frac{P}{Q}\right) = \frac{\partial P}{\partial c_1}\cdot\frac{1}{Q} + P\cdot\left(\frac{-1}{Q^2}\right)\frac{\partial Q}{\partial c_1} = 0$$

$$\Rightarrow \quad \frac{\partial P}{\partial c_1} - \frac{P}{Q}\frac{\partial Q}{\partial c_1} = 0$$

$$\frac{\partial P}{\partial c_1} - \omega^2 \frac{\partial Q}{\partial c_1} = 0$$

$$P = \int_0^l EI(c_1 y_1'' + c_2 y_2'')^2\,dx$$

$$= \int_0^l EI(c_1^2 y_1''^2 + 2c_1 c_2 y_1'' y_2'' + c_2^2 y_2''^2)\,dx$$

$$= c_1^2 \int_0^l EI y_1''^2\,dx + 2c_1 c_2 \int_0^l EI y_1'' y_2''\,dx + c_2^2 \int_0^l EI\, y_2''\,dx$$

$$\rightarrow \quad \frac{\partial P}{\partial c_1} = 2c_1 \int_0^l EI y_1''^2\,dx + 2c_2 \int_0^l EI y_1'' y_2''\,dx + 0$$

$$Q = \int_0^l \rho A (c_1 y_1 + c_2 y_2)^2\,dx$$

$$= c_1^2 \int_0^l \rho A y_1^2\,dx + 2c_1 c_2 \int_0^l \rho A y_1 y_2\,dx + c_2 \int_0^l \rho A y_2^2\,dx$$

$$\rightarrow \quad \frac{\partial Q}{\partial c_1} = 2c_1 \int_0^l \rho A y_1^2\,dx + 2c_2 \int_0^l \rho A\, y_1 y_2\,dx + 0$$

$$\cancel{2}C_1\int_0^l EI\,{y_1''}^2\,dx + \cancel{2}C_2\int_0^l EI\,y_1''y_2''\,dx - \omega^2\left(\cancel{2}C_1\int_0^l \rho A y_1^2\,dx + \cancel{2}C_2\int_0^l \rho A y_1y_2\,dx\right)=0$$

where $\dfrac{\partial \omega^2}{\partial C_2} = 0 \rightarrow \dfrac{\partial P}{\partial C_2} - \omega^2\dfrac{\partial Q}{\partial C_2} = 0$

$$C_1\int_0^l EI\,y_1''y_2''\,dx + C_2\int_0^l EI\,{y_2''}^2\,dx \cdots$$
$$\cdots - \omega^2\left(C_1\int_0^l \rho A y_1y_2\,dx + C_2\int_0^l \rho A y_2^2\,dx\right) = 0$$

Matrix form:

$$\left([K] - \omega^2[M]\right)\begin{Bmatrix} C_1 \\ C_2 \end{Bmatrix} = 0$$

Here:

same

$$[K] = \begin{bmatrix} \int_0^l EI\,{y_1''}^2\,dx & \int_0^l EI\,y_1''y_2''\,dx \\ \int_0^l EI\,y_1''y_2''\,dx & \int_0^l EI\,{y_2''}^2\,dx \end{bmatrix}$$

same

$$[M] = \begin{bmatrix} \int_0^l \rho A y_1^2\,dx & \int_0^l \rho A y_1y_2\,dx \\ \int_0^l \rho A y_1y_2\,dx & \int_0^l \rho A y_2^2\,dx \end{bmatrix}$$

Since $A = \left(\dfrac{h}{l}\right)x$; $I = \dfrac{1}{12}\left(\dfrac{hx}{l}\right)^3$

$$\Rightarrow [K] = \begin{bmatrix} 0.0833333 & 0.0333333 \\ 0.0333333 & 0.0333333 \end{bmatrix}\frac{Eh^3}{l}$$

$$[M] = \begin{bmatrix} 0.0333333 & 0.00952381 \\ 0.00952381 & 0.00357143 \end{bmatrix}\rho h$$

$\Rightarrow$ Solution:

$$\omega_1^2 = 2.35741\,\frac{Eh^2}{\rho l^4}$$

$$\omega_2^2 = 24.9426\,\frac{Eh^2}{\rho l^4}$$

The First natural Frequency:

$$\omega_1 = 1.5353\sqrt{Eh^2/\rho l^4}$$

For one term, $C_2 = 0$

$$\omega_1^2 = 2.50000\,\frac{Eh^2}{\rho l^4}$$

$$\omega_1 = 1.5811\sqrt{Eh^2/\rho l^4}$$

The exact solution

$$\omega_{1,exact} = 1.5343\sqrt{Eh^2/\rho l^4}$$

Take more terms

$y_3 = (1 - x/\ell)^2 (x/\ell)^2$; $y_4 = (1 - x/\ell)^2 (x/\ell)^3$

$y_5 = (1 - x/\ell)^2 (x/\ell)^4$

$y(x) = C_1 y_1 + C_2 y_2 + C_3 y_3 + C_4 y_4 + C_5 y_5$

$\Rightarrow \quad \omega_1^2 = 2.354190 \; Eh/\rho\ell^4$

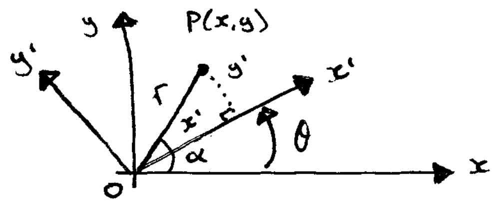
$\omega_1 = 1.53434 \sqrt{Eh/\rho\ell^4}$

