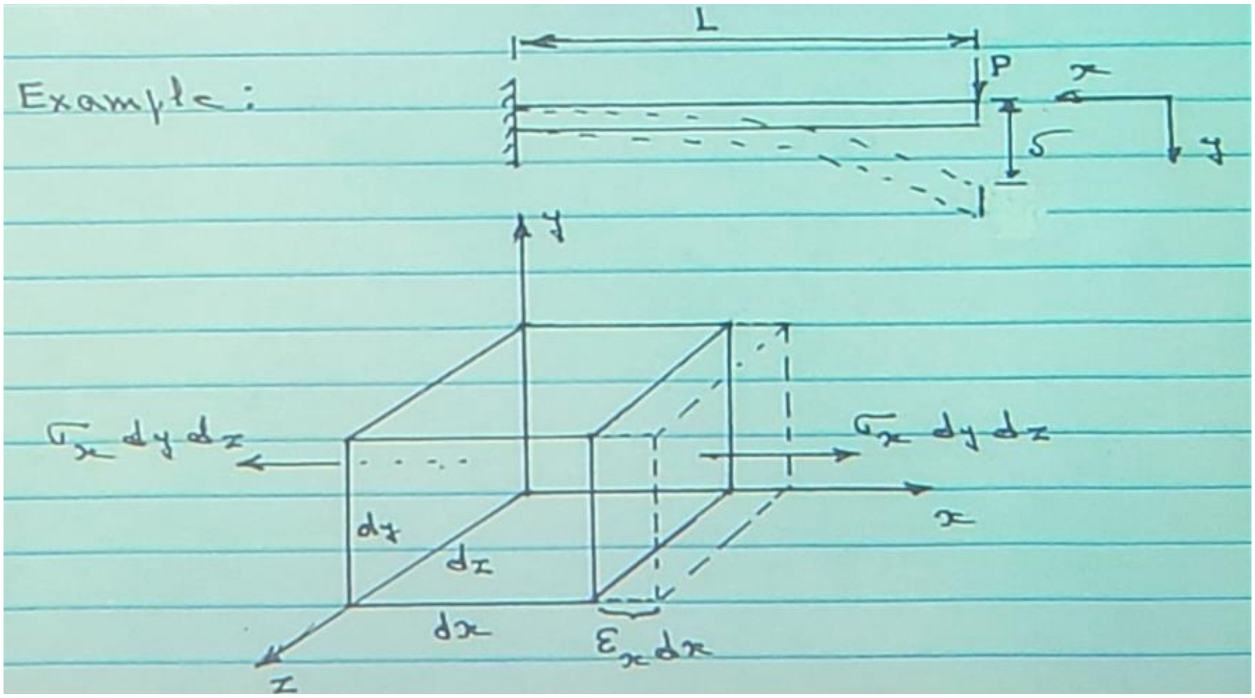
Lecture (Mar. 5th, 2019)



$$Du = \left(\frac{1}{2}\right)(\sigma_x\;dy\;dx)(\varepsilon_x\;dx)$$

$$Du = \left(\frac{1}{2}\right)(\sigma_x\varepsilon_x)(dx\;dy\;dz)$$

The strain energy per unit volume is:

$$du = \left(\frac{1}{2}\right)\sigma_x\varepsilon_x(dx\;dy\;dz)/(dx\;dy\;dz)$$

$$du = \left(\frac{1}{2}\right)\sigma_x\varepsilon_x = \frac{\sigma_x^2}{2E} \quad ; \quad Note\ that\ \varepsilon_x = \frac{\sigma_x}{E}$$

And:

$$U = \int^{v} du \quad = \quad \left(\frac{1}{2}\right)\int_L \int_A \frac{\sigma_x^2}{E}\,dA\,dx$$

$$= \quad \left(\frac{1}{2}\right)\int_L \int_A \left(\frac{1}{E}\right)\left(\frac{My}{I}\right)^2 dA\,dx$$

$$U = \left(\frac{1}{2}\right)\int_L \frac{1}{E}\frac{M^2}{I^2}dx\int_A y^2\,dA = \left(\frac{1}{2}\right)\int_0^L \frac{M^2}{EI}dx$$

From Castegliano's Theorem:

$$\delta = \frac{\partial u}{\partial P}$$

$$\delta = \frac{\partial u}{\partial P} = \left(\frac{1}{2}\right) \int_L \frac{\partial}{\partial P} \left(\frac{M^2}{EI}\right) dx$$

$$= \int_L M \frac{\partial M / \partial P}{EI} dx$$

$$\delta = \int_0^L \frac{(-Px)(-x)}{EI} dx = \frac{PL^3}{3EI}$$

The deflection is always in the direction of the force.