(*) Correction: units of $[K]$: $\frac{Eh^3}{l} \rightarrow \frac{Eh^3}{l^3}$
(For last lecture)

$[M]$: $\rho h \rightarrow \rho h l$

Jacobi's Method:

Coordinate transformation



$x = r\cos\alpha$

$y = r\sin\alpha$

$P(x,y) \rightarrow P(x',y')$

$x' = r\cos(\alpha-\theta)$

$y' = r\sin(\alpha-\theta)$

$\rightarrow x' = r\cos\alpha\cos\theta + r\sin\alpha\cdot\sin\theta = x\cos\theta + y\sin\theta$

$y' = r\sin\alpha\cos\theta - r\cos\alpha\cdot\sin\theta = -x\sin\theta + y\cos\theta$

$$\rightarrow \begin{Bmatrix} x' \\ y' \end{Bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{Bmatrix} x \\ y \end{Bmatrix}$$

$$\begin{Bmatrix} x \\ y \end{Bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{Bmatrix} x' \\ y' \end{Bmatrix}$$

Define: $$[Q] = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$[Q]^{-1} = [Q]^T$$

Given $[D] = \begin{bmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{bmatrix}$; where $d_{12} = d_{21}$

$$\rightarrow [Q]^T[D][Q] = [D_1]$$

$$[D_1] = \begin{bmatrix} d_{11}^{(1)} & d_{12}^{(1)} \\ d_{21}^{(1)} & d_{22}^{(1)} \end{bmatrix}$$

Here: $d_{12}^{(1)} = (\cos^2\theta - \sin^2\theta)d_{12} + (d_{22} - d_{11})\sin\theta\cos\theta$

$d_{11}^{(1)} = d_{11}\cos^2\theta + 2d_{12}\cos\theta\sin\theta + d_{22}\sin^2\theta$

$d_{22}^{(1)} = d_{11}\sin^2\theta - 2d_{12}\cos\theta\sin\theta + d_{22}\cos^2\theta$

Let $\quad d_{12}^{①} = 0$

$$\Rightarrow \quad \tan(2\theta) = \frac{2 d_{12}}{d_{11} - d_{22}}$$

$$[D] = \begin{bmatrix} d_{11} & d_{12} & d_{13} \\ d_{21} & d_{22} & d_{23} \\ d_{31} & d_{32} & d_{33} \end{bmatrix}$$

$$[Q_1] = \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 & 0 \\ \sin\theta_1 & \cos\theta_1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$[Q_1]^T [D][Q_1] = [D_1] = \begin{bmatrix} d_{11}^{①} & 0 & d_{13}^{①} \\ 0 & d_{22}^{①} & d_{23}^{①} \\ d_{31}^{①} & d_{32}^{①} & d_{33}^{①} \end{bmatrix}$$

$$[Q_2] = \begin{bmatrix} \cos\theta_2 & 0 & -\sin\theta_2 \\ 0 & 1 & 0 \\ \sin\theta_2 & 0 & \cos\theta_2 \end{bmatrix}$$

$$[Q_2]^T [D_1][Q_2] = \begin{bmatrix} d_{11}^{②} & d_{12}^{②} & 0 \\ d_{21}^{②} & d_{22}^{②} & d_{23}^{②} \\ 0 & d_{32}^{②} & d_{33}^{②} \end{bmatrix} = [D_2]$$

$\rightarrow$ Repeat many times ...

$$[Q_m]^T \cdots [Q_2]^T [Q_1]^T [D][Q_1][Q_2] \cdots [Q_m]$$

$$= \begin{bmatrix} d_{11}^m & 0 & 0 \\ 0 & d_{22}^m & 0 \\ 0 & 0 & d_{33}^m \end{bmatrix}$$

Define: $[U] = [Q_1][Q_2] \cdots [Q_M]$

$$\rightarrow \quad [D][U] = [U][\Lambda]$$

$$[\Lambda] = \text{diag}(d_{11}^{ⓜ}, d_{22}^{ⓜ}, d_{33}^{ⓜ})$$

$\boxed{\text{Example}}$ Find the eigenvalues by using Jacobi's method

$$[D] = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$$

Solution: $\rightarrow \tan(2\theta) = \dfrac{2d_{12}}{d_{11}-d_{22}} = \dfrac{2\times 1}{1-2} = -2$

$$\theta = -0.553574$$

$$[Q_1] = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0.8506 & 0.5257 & 0 \\ -0.5257 & 0.8506 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$[Q_1]^T[D][Q_1] = \begin{bmatrix} 0.3820 & 0 & -0.2008 \\ 0 & 2.618 & 2.227 \\ -0.2008 & 2.227 & 3 \end{bmatrix} = D_1$$

$\rightarrow \tan(2\theta) = \dfrac{2d_{23}}{d_{22}-d_{33}} = \dfrac{2\times 2.227}{2.618-3}$

$$\theta = -0.7426$$

$$[Q_2] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.7367 & -0.6762 \\ 0 & -0.6762 & 0.7367 \end{bmatrix}$$

$$[Q_2]^T[D_1][Q_2] = \begin{bmatrix} 0.3820 & 0.1358 & -0.1479 \\ & 0.5738 & 0 \\ & & 5.044 \end{bmatrix} = D_2$$

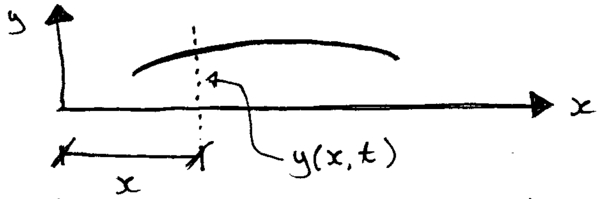
$$\rightarrow [Q_7]^T \cdots [Q_1]^T[D][Q_1] \cdots [Q_7] = \begin{bmatrix} 0.3080 & 0.1762\times 10^{-6} & 0 \\ & 0.6431 & 7.02\times 10^{-6} \\ \text{Sym.} & & 5.049 \end{bmatrix}$$

and $[U] = [Q_1][Q_2]\cdots[Q_7]$

$$= \begin{bmatrix} 0.5910 & 0.7370 & 0.3280 \\ -0.7370 & 0.3280 & 0.5910 \\ 0.3280 & -0.5910 & 0.7370 \end{bmatrix}$$

(Column Corresponds to): $\uparrow 0.3820$ $\quad \uparrow 0.6431$ $\quad \uparrow 5.049$

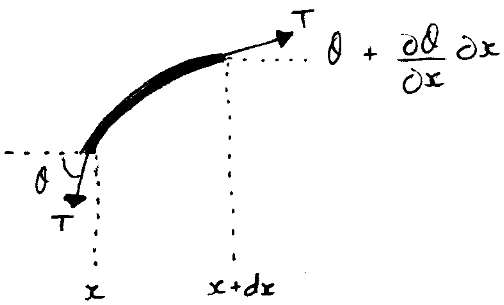
Vibrating string :



: the displacement is a function of both space and time.

Mass density per unit length : ρ

small vibration, the tension : T = const.



$T \sin\left(\theta + \frac{\partial \theta}{\partial x}\partial x\right)$

$$T\sin\left(\theta + \partial\theta/\partial x \, dx\right) - T\sin\theta = \rho\, dx \frac{\partial^2 y}{\partial t^2}$$

$\theta \ll 1$, then $\sin\theta \approx \theta$

$$T\left(\theta + \partial\theta/\partial x \, dx\right) - T\theta = \rho \frac{\partial^2 y}{\partial t^2}\partial x$$

$$T\left(\partial\theta/\partial x\right) = \rho\left(\partial^2 y/\partial t^2\right)$$

$$\theta \approx \tan\theta = \partial y/\partial x$$

$\rightarrow$ $T\left(\frac{\partial^2 y}{\partial x^2}\right) = \rho\left(\partial^2 y/\partial t^2\right)$

$\rightarrow$ $\boxed{\left(\partial^2 y/\partial x^2\right) = \left(1/c^2\right)\left(\partial^2 y/\partial t^2\right) \quad ; \quad c = \sqrt{T/\rho}}$ wave speed

$f(x - ct) \longrightarrow$, $g(x + ct) \longleftarrow$

$\rightarrow$ $y(x,t) = Y(x) \cdot T(t)$

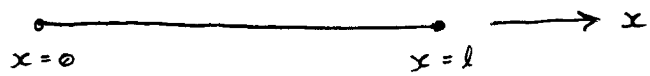
$\rightarrow$ Harmonic solution of time t

$$y(x,t) = Y(x)e^{i\omega t}$$

$$\frac{\partial^2 y}{\partial x^2} = Y''(x)e^{i\omega t}$$

$$\frac{\partial^2 y}{\partial x^2} = -\omega^2 Y(x)e^{i\omega t}$$

$$Y'' e^{i\omega t} = -\frac{\omega^2}{c^2} Y(x) e^{i\omega t} \longrightarrow Y'' + \frac{\omega^2}{c^2} Y = 0 \qquad \text{I}$$



At $x = 0$, $\quad y(0, t) = Y(0)\, e^{i\omega t} = 0$

$$Y(0) = 0 \qquad \text{II}$$

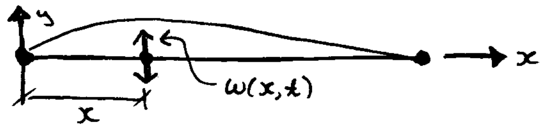
At $x = l$, $\quad y(l, t) = Y(l)\, e^{i\omega t} = 0$

$$Y(l) = 0 \qquad \text{III}$$

Boundary value problem.