If an applied force does not exist at the pint where the deflection is to be determined, then a fictitious force Q must be applied. After the strain energy equation has been differentiated with respect to Q, the force Q is set equal to zero. The resulting expression is the displacement at the point of application of Q and Is in the same direction as Q was assumed to be acting.

The following provided strain energy expressions for various types of loading:

Tension and Compression:

$$U = \frac{F^2 L}{2AE}$$

Torsion:

$$U = \frac{T^2 L}{2AG}$$

Direct Shear:

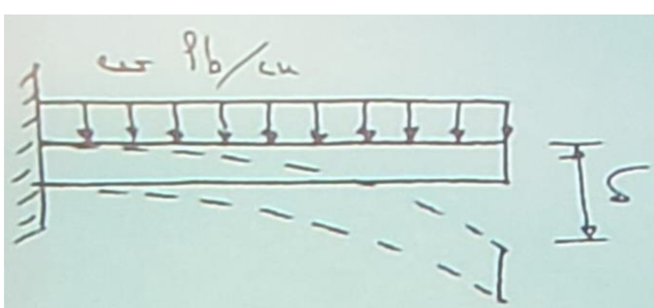
$$U = \frac{F^2 L}{2AG}$$

Bending:

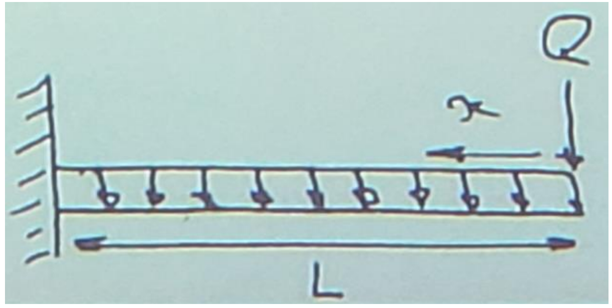
$$U = \int \frac{M^2}{2EI} dx$$

Example:

Determine the end deflection of a uniformly loaded cantilever beam.

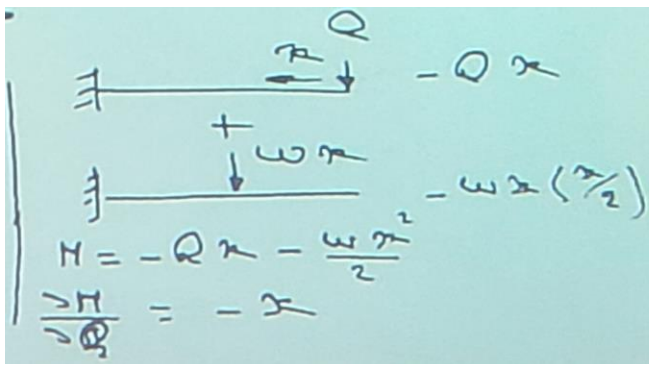


Assume a fictitious force Q acting as shown at the point where the deflection is required.



$$M = -Qx - \frac{wx^2}{2}$$

$$\frac{\partial M}{\partial Q} = -x$$



$$\delta = \frac{\partial u}{\partial Q} = \int_0^L M \frac{(-Px)(-x)}{EI} dx = \frac{wL^4}{8EI}$$

Alternative solution:

Recall that if y is the beam deflection function and q is the load per unit length, then:

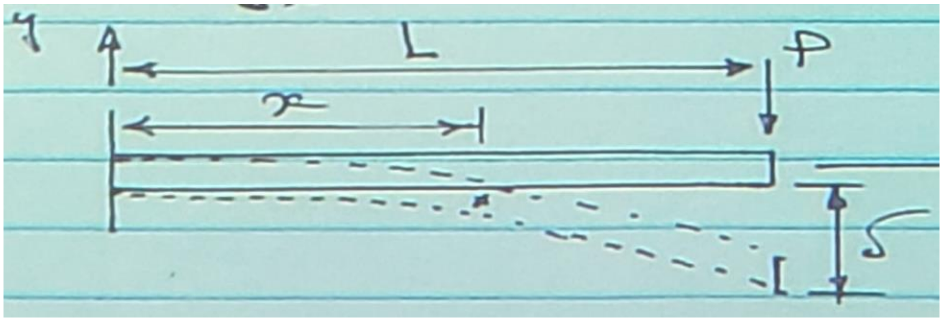
$$y = f(x)$$

$$\theta = \frac{dy}{dx} = slope$$

$$M = \frac{d^2y}{dx^2} EI$$

$$V = \frac{d^3y}{dx^3} EI$$

$$q = \frac{d^4y}{dx^4} EI$$



$$M = \frac{EI\, d^2y}{dx^2} = -P(L-x)$$

$$\frac{d^2y}{dx^2} = -\frac{1}{EI}[P(L-x)]$$

$$\frac{dy}{dx} = -\frac{P}{EI}\int (L-x)\,dx = -\frac{P}{EI}\left(Lx - \frac{x^2}{2} + C_1\right)$$

$$y = -\frac{P}{EI}\int \left(Lx - \frac{x^2}{2} + C_1\right)dx$$

$$y = -\frac{P}{EI}\left(\frac{Lx^2}{2} - \frac{x^3}{6} + C_1x + C_2\right)$$

BC: $at\ x = 0;$ $y' = 0 \ \therefore C_1 = 0$

$y = 0 \ \therefore C_2 = 0$

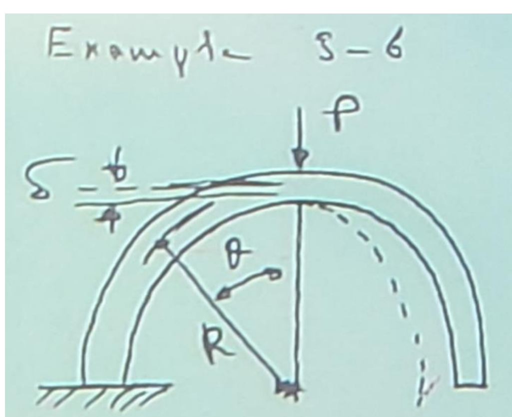
And:

$$y = -\frac{P}{EI}\left(\frac{Lx^2}{2} - \frac{x^3}{6}\right)$$

At $x = L$;

$$y = \delta = -\frac{P}{EI}\left(\frac{L^3}{2} - \frac{L^3}{6}\right) = -\frac{PL^3}{3EI} = \frac{PL^3}{3EI}\downarrow$$

Deflection by use of singularity functions:



$$For\ 0 \le x \le L \quad ; \quad \frac{EI\,d^4y}{dx^4} = q = -P < x - L >^{-1}$$

$$V = -P < x - L >^{0} + C_1$$

$$M = -P < x - L >^{1} + C_1 x + C_2$$

$$EI\,\theta = -\frac{P}{2} < x - L >^{2} + \frac{C_1}{2}x^2 + C_2 x + C_3$$

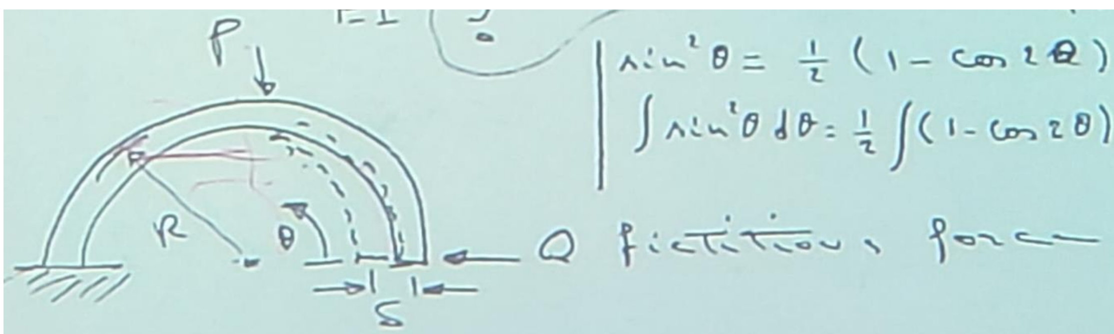
$$EI\,y = -\frac{P}{6} < x - L >^{3} + \frac{C_1}{6}x^3 + \frac{C_2}{2}x^2 + C_3 x + C_4$$