NOV. 26/19

$\boxed{\text{Example}}$ determine the frequencies of a string (G3)



Tension $\Rightarrow$ $T = \text{const.}$

mass density $\Rightarrow$ $\rho = \text{const.}$ (mass per unit length)

Solution: $c^2 \dfrac{\partial^2 w}{\partial x^2} = \dfrac{\partial^2 w}{\partial x^2}$; $c = \sqrt{\dfrac{T}{\rho}}$ ← tension, ← mass density

$x = 0$, $w(0,t) = 0$
$x = l$, $w(l,t) = 0$ } Boundary conditions

$w(x,t) = W(x)\, e^{i\omega t}$

$\rightarrow$ $c^2 W'' = -\omega^2 W$

$W'' + (\omega^2/c^2)\, W = 0$

$\rightarrow$ $W(x) = a_1 \sin(\omega/c)x + a_2 \cos(\omega/c)x$

@ $x = 0$, $w(0,t) = W(0)e^{i\omega t} = 0$
$W(0) = 0$

@ $x = l$, $w(l,t) = W(l)e^{i\omega t} = 0$
$W(l) = 0$

$\rightarrow$ $\begin{cases} a_1(0) + a_2(1) = 0 \longrightarrow a_2 = 0 \\ a_1 \sin(\omega l/c) + a_2 \cos(\omega l/c) = 0 \end{cases}$

$\hookrightarrow$ $a_1 \sin(\omega l/c)$

since $a_1 \neq 0$

$\boxed{\sin(\omega l/c) = 0}$

The solution:

$(\omega l/c) = \pi, 2\pi, 3\pi, \ldots$

or $(\omega_n l/c) = n\pi$, $n = 1, 2, 3, \ldots$

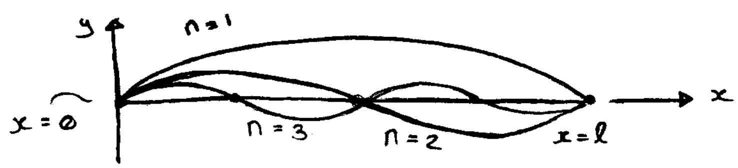
$\hookrightarrow$ $\omega_n = \dfrac{n\pi l}{c}$, $n = 1, 2, 3, \ldots$

The Fundamental Frequency is :

$$\boxed{\omega_1 = \frac{\pi c}{l}}$$

The mode shapes (eigenFunctions)

$$W_n(x) = a_1 \sin\left(\frac{\omega_n x}{c}\right) \quad ; n = 1, 2, 3, \ldots$$
$$= a_1 \sin\left(\frac{n\pi x}{l}\right)$$



→ For the Frequency ω_n

$$W_n(x) \cdot (A_n \sin(\omega_n t) + B_n \cos(\omega_n t))$$

→ For the response of the string

$$w(x,t) = \sum_{n=1}^{\infty} W_n(x)(A_n \sin(\omega_n t) + B_n \cos(\omega_n t))$$

For G3 :

Tension : $T = 30$ lbs $= 133.447$ N

mass density : $\rho = 0.00011602$ lbs/in

$= 0.00207510$ kg/m

The wave (phase) velocity :

$$c = \sqrt{\frac{T}{\rho}} \rightarrow c = \sqrt{\frac{133.447}{0.00207510}} \rightsquigarrow c = 253.592 \text{ m/s}$$

The length of the string is :

$l = 25.5$ in $= 0.65$ m

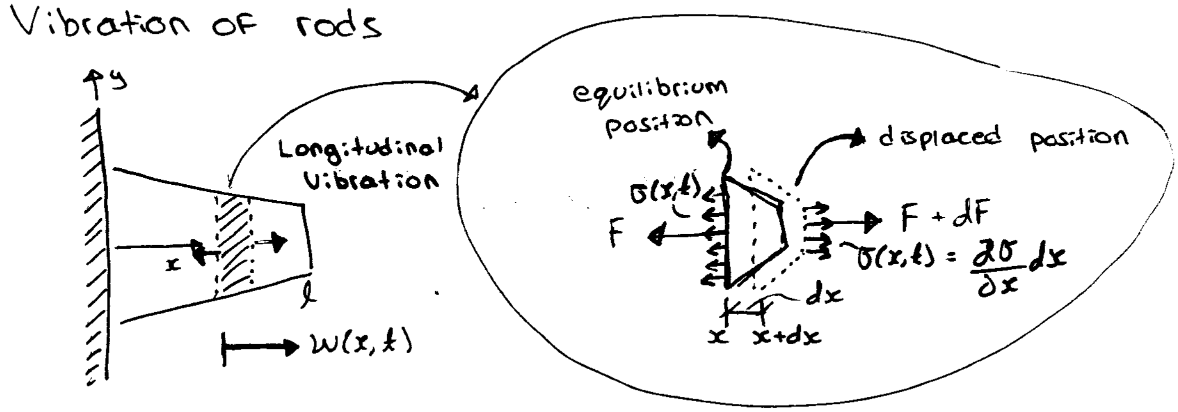
The Fundamental Freq :

$$\omega_1 = \frac{\pi c}{l} \qquad \text{(but we use Hz)}$$

or

$$f_1 = \frac{\omega_1}{2\pi} = \frac{c}{2l} = \frac{253.592}{2(0.65)} = 195.1 \text{ Hz}$$