BC:	$At\ x = 0;$	$EI\,\theta = EI\,y = 0$	$\therefore C_3 = C_4 = 0$
	$At\ x = 0;$	$V = R_1$	$\therefore C_1 = R_1$
	$at\ x = L;$	$M = 0$	$\therefore C_2 = -R_1 L$

$$\therefore EI\,y = -\frac{P}{6} < x - L >^{3} +$$

Beam Deflection by Superposition

The results of many simple load cases and boundary conditions have been solved and tabulated. A limited number of these cases is presented in Table A-9. The effect of a combined loading on a structure can be obtained by adding the effects of each individual loading algebraically.



$$M = -PRsin\theta$$

$$\frac{\partial M}{\partial P} = -Rsin\theta$$

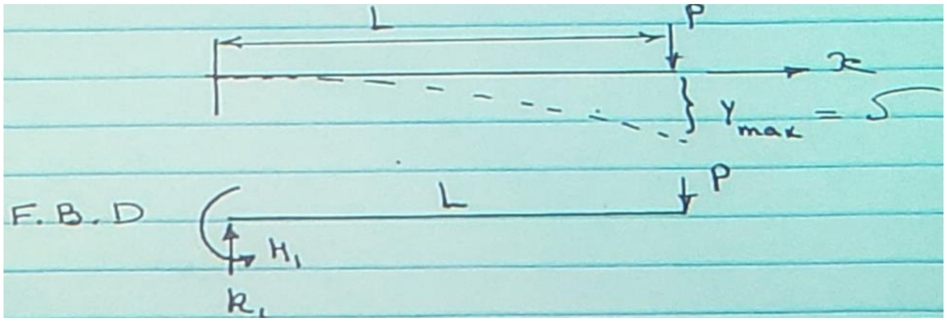
$$\delta = \frac{\partial u}{\partial P} = \int_0^{\frac{\pi}{2}} \frac{M\left(\frac{\partial M}{\partial P}\right)}{EI} Rd\theta$$

$$= \int_0^{\frac{\pi}{2}} \frac{(-PRsin\theta)(-Rsin\theta)}{EI} Rd\theta$$

$$= \frac{PR^3}{EI}\int_0^{\frac{\pi}{2}} \sin^2\theta\, d\theta = \frac{\pi PR^3}{4EI}$$

$$\sin^2\theta = \left(\frac{1}{2}\right)(1-\cos 2\theta)$$

$$\int \sin^2\theta \; d\theta = \left(\frac{1}{2}\right)\int (1-cos2\theta)d\theta = \left(\frac{1}{2}\right)\theta - \left(\frac{1}{4}\right)sin2\theta + C$$



$$M = QRsin\theta \qquad for \quad 0 \le \theta \le \pi/2$$

$$M = QRsin\theta + PRsin\left(\theta - \frac{\pi}{2}\right) \qquad \frac{\pi}{2} \le \theta \le \pi$$

$$= QRsin\theta - PRcos\theta$$

$$\frac{\partial M}{\partial Q} = Rsin\theta \qquad for \quad 0 \le \theta \le \pi$$

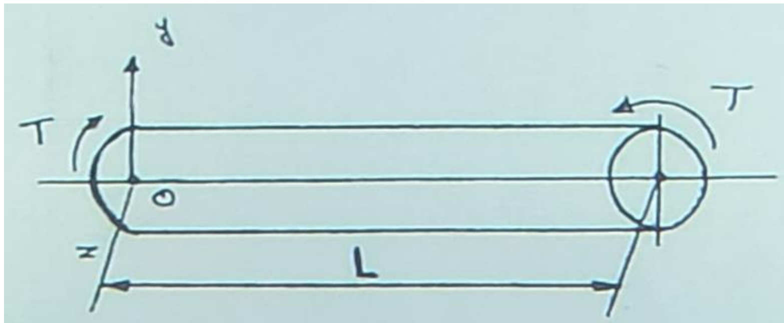
$$\delta_q = \frac{\partial u}{\partial Q} = \int_0^{\frac{\pi}{2}} \frac{(QRsin\theta)(Rsin\theta)}{EI} R\,d\theta + \int_{\pi/2}^{\pi} \frac{\pi(QRsin\theta - PRcos\theta)(Rsin\theta)}{EI} R\,d\theta$$

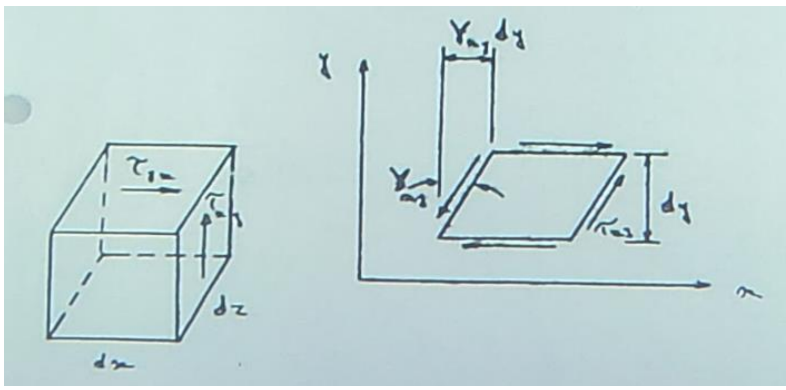
$$= -\frac{PR^3}{EI}\int_{\frac{\pi}{2}}^{\pi} sin\theta\; cos\theta\; d\theta = -\frac{PR^3\sin^2\theta}{2EI}\Big|_{\frac{\pi}{2}}^{\pi}$$

$$\delta_q = \frac{PR^3}{2EI}$$

Lecture (Mar. 7th, 2019)

Castigliano's Theorem may also be employed to calculate the angle of twist in members subject to torsion.





$$dU_{shear} = \left(\frac{1}{2}\right) \tau_{xy}\, dx\, dz \cdot \gamma_{xy}\, dy = \left(\frac{1}{2}\right) \tau_{xy}\, dv\, \gamma_{avg}$$

But $\gamma_{avg} = \tau / G$

$$\therefore du = \frac{\tau^2}{2g}\, dv$$

And:

$$U = \int_L \int_A \frac{\tau^2}{2G}\, dx\, dA$$

But:

$$\tau = \frac{Tr}{J}$$

$$\therefore U = \int_L \int_A \frac{T^2 \tau^2}{2GJ^2}\, dx\, dA$$

$$U = \int_L \frac{T^2}{2GJ^2}\, dx = \int_A r^2\, dA = \int_0^L \frac{T^2}{2JG}\, dx$$

Where:
τ = shear stress, psi
γ =shear strain, in/in
A = cross-sectional area, in^2
T = torque, $in - lb$
J =polar moment of inertia, in^4

$(for\ circular\ shaft) = \int_A r^2\, dA$

$$\theta = \frac{\partial U}{\partial T} = \int_0^L \frac{2T}{2JG}\, dx = \int_0^L \frac{T\, dx}{JG}$$

If the torque is uniform along the length of the shaft:

$$\theta = \frac{TL}{JG} = total\ angle\ of\ twist$$

The rotation of a section of a beam at a particular section is found to be:

$$\theta = \frac{\partial U}{\partial C} = \int_0^L \frac{M(\partial M/\partial C)}{EI}\, dx \quad (C = couple)$$

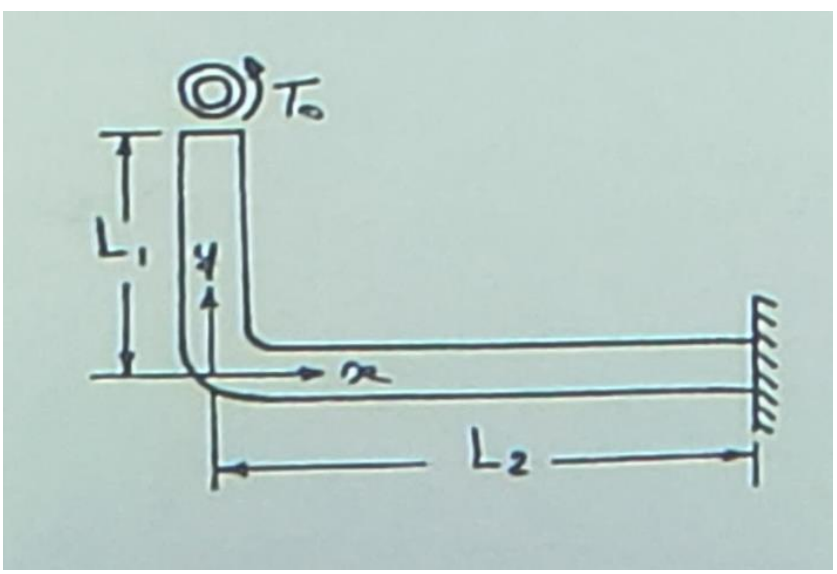
Where C is the couple at the section of interest.

Case of pure bending where $M = C$ throughout the length of the beam:

$$\theta = ML/EI$$

Where E and I are assumed to be constant

Example: Determine the rotation θ of the free end of a tube in the plane of a torque T_o; see Fig. Both portions of the tube lie in the same plane. Neglect the effect of deflection of the radius of the quarter load.