Vibration of rods



$$F + dF - F = ma$$

$$F + dF = A + (dA/dx)dx\left(\sigma + \frac{\partial\sigma}{\partial x}dx\right)$$

$$F = A\sigma$$

$$m = \rho A\,dx \qquad a = (\partial^2 w/\partial t^2)$$

$$\longrightarrow \quad (A + dA)(\sigma + (\partial\sigma/\partial x)dx) - A\sigma = \rho A\,dx \cdot (\partial^2 w/\partial t^2)$$

$$A(\partial\sigma/\partial x)dx + \sigma\,dA = \rho A(\partial^2 w/\partial t^2)dx$$

$$\partial/\partial x(A\sigma) = \rho A(\partial^2 w/\partial t^2)$$

From Hooke's Law: $\sigma = E\varepsilon = E(\partial w/\partial x)$

$$\longrightarrow \quad (\partial^2 w/\partial x^2)AE = \rho A(\partial^2 w/\partial t^2)$$

$$\Rightarrow \quad \boxed{\frac{\partial}{\partial x}\left(AE\frac{\partial w}{\partial x}\right) = \rho A\left(\frac{\partial^2 w}{\partial t^2}\right)}$$

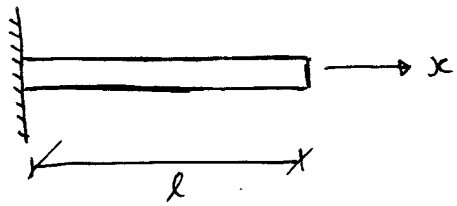
Consider: $A = \text{const.}$, $E = \text{const.}$

$$\frac{E}{\rho}\left(\frac{\partial^2 w}{\partial x^2}\right) = \left(\frac{\partial^2 w}{\partial t^2}\right)$$

Define: $c = \sqrt{\dfrac{E}{\rho}}$

$$\Rightarrow \quad \boxed{c^2\left(\frac{\partial^2 w}{\partial x^2}\right) = \left(\frac{\partial^2 w}{\partial t^2}\right)}$$

For the Fixed-Free bar :



E, ρ : constant

Let : $w(x,t) = W(x) e^{i\omega t}$

$$\Rightarrow \frac{\partial^2 w}{\partial x^2} + \frac{\omega^2}{c^2} W = 0$$

$$W(x) = a_1 \sin\left(\tfrac{\omega}{c}x\right) + a_2 \cos\left(\tfrac{\omega}{c}x\right)$$

Boundary conditions :

$x = 0$, $w(0,t) = 0 \Rightarrow W(0) = 0$

$x = \ell$, $\sigma(\ell, t) = 0$

$\sigma(x,t) = E\epsilon(x,t) = E(\partial w/\partial x)$

$\sigma(\ell,t) = E\left(\partial w/\partial x \,\big|_{x=\ell}\right) = E\left(\partial W/\partial x \big|_{x=\ell}\right) e^{i\omega t} = 0$

$$\Rightarrow \frac{\partial W}{\partial x} = 0$$

$$W(0) = a_1 \cdot 0 + a_2 \cdot 1 = a_2 = 0$$

$$\frac{\partial W}{\partial x}\bigg|_{x=\ell} = a_1\left(\frac{\omega}{c}\right)\cos\left(\frac{\omega \ell}{c}\right) = 0$$

$a_1 \neq 0$, $\omega \neq 0$

$$\boxed{\cos\left(\frac{\omega \ell}{c}\right) = 0}$$

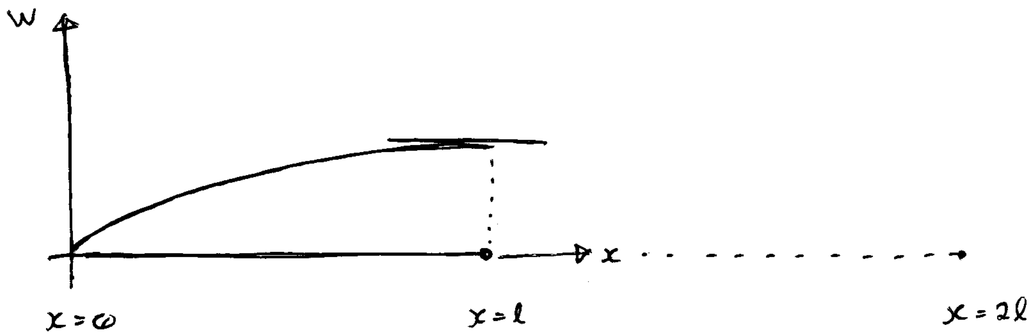
$$\Rightarrow \frac{\omega \ell}{c} = \frac{\pi}{2},\ \pi + \frac{\pi}{2},\ 2\pi + \frac{\pi}{2},\ \cdots\cdots,\ n\pi + \frac{\pi}{2}$$

$$\frac{\omega_n \ell}{c} = (n-1)\pi + \frac{\pi}{2} = (n - 1/2)\pi, \quad n = 1, 2, 3 \ldots$$

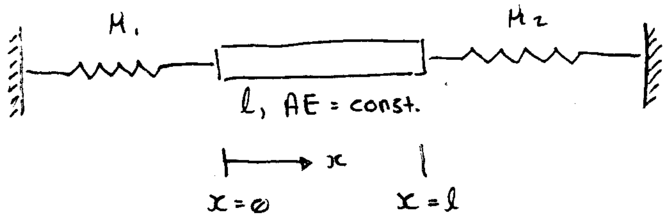
$$\omega_n = \left(\frac{c}{\ell}\right)\left(\frac{2n-1}{2}\right)\pi, \quad n = 1, 2, 3, \ldots$$

$$W_n(x) = a_1 \sin\left(\frac{\omega_n x}{c}\right) = a_1 \sin\left[\frac{(2n-1)\pi}{2} \cdot \frac{x}{\ell}\right]$$

$n = 1$: $\omega_1 = \dfrac{c\pi}{2\ell}$; and $W_1 = a_1 \sin(\pi x / 2\ell)$

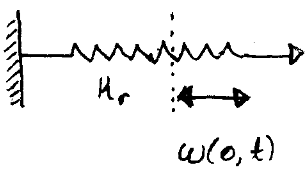


w
x
x = 0
x = l
x = 2l



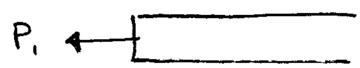
$$C^2\left(\frac{\partial^2 w}{\partial x^2}\right) = \left(\frac{\partial^2 w(x,t)}{\partial x^2}\right) \qquad 0 < x < l$$

Left end has a displacement $w(0,t)$



$P_1 = K_1 w(0,t)$

(Equivalent)



$$P_1 = A\sigma(0,t) = AE\varepsilon(0,t) = AE\,\frac{\partial w(0,t)}{\partial x}$$

$$\Rightarrow \quad K_1 w(0,t) = AE\left(\frac{\partial w(0,t)}{\partial x}\right)$$

At right end, $x = l$

$$K_2\, w(l,t) = -AE\left(\frac{\partial w(l,t)}{\partial x}\right)$$

Free-vibration :

$$w(x,t) = W(x)\,e^{i\omega t}$$

Eq. of motion :

$$C^2\,\frac{d^2 W(x)}{dx^2} = -\omega^2 W$$

$$\Rightarrow \quad \frac{d^2 W(x)}{dx^2} + \frac{\omega^2}{C^2}W = 0$$

$$\Rightarrow \quad W(x) = a_1 \sin(\omega/c)x + a_2\cos(\omega/c)x$$

Boundary conditions

$x = 0: \quad K_1 w(0,t) = \dfrac{AE\,\partial w(0,t)}{\partial x}$

$K_1 W(0) = AE\,W'(0)$

$x = l: \quad K_2 W(l) = -AE\,W'(l)$

Since $W'(x) = a_1\,\omega/c \cos \omega/c\, x - a_2\,\omega/c \sin \omega/c\, x$

$x = 0: \quad K_1 a_2 = AE \cdot a_1\,\omega/c$

$x = l: \quad K_2 (a_1 \sin(\omega l/c) + a_2 \cos(\omega l/c)) = AE(\omega/c)(a_1 \cos(\omega l/c) - a_2 \sin(\omega l/c))$

$$\rightarrow \begin{bmatrix} \times & \times \\ \times & \times \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix} = 0$$

$$\rightarrow \det \begin{bmatrix} \times & \times \\ \times & \times \end{bmatrix} = 0$$

$$a_1 = a_2 \frac{K_1}{AE(\omega/c)}$$

Sub $K_2\left(a_2 \dfrac{K_1}{AE(\omega/c)} \sin(\omega l/c) + a_2 \cos(\omega l/c)\right)$

$= -AE(\omega/c)\left(a_2 \dfrac{K_1}{AE(\omega/c)} \sin(\omega l/c) + a_2 \cos(\omega l/c)\right)$

Since $a_2 \neq 0$

$$\tan(\omega l/c) = \frac{K_1 + K_2}{(AE/l)(\omega l/c) - \dfrac{(K_1 K_2)}{(AE/l)\cdot(\omega l/c)}}$$

Define $\alpha = \dfrac{\omega l}{c}$; $K = \dfrac{AE}{l}$

$$\tan\alpha = \frac{K_1 + K_2}{K\alpha - \dfrac{K_1 K_2}{K\alpha}} = \frac{(K_1 + K_2) K\alpha}{K^2\alpha^2 - K_1 K_2}$$

$$\tan\alpha = \frac{(K_1/K + K_2/K)\,\alpha}{\alpha^2 - (K_1/K \cdot K_2/K)}$$

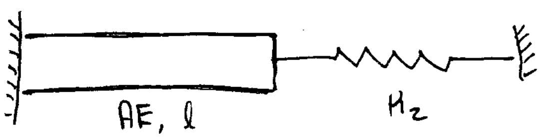
$$\boxed{\tan\alpha = \frac{\left(\frac{H_1}{H} + \frac{H_2}{H}\right)\alpha}{\alpha^2 - \frac{H_1}{H}\cdot\frac{H_2}{H}}}$$