Length L_1 of pipe is subjected to torque T_o
Length L_2 of pipe is subjected to bending moment T_o (about $y - axis$)

$$\therefore U_1 = \int_0^{L_1} \frac{T^2}{2JG}\, dy \quad ; \quad U_2 = \int_0^{L_2} \frac{M^2}{2EI}\, dx$$

Where:
$M = T = T_o$

$$U = U_1 + U_2 = \int_0^{L_1} \frac{T_o^2}{2JG}\, dy + \int_0^{L_2} \frac{T_o^2}{2EI}\, dx$$

And:

$$\theta = \frac{\partial U}{\partial T_o} = \int_0^{L_1} \frac{T_o}{JG}\, dy + \int_0^{L_2} \frac{T_o}{EI}\, dx = \frac{T_o L_1}{JG} + \frac{T_o L_2}{EI}$$

<u>Failure Preventions = Static Loading</u>

Stress Concentration: Stress concentration is a localized effect that may be caused by a surface scratch, variation in material properties, localized high pressure points, or abrupt changes of section.

The stress at a point in a member influenced by one or more of these causes is, in general, grater than the nominal stress determined by elementary strength of materials.

The definition of geometric or theoretical stress concentration factor for normal stress (k_t) and shear stress (k_{ts}) is given by:

$$\sigma_{max} = k_t \sigma_{nom} \quad ; \quad \tau_{max} = k_{ts} \tau_{nom}$$

Table A.15 provides charts for the theoretical stress concentration factors for several load conditions and geometry.

Material	Static Load	Cyclic Load
Brittle	Serious	Very Serious
Ductile	Not Serious	Serious

Failure Theories: The generally accepted failure theories for ductile materials (yield criteria) are:
- Maximum Shear Stress theory (MSS)
- Distortion Energy theory (DE)
- Ductile Coulomb-Mohr theory (DCM)

And for brittle materials (fracture criteria) are:
- Maximum normal stress theory (MNS)
- Brittle Coulomb-Mohr Theory (BCM)
- Modified Mohr Theory (MM)

Maximum Shear-Stress Theory (For Ductile Materials)

This theory assumes that failure occurs for a combined stress condition when the maximum shear stress equals the value of a critical shear stress produced in an element subjected to simple tension, which is:

$$(S_s)_{yp} = \frac{S_{yp}}{2}$$

For $3D$ stressed, the maximum shear stress is given by one of the following, whichever is largest:

$$\frac{(\sigma_1 - \sigma_2)}{2} \quad ; \quad \frac{(\sigma_2 - \sigma_3)}{2} \quad ; \quad \frac{(\sigma_3 - \sigma_1)}{2}$$

Or:

$$\frac{S_{yp}}{2} = \begin{cases} \frac{(\sigma_1 - \sigma_2)}{2} \\ \frac{(\sigma_2 - \sigma_3)}{2} \\ \frac{(\sigma_3 - \sigma_1)}{2} \end{cases} \quad or\ S_{yp} = \begin{cases} \sigma_1 - \sigma_2 \\ \sigma_2 - \sigma_3 \\ \sigma_3 - \sigma_1 \end{cases}$$

For $2D$ stresses, $\sigma_3 = 0$, then:

If σ_1 and σ_2 are of **opposite** sign:
$\sigma_1 - \sigma_2 = \pm S_{yp}$
[or $n_d = S_{yp}/(\sigma_1 - \sigma_2)$]

If σ_1 and σ_2 are of the **same** sign:
$\sigma_1 = \pm S_{yp}$ if $|\sigma_1| > |\sigma_2|$
[or $n_d = S_{yp}/\sigma_1$]

$\sigma_2 = \pm S_{yp}$ if $|\sigma_2| > |\sigma_1|$
[or $n_d = S_{yp}/\sigma_2$]

Distortion-Energy Theorem (For Ductile Materials)

This theory assumes that yielding will occur when the strain energy of distortion per unit volume equals the strain energy of distortion per unit volume for a specimen in uniaxial tension or compression strained to the yield stress. This energy is found to be for the body under $3D$ stress.

$$U_s = \frac{(1 + v)}{6E}[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2]$$

For the specimen,

$$U_s = \frac{(1 + v)}{6E}(2S_{yp}^2)$$

$$\therefore (\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 = 2S_{yp}^2$$