(*) For the case $\frac{H_1}{H} = 1$, $\frac{H_2}{H} = 1$

$$\boxed{\tan\alpha = \frac{2\alpha}{\alpha^2 - 1}}$$

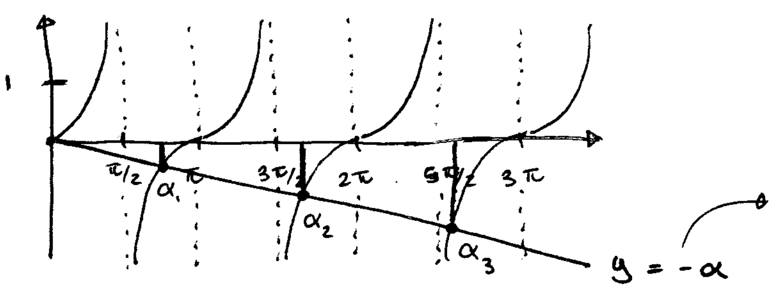
(*) For the case $H_1 \rightarrow \infty$

$$\boxed{\tan\alpha = \left(\frac{-H}{H_2}\right)\alpha}$$



considering $H_2 = H = \frac{AE}{l}$

$$\tan\alpha = -\alpha$$

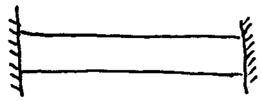


$(2n-1)\cdot\pi/2$

$n \rightarrow \infty$

(*) $H_2 \to \infty$

$\tan\alpha = 0 \Rightarrow \sin\alpha = 0$

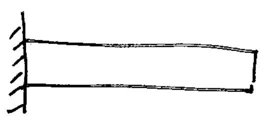


(*) $H_2 \to 0$

$H_2 \sin\alpha = -K\alpha\cos\alpha$

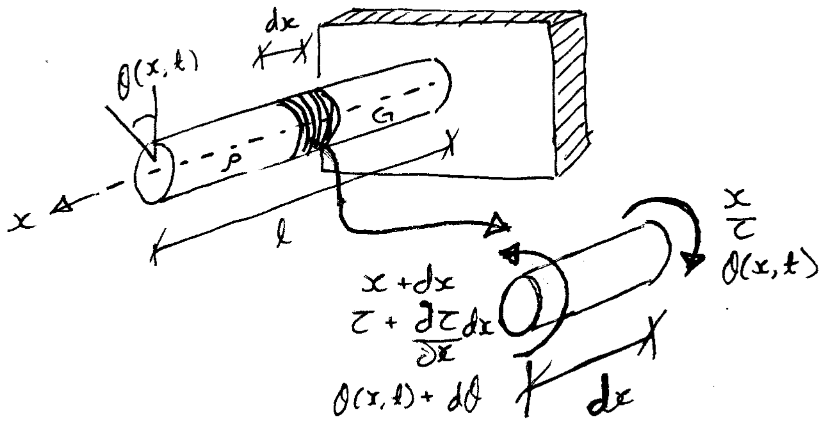
$\alpha\cos\alpha = 0$

$\alpha \neq 0$



$\cos\alpha = 0$

Torsional Vibration



→ $\boxed{\Sigma M_x = I_x \alpha}$ (Here τ is torsion)

$$\left(\tau + \frac{\partial \tau}{\partial x} dx\right) - \tau = (\rho J dx)\left(\frac{\partial^2 \theta}{\partial t^2}\right)$$

$$\Rightarrow \frac{\partial \tau}{\partial x} = \rho J \frac{\partial^2 \theta}{\partial t^2}$$

Mechanics of Materials :

$$\tau = GJ(\partial\theta/\partial x)$$

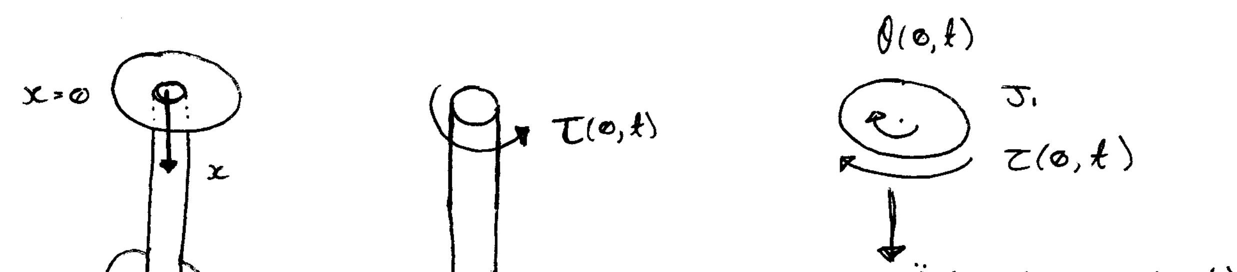
$$\Rightarrow GJ \frac{\partial^2 \theta}{\partial t^2} = \rho J \frac{\partial^2 \theta}{\partial t^2}$$

$$\Rightarrow \frac{G}{\rho} \frac{\partial^2 \theta}{\partial x^2} = \frac{\partial^2 \theta}{\partial t^2}$$

OR $C = \sqrt{\dfrac{G}{\rho}}$

$$\boxed{C^2 \frac{\partial^2 \theta}{\partial x^2} = \frac{\partial^2 \theta}{\partial t^2}}$$

→ From textbook, Figure 6.8



$$J_1\ddot{\theta}(0,t) = \tau(0,t)$$

$$\tau(0,t) = GJ\left(\partial\theta/\partial x\big|_{x=0}\right)$$

At $x=0$:

$$\Rightarrow \quad GJ\left(\frac{\partial\theta}{\partial x}\bigg|_{x=0}\right) = J_1\left(\frac{\partial^2\theta}{\partial t^2}\bigg|_{x=0}\right)$$

At $x=l$:

$$\Rightarrow \quad GJ\left(\partial\theta/\partial x\big|_{x=l}\right) = -J_2\left(\frac{\partial^2\theta}{\partial t^2}\bigg|_{x=l}\right)$$

Assume

$$\theta(x,t) = \Theta(x)\,e^{i\omega t}$$

$$\frac{\partial^2\Theta}{\partial x^2} + \frac{\omega^2}{c^2}\Theta = 0 \qquad (0 < x < l) \qquad (1)$$

$x=0$:

$$\Rightarrow \quad GJ\Theta' + J_1\omega^2\Theta = 0 \qquad (2)$$

$x=l$:

$$\Rightarrow \quad GJ\Theta' - J_2\omega^2\Theta = 0 \qquad (3)$$

$$(1) \rightarrow \quad \Theta(\omega) = a_1\sin(\omega/c\, x) + a_2\cos(\omega/c\, x) \qquad (4)$$

Since $\quad \Theta'(x) = a_1(\omega/c)\cos(\omega/c\, x) - a_2(\omega/c)\sin(\omega/c\, x) \qquad (5)$

Sub (4) & (5) into (2) & (3):

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}\begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix} = 0$$

$$\rightarrow \quad \det\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = 0$$

$$\rightarrow \quad \boxed{\tan\left(\frac{\omega}{c}l\right) = \frac{\frac{\omega}{c}l}{b\left(\frac{\omega}{c}l\right)^2 - a}}$$

$$\boxed{a = \rho l\frac{J}{J_1+J_2} \qquad b = \frac{J_1J_2}{\rho J l(J_1+J_2)}}$$

J: Polar moment of inertia of shaft

Given : $J_1 = 10 \ kg \cdot m^2$

$J_2 = 10 \ kg \cdot m^2$

$J = 5 \ m^4$

$l = 0.425 \ m$

$\rho = 7870 \ kg/m^3$

Then : $a = \dfrac{7870 \times 0.425}{10 + 10} \longrightarrow a = 836.1875$

$b = \dfrac{10 \times 10}{7870 \times 5 \times 0.425 \times (10+10)} \longrightarrow b = 0.000298976$

Freq. Eqn. :

$$\tan\left(\frac{\omega}{c} l\right) = \frac{\left(\frac{\omega}{c} l\right)}{0.000298976\left(\frac{\omega l}{c}\right)^2 - 836.1875}$$

1^{st} : $\frac{\omega l}{c} = 0 \longrightarrow \omega = 0$ is a Freq.

2^{ND} : $f_1 = \dfrac{\omega_1}{2\pi} = 0$

$f_2 = \dfrac{\omega_2}{2\pi} = 3813 \ Hz$

$f_3 = 76026 \ Hz$

Summary of Concepts

Vibration

* Potential energy, spring, elasticity [K]
* Kinetic energy: mass / inertia [M]
* energy lost : damper [C]

Modeling :

- Newton's law

  Influence coefficients — Flexibility / Stiffness

- Energy Method — only for conservative (no damping)
- Lagrange Method

Free-vibration : (Free-response)

- 1 DOF : Natural freq. $\omega_n = \sqrt{k/m}$
- multiple DOF : Natural Frequencies (eigenvalue) / modal shapes (eigenvector) *

  Modal expansion

  → decouple eq. of motion

* response due to initial conditions (for single DOF)
* Find the natural freq / mode shapes
  * eigenvalue / eigenvector
  * approximate method

    Dunkerley's Method

    Rayleigh's Method

    Matrix iteration (Power method)

    Jacobi's Method

Forced Response : 1 DOF

* resonance *
* beat (2 DOF)
* base excitation
* rotating unbalance

disp. Forced response

- Forced response
ω_n multiple DOF - approx Method
(never exceed 3)
Free response
3×3 for sure (for other Q's)
5Q's (roughly)
closed